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STATISTICAL APPLICATIONS

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Trivial resonances for a system of Klein-Gordon equations and statistical applications

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Abstract

These notes are about a talk I gave at séminaire Laurent Schwartz on the third of February 2026. In this talk, I presented a result from [dSST25], which is a joint work with Annalaura Stingo (from École polytechnique) and Arthur Touati (from CNRS/ Université de Bordeaux). The result is an obstruction to a kinetic regime, in the framework of wave turbulence, for a non Hamiltonian equation, namely a system of Klein–Gordon equations with generic quadratic nonlinearities; and the derivation of an effective equation in the weak turbulence asymptotics.

1 Introduction

Wave turbulence is the study of the dynamics of interacting waves at a mesoscopic scale. Let us make a parallel with the well-known problem of the derivation of the Boltzmann equation.

To derive the Boltzmann equation, one considers $N \gg 1$ particles modeled as hard spheres that interact through elastic shocks. As one should eventually let N go to infinity, it is important to control the size of the interaction between particles, that is the number of elastic shocks per unit time; in other words, one should control the size $\varepsilon \ll 1$ of particles. In fact, the kinetic regime, that is to say the relation between ε and N that allows to observe a mesoscopic behavior, is obtained when ε is comparable to $N^{-1/2}$ (in dimension 3). One important feature of the derivation of the Boltzmann equation is *propagation of chaos*: it is the fact that if particles are initially independent, even though they lose their independence through their interaction, they regain it in the asymptotical regime $N \rightarrow \infty$. The Boltzmann equation describes the dynamics of the marginal law of a particle and, since particles are asymptotically independent, that describes the full law of the system of particles. The literature about the derivation of the Boltzmann equation is paramount: not to be exhaustive, we mention here the celebrated works of Lanford [Lan75, Lan76, LI81], the equally celebrated works of Bodineau, Gallagher, Saint-Raymond, Simonella and Texier [GSRT13, BGS16, BGSRS23] and the recent works of Deng, Hani and Ma [DHM24, DHM25].

In wave (weak) turbulence, the analog of particles are waves, namely the Fourier coefficients of solutions to a dispersive equation. To control the size of the interaction, one multiplies the non linearity, through which waves interacts, by a parameter $\varepsilon \ll 1$, as in the generic form

$$i\partial_t U = WU + \varepsilon F(U) \tag{1}$$

where U is the unknown, where W is some Fourier multiplier by a (real-valued) *dispersion relation* ω , and where F is some polynomial nonlinearity. The number of waves is controlled in the following way. Usually, these dispersive equations as in (1), as posed on a torus of size $L \gg 1$, that is $L\mathbb{T}^d$, integer d being the dimension. In Fourier mode, the phase space being $\frac{1}{L}\mathbb{Z}^d$, the initial datum writes

$$U_{\text{in}} = \sum_{k \in (1/L)\mathbb{Z}^d} \chi(k) \frac{e^{ikx}}{L^{d/2}} g_k$$

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where $\chi(k)g_k$ are the Fourier coefficients. Assume that χ is a function of $\mathbb{R}^d$ supported in the ball of center 0 and radius 1 – we will go back to the family $(g_k)_k$ soon enough. Then, the number of activated Fourier modes is of order L^d . We say in this context that there are L^d waves. There are different behaviors and effective equations in the limit $L \rightarrow \infty$ and $\varepsilon \rightarrow 0$ depending on the regime we chose between ε and L , we describe this at length in this text. For the sake of this introduction, let us say that it is usually a power regime, as in $\varepsilon = L^{-\beta}$ with $\beta > 0$. There is also a phenomenon of propagation of chaos, that is, if initially waves are supposed independent, in other words that $(g_k)_k$ is a family of independent random variables (traditionally Gaussians), then waves will end up independent in the asymptotic regime. Moreover, some sort of central limit theorem applies and at the limit the family of waves $(\widehat{U}(k))_k$ behaves like a family of independent Gaussian variables. Effective equations describe the behavior of moments of order 2 of waves:

$$\mathbb{E}(\overline{B(\widehat{U}(k, tT), \widehat{U}(k, tT))})$$

where k is a wave number, where $T \gg 1$ is a large timescale and where B is some bilinear map. Since $(\widehat{U}(k))_k$ are believed to be independent Gaussians, this should suffice to describe the dynamics of their full law. A well-known effective equation is the so-called *wave kinetic equation* derived successfully by Deng and Hani [DH26] from the microscopic cubic Schrödinger dynamics. We mention also [LS11, ST21]. Again, this list is not exhaustive, we refer to [dSST25] for a more comprehensive list.

We sum up this analogy in the following table.

Boltzmann	Weak turbulence
N particles	L^d waves $U_{\text{in}} = \sum_{k \in (1/L)\mathbb{Z}^d} \chi(k)(e^{ikx}/L^{d/2})g_k$
size of particles $\varepsilon \rightarrow 0$	size of the NL ε
few interactions	$i\partial_t U = WU + \varepsilon F(U)$
Regime $\varepsilon = N^{-1/2}$	$\varepsilon = L^{-\beta}$, $\beta > 0$
Propagation of chaos	Propagation of chaos
Marginal law of a particle	$\mathbb{E}(\overline{B(\widehat{U}(k, tT), \widehat{U}(k, tT))})$
Boltzmann equation	EX : Wave kinetic equation

Let us now go back to the motivation behind the choice of setting equations on the torus of size $L \gg 1$. It goes back to the modeling of the problem. The main idea is to describe *chaos at equilibrium* not for particles but for waves as we have already mentioned. Laws of physics are often invariant under the action of space translations, and thus so should be the model behind these equilibria. Unfortunately, dispersive equations are not so well-posed on homogeneous data, as their energy space is often a space of localized functions. The idea to overcome this difficulty is to sample the full space by tori of increasing size, under the conviction that at equilibrium, what happens on a window of size L should be the same as on the whole space.

2 Framework

We consider a system of Klein–Gordon equations

$$\begin{cases} \partial_t^2 u - \Delta u + u = \varepsilon Q^0(v, v) \\ \partial_t^2 v - \Delta v + v = \varepsilon Q^1(u, u) \end{cases} \quad (2)$$

where $u : \mathbb{R} \times L\mathbb{T}^d \rightarrow \mathbb{R}$ and $v : \mathbb{R} \times L\mathbb{T}^d \rightarrow \mathbb{R}$ are the unknowns, where $\varepsilon = L^{-\beta}$ with $\beta \in (0, 1/2)$ and where the generic nonlinearities are described in Fourier mode as

$$\widehat{Q}^\eta(v, w)(k) = L^{-d/2} \sum_{k_1 + k_2 = k} Q^\eta(k_1, k_2) \widehat{v}(k_1) \widehat{w}(k_2)$$

for $\eta = 0, 1$ and where $Q^\eta : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ serves as an abuse of notation.

Note that if the function Q^η is constant to 1, then the nonlinearity Q^η is the product in the physical space, that is $Q^\eta(v, w) = vw$.

The motivation behind this system comes a long way. Initially, with Stingo and Touati, we wanted to study a toy model for general relativity. Touati suggested the $U(1)$ symmetry system. The nonlinearity for this system is rather complicated (not polynomial) and I suggested that, to get a nicer equation in the limit, we focused on a simpler toy model, such as a system of wave equations with nonlinearity with null structure. The three of us figured out relatively quickly that there was an obstruction to a kinetic regime that we wanted to highlight. As it happened, there was a lot of difficulties for this model due to singularities in low frequencies, this is why we turned to a regularization in low frequencies of the wave equation, that is the Klein–Gordon equation. The Klein–Gordon dispersion relation does not require specific null structure to avoid resonances of order 1 as we will see. This is why the nonlinearities are generic.

Assumption 2.1. We assume that the functions Q^η satisfy:

- for $\eta = 0, 1$, we have that Q^η is symmetric;
- for $\eta = 0, 1$ and for all $k_1, k_2 \in \mathbb{R}^d$, we have $\overline{Q^\eta(k_1, k_2)} = Q^\eta(-k_1, -k_2)$;
- for $\eta = 0, 1$, we have that Q^η is bounded and of class $\mathcal{C}^1$;
- for $\eta = 0, 1$ and for all $k \in \mathbb{R}^d$, we have $Q^\eta(k, -k) = 0$.

Remark 2.1. The symmetry assumption is unnecessary, functions Q^η can always be chosen symmetric. The second assumption is here to ensure the real-valued character of the solution. The third one is technical. The last one is not really necessary to conduct our analysis, but it highly simplifies the expression of our theorem.

There is a standard way to transform the Klein–Gordon equation, a second order equation with real-valued solutions, into a first order equation with complex-valued solutions: the half-Klein–Gordon equation. Set $U = \langle \nabla \rangle u + i\partial_t u$ and $V = \langle \nabla \rangle v + i\partial_t v$ with $\langle \nabla \rangle = \sqrt{1 - \Delta}$. Functions u and v satisfy (2) if and only if functions U and V satisfy

$$\begin{cases} i\partial_t U = \langle \nabla \rangle U + \varepsilon Q^0(\langle \nabla \rangle^{-1} \text{Re} V, \langle \nabla \rangle^{-1} \text{Re} U) \\ i\partial_t V = \langle \nabla \rangle V + \varepsilon Q^1(\langle \nabla \rangle^{-1} \text{Re} U, \langle \nabla \rangle^{-1} \text{Re} V) \\ U(t=0) = U_{\text{in}}, \quad V(t=0) = V_{\text{in}}. \end{cases} \quad (3)$$

We consider the Cauchy problem (3) posed on the torus $L\mathbb{T}^d$ with initial datum

$$\begin{aligned} U_{\text{in}} &= L^{-d/2} \sum_{k \in \frac{1}{L}\mathbb{Z}^d} e^{ikx} \mu_k^0 \\ V_{\text{in}} &= L^{-d/2} \sum_{k \in \frac{1}{L}\mathbb{Z}^d} e^{ikx} \mu_k^1 \end{aligned}$$

where $((\mu_k^0, \mu_k^1))_k$ is a family of independent centred $\mathbb{C}^2$ Gaussian variables such that there exists continuous functions with compact support $M^\eta, M^\times$ such that $\mathbb{E}(|\mu_k^\eta|^2) = M^\eta(k)$ and $\mathbb{E}(\overline{\mu_k^0} \mu_k^1) = M^\times(k)$.

Our objective is to derive an effective dynamics for the correlations of the Fourier coefficients of U and V at an appropriate time scale: namely, to derive an equation for the dynamics of

$$\mathbb{E}(|\widehat{U}(Tt, k)|^2), \quad \mathbb{E}(|\widehat{V}(Tt, k)|^2), \quad \mathbb{E}(\widehat{U}(Tt, k) \overline{\widehat{V}(Tt, k)}).$$

as $\varepsilon \rightarrow 0$ where T is appropriate (we will see why we choose the timescale $T = \varepsilon^{-2}$).

Remark 2.2. These quantities are relevant for the following reasons. First, the law of the initial datum is invariant under the action of space translations. This implies that $\mathbb{E}(\widehat{U}(t, k) \overline{\widehat{U}(t, k')}) = 0$ if $k \neq k'$. Secondly, *a priori*, there are not enough invariances to ensure that $\mathbb{E}((\widehat{U}(t, k))^2) \equiv 0$ but at the limit, the dynamics for these quantities is trivial. The law of the initial datum is invariant under the action of the linearized flow, hence T must be bigger than the nonlinear time.

Remark 2.3. Remark that the law of the initial datum being invariant under space translations, the L^2 norm (or any Sobolev norm) of the initial datum is of size $L^{d/2}$.

3 Timescales

3.1 Notation

We set $X^0(t) = e^{it\langle \nabla \rangle} U(t)$, $X^1(t) = e^{it\langle \nabla \rangle} V(t)$. For $\eta \in \{0, 1\}$, we set $\bar{\eta} = 1 - \eta$. For all $A \in \mathbb{C}$, we set $A^+ = A$ and $A^- = \bar{A}$, the complex conjugate of A .

Equation (3) writes in Duhamel form and in Fourier mode

$$\widehat{X}^\eta(t, k) = \widehat{X}_{\text{in}}^\eta(k) - i \frac{\varepsilon}{L^{d/2}} \int_0^t \sum_{\substack{k_1+k_2=k \\ (\iota_1, \iota_2) \in \{\pm\}^2}} e^{is\Delta_{k_1, k_2}^{\iota_1, \iota_2}} q^\eta(k_1, k_2) \widehat{X}^{\bar{\eta}}(s, \iota_1 k_1)^{\iota_1} \widehat{X}^{\bar{\eta}}(s, \iota_2 k_2)^{\iota_2} \quad (4)$$

where $X_{\text{in}}^0 = U_{\text{in}}$, where $X_{\text{in}}^1 = V_{\text{in}}$ and where

$$\Delta_{k_1, k_2}^{\iota_1, \iota_1} = \langle k_1 + k_2 \rangle - \iota_1 \langle k_1 \rangle - \iota_2 \langle k_2 \rangle, \quad q^\eta(k_1, k_2) = \frac{Q^\eta(k_1, k_2)}{\langle k_1 \rangle \langle k_2 \rangle}.$$

3.2 Resonances of order 1

In the wave turbulence literature, there are many regimes possible and it is driven by the resonances of the equation. If exact resonances, that is couples k_1, k_2 satisfying $\Delta_{k_1, k_2}^{\iota_1, \iota_1} = 0$, drives the asymptotic dynamics, we say that the regime is discrete (for instance [FGH16, DKMV23]). If quasi-resonances ($0 \neq \Delta_{k_1, k_2}^{\iota_1, \iota_1} \ll 1$) takes precedence over exact resonances, the regime is said kinetic (see [DH26, CG25, BGHS21]).

In our case, for all $\iota_1, \iota_2, k_1, k_2$, we have that $\Delta_{k_1, k_2}^{\iota_1, \iota_1} \neq 0$ and

$$|\Delta_{k_1, k_2}^{\iota_1, \iota_2}| \geq \langle k_1 \rangle^{-1/2} \langle k_2 \rangle^{-1/2}.$$

There are neither exact nor quasi-resonances!

3.3 Poincaré–Dulac normal form

We perform a Poincaré–Dulac normal form on the system (3), that is more or less an integration by parts on the integral in (4). This transforms the fixed point problem (4) into the following fixed point problem

$$\begin{aligned} \widehat{X}^\eta(t, k) &= \widehat{X}_{\text{in}}^\eta(k) \\ &\quad - \frac{\varepsilon}{L^{d/2}} \sum_{\substack{k_1+k_2=k \\ (\iota_1, \iota_2) \in \{\pm\}^2}} e^{it\Delta_{k_1, k_2}^{\iota_1, \iota_2}} \frac{q^\eta(k_1, k_2)}{\Delta_{k_1, k_2}^{\iota_1, \iota_2}} \widehat{X}^{\bar{\eta}}(t, \iota_1 k_1)^{\iota_1} \widehat{X}^{\bar{\eta}}(t, \iota_2 k_2)^{\iota_2} \\ &\quad + \frac{\varepsilon}{L^{d/2}} \sum_{\substack{k_1+k_2=k \\ (\iota_1, \iota_2) \in \{\pm\}^2}} \frac{q^\eta(k_1, k_2)}{\Delta_{k_1, k_2}^{\iota_1, \iota_2}} \widehat{X}_{\text{in}}^{\bar{\eta}}(\iota_1 k_1)^{\iota_1} \widehat{X}_{\text{in}}^{\bar{\eta}}(\iota_2 k_2)^{\iota_2} \\ &\quad - i \frac{\varepsilon^2}{L^d} \int_0^t \sum_{\substack{k_1+k_2+k_3=k \\ (\iota_1, \iota_2, \iota_3) \in \{\pm\}^3}} e^{is\Delta_{k_1, k_2, k_3}^{\iota_1, \iota_2, \iota_3}} q_{\iota_2}^\eta(k_1, k_2, k_3) \widehat{X}^\eta(s, \iota_1 k_1)^{\iota_1} \widehat{X}^{\bar{\eta}}(s, \iota_2 k_2)^{\iota_2} \widehat{X}^\eta(s, \iota_3 k_3)^{\iota_3} \end{aligned} \quad (5)$$

where $\Delta_{k_1, k_2, k_3}^{\iota_1, \iota_2, \iota_3} = \langle k_1 + k_2 + k_3 \rangle - \iota_1 \langle k_1 \rangle - \iota_2 \langle k_2 \rangle - \iota_3 \langle k_3 \rangle$ and where $q_{\iota_2}^\eta(k_1, k_2, k_3)$ has an expression similar to $q_{\iota_2}^\eta(k_1, k_2)$.

Remark 3.1. Note that since functions Q^η are bounded, the functions $(k_1, k_2) \mapsto q^\eta(k_1, k_2)/\Delta_{k_1, k_2}^{\iota_1, \iota_2}$ are also bounded. This implies that admittedly the second and third lines of (5) will vanish at the limit $\varepsilon \rightarrow 0$.

Remark 3.2. We have that $\Delta_{k_1, k_2, k_3}^{\iota_1, \iota_2, \iota_3} = 0$ at least in trivial cases: $k_1 + k_2 = 0$, $\iota_1 + \iota_2 = 0$, $\iota_3 = +$, and those obtained by permuting indices. We call these trivial cases *trivial resonances*.

4 Result

4.1 Main result

Heuristic computations suggest that regardless of the regime, the asymptotical dynamics of

$$\mathbb{E}(|\widehat{U}(Tt, k)|^2), \mathbb{E}(|\widehat{V}(Tt, k)|^2), \mathbb{E}(\overline{\widehat{U}(Tt, k)}\widehat{V}(Tt, k))$$

are driven by the trivial resonances and by the effective equation

$$\left\{ \begin{array}{l} \partial_t \rho^0(t, \xi) = 4\rho^0(t, \xi) \int_{\mathbb{R}^d} \operatorname{Im} \left(N^0(\xi, \zeta) \overline{\rho^\times(t, \zeta)} \right) d\zeta, \\ \partial_t \rho^1(t, \xi) = 4\rho^1(t, \xi) \int_{\mathbb{R}^d} \operatorname{Im} \left(N^1(\xi, \zeta) \rho^\times(t, \zeta) \right) d\zeta, \\ \partial_t \rho^\times(t, \xi) = 2i\rho^\times(t, \xi) \int_{\mathbb{R}^d} \left(P_+(\xi, \zeta) \overline{\rho^\times(t, \zeta)} + P_-(\xi, \zeta) \rho^\times(t, \zeta) \right) d\zeta, \\ (\rho^0, \rho^1, \rho^\times)|_{\{t=0\}} = (M^{0,0}, M^{1,1}, M^{0,1}), \end{array} \right. \quad (6)$$

where

$$4(2\pi)^d P_\pm(\xi, \zeta) = \frac{\overline{Q^0(\xi, \mp\zeta)Q^1(\xi \mp \zeta, \pm\zeta)} - Q^1(\xi, \mp\zeta)Q^0(\xi \mp \zeta, \pm\zeta)}{\langle \xi \rangle \langle \zeta \rangle^2 (m \pm 2(\xi \cdot \zeta - \langle \xi \rangle \langle \zeta \rangle))}.$$

Remark 4.1. Note that :

- the quantities $\rho^0(\xi)\rho^1(\xi)/|\rho^\times(\xi)|^2$ are conserved;
- If $P_+ = \overline{P_-}$ then $|\rho^\times(\xi)|$ is conserved;
- If $Q^0 = \overline{Q^1}$, then $\rho^\times$ is conserved, but for a given $\xi \in \mathbb{R}^d$, either $\rho^0(\xi)$ or $\rho^1(\xi)$ grows exponentially fast to ∞ as time goes to ∞ while the other decays exponentially fast to 0.

Theorem 4.1. Assume $d \geq 3$. And let $s > d/2$. There exists $\delta \ll 1$ such that the effective system (6) is well-posed with a solution in $\mathcal{C}([0, \delta], \mathcal{C}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$. For all L , there exists a set $\mathcal{E}_L$ whose measure is going to 1 as L goes to ∞ such that the half Klein-Gordon system is well-posed in $\mathcal{C}([0, \varepsilon^{-2}\delta], H^s)$. And we have

$$\lim_{L \rightarrow \infty} \sup_{k \in (1/L)\mathbb{Z}^d} \left| \frac{\mathbb{E}(\mathbf{1}_{\mathcal{E}_L} |\widehat{X}^\eta(\varepsilon^{-2}t, k)|^2) - \rho^\eta(t, k)}{\mathbb{E}(\mathbf{1}_{\mathcal{E}_L} \overline{\widehat{X}^0(\varepsilon^{-2}t, k)} \widehat{X}^1(\varepsilon^{-2}t, k)) - \rho^\times(t, k)} \right| = 0.$$

Remark 4.2. • The time threshold δ is small but independent from L (or equivalently from ε).

- As we have already discussed in our analysis of resonances of order 1, timescale ε^{-2} is in fact the nonlinear time.
- The size of the initial datum is growing with L in such a way that from standard deterministic methods we would get a time of well-posedness of (3) going to 0 rather than ∞ as ε is going to 0. Therefore, it is crucial to exploit the probabilistic structure of the initial datum to conduct our analysis.
- The effective equation (6) is not kinetic: it is not driven by quasi-resonances but by exact trivial resonances.
- Recall that we are under the regime $\varepsilon = L^{-\beta}$ with $\beta \in (0, \frac{1}{2})$. Our point is that β may be as small as possible. The threshold $1/2$ is there for technical reasons, namely to make a Riemann sum converge to an integral. However, heuristics suggest that this threshold is irrelevant, which makes us hopeful that this convergence could be numerically tested.

4.2 Comparison with the Schrödinger equation

We compare our result on Klein–Gordon systems with derivation of mesoscopic dynamics when the microscopic dynamics is the cubic Schrödinger equation

$$i\partial_t u = -\Delta u + \varepsilon|u|^2 u.$$

In the physics literature (for instance [Naz11, ZLF25]), there is a discussion between two¹ possible regimes either the size of the torus (or sample) L is taken to ∞ faster or rather before ε is taken to 0 or the reverse. In the first case, we talk of a kinetic regime, in the latter, of a discrete regime. For the Schrödinger equation, this amounts to taking the regime $\varepsilon = L^{-\beta}$, with $\beta \in (0, 1)$ in the kinetic case and $\beta > 1$ in the discrete one. Kinetic regimes lead to the kinetic wave equation (called after kinetic equations due to its resemblance with the [static] Boltzmann equation). Discrete regimes lead to the *continuous resonant equation*, see [FGH16]. Wave kinetic equations admit steady-states called KZ spectra modeling transfers of energy between frequencies, see [ZLF25], and the one derived from the Schrödinger equation is irreversible, see [DH23b]. In the derivation of the continuous resonant equation, one finds obstructions to the kinetic regime and no mechanism to trigger irreversibility, therefore, we may assume that this is a reversible equation. While the physical quantities involved in discriminating a kinetic versus a discrete regime are well-identified, see Chapter 10 of [Naz11], their mathematical meaning was unclear until Deng and Hani figured out in [DH23a] that it came from a competition between exact and quasi-resonances. We give a summary in the following table.

Kinetic	Discrete
$\varepsilon = L^{-\beta}, \beta \in (0, 1)$	$\varepsilon = L^{-\beta}, \beta > 1$
$L \rightarrow \infty$ then $\varepsilon \rightarrow 0$	$\varepsilon \rightarrow 0$ then $L \rightarrow \infty$
Wave kinetic equation	Continuous resonant equation
KZ spectra	Obstructions to the kinetic regime
Irreversibility	Reversibility (?)
Quasi-resonances	Exact resonances

In our case, that is for (2) and Theorem 4.1, we see that the regime is $\varepsilon = L^{-\beta}$, with $\beta \in (0, 1/2)$. In other words, L is taken to ∞ faster than ε is taken to 0. This should lead to a kinetic regime. However, in our derivation of (6), there is no mechanism to trigger irreversibility, and Equation (6) is driven by exact (and even trivial) resonances. We conclude that there exists no kinetic regime, at least in our setting.

4.3 Quasi- and exact resonances

In the heuristic derivation of effective dynamics, there is a competition between exact resonances

$$E = \{(\xi_1, \xi_2, \xi_3) \in (1/L)\mathbb{Z}^d \mid \omega(\xi_1 - \xi_2 + \xi_3) - \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) = 0\},$$

where ω is the dispersion relation, and quasi-resonances

$$Q = \{(\xi_1, \xi_2, \xi_3) \in (1/L)\mathbb{Z}^d \mid 0 \neq |\omega(\xi_1 - \xi_2 + \xi_3) - \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3)| \leq \varepsilon^2\}.$$

The question is which set E or Q is bigger. If Q is bigger, the regime is kinetic; if E is bigger, the regime is discrete². The size of E is big, because the trivial resonances are many, it is always bigger than that of Q .

For the Schrödinger equation, however, invariance under $U(1)$ yields that the contribution of trivial resonances cancel out and we have to replace E with

$$\tilde{E} = \{(\xi_1, \xi_2, \xi_3) \in (1/L)\mathbb{Z}^d \mid \omega(\xi_1 - \xi_2 + \xi_3) - \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) = 0 \wedge \xi_2 \neq \xi_1, \xi_3\}.$$

¹I am voluntarily being a bit reductive here but more information may be found in Chapter 10 of [Naz11]

²If they are of comparable sizes, the regime is called mesoscopic, which seems not a very good choice of words retrospectively, and the asymptotical dynamics is quite involved, see Section 10.3 of [Naz11]

The size of $\tilde{E}$ is a lot smaller than that of E , and it is smaller than that of Q when $\varepsilon = L^{-\beta}$ with $\beta \in (0, 1)$.

Note that if L is fixed and ε goes to 0, since the phase space $(1/L)\mathbb{Z}^d$ is discrete, set Q will eventually become empty under a certain threshold for ε .

5 Ideas of proof

5.1 General strategy

The proof follows the following guide line: first find an appropriate Ansatz for the solution X to the fixed point (5), then identify the resonant structures in the Ansatz. Again, one difficult feature is that the norm of the initial datum grows with L as $\|X_{\text{in}}\|_{H^s} \sim L^{d/2}$.

To get this Ansatz, we iterate the Duhamel formula. Schematically, ignoring the quadratic terms for the sake of simplicity, the solution X satisfies the fixed point

$$X = X_{\text{in}} + \varepsilon^2 C(X, X, X)$$

where C is some 3-linear map.

Define X_n by induction on $n \in \mathbb{N}$ as $X_0 = X_{\text{in}}$, and

$$X_{n+1} = \varepsilon^2 \sum_{n_1+n_2+n_3=n} C(X_{n_1}, X_{n_2}, X_{n_3}).$$

The Ansatz we chose is $Y = \sum_{n \leq \ln L} X_n$. Our analysis consists in proving that X is indeed well approximated by Y . In this view, set $Z = X - Y$, it solves

$$Z = \mathfrak{S} + \mathfrak{L}Z + \mathfrak{Q}(Z, Z) + \mathfrak{C}(Z, Z, Z)$$

with

$$\mathfrak{S} = \varepsilon^2 \sum_{\substack{n_1+n_2+n_3 > \ln L \\ n_1, n_2, n_3 \leq \ln L}} C(X_{n_1}, X_{n_2}, X_{n_3}) \quad (7)$$

$$\mathfrak{L}Z = \varepsilon^2 C(Y, Y, Z) + \varepsilon^2 C(Y, Z, Y) + \varepsilon^2 C(Z, Y, Y), \quad (8)$$

where Q is some quadratic term, and with initial datum 0. The difficulties lie in proving that the equation for the remainder Z is well-posed up to times of order ε^{-2} and that the solution is small. The quadratic and cubic terms are treated in a standard deterministic way, we do not insist on them. The key feature is that the source term $\mathfrak{S}$ is small and $1 - \mathfrak{L}$ is invertible. The ideas behind the treatment of the linear term $\mathfrak{L}$ are quite similar to the ones behind the treatment of the source term $\mathfrak{S}$. These latter ones require some technical adjustments that we will not comment upon in these notes.

We therefore turn to the source term $\mathfrak{S}$. There is an ε^2 in (7) that looks like good news but unfortunately it is not good enough. The invariance under space translations is preserved by the cubic term C in the sense that if τ is a space translation then

$$C(\tau \cdot, \tau \cdot, \tau \cdot) = \tau C(\cdot, \cdot, \cdot).$$

This ensures that X_n is homogeneous: its law is invariant under the action of space translations. Hence the dependence over L of its norm is of order $L^{d/2}$, which is too much to be compensated by ε^2 . What is more, the timescale ε^{-2} needs be considered. The key point is that the dependance in n of the norm of X_n is much more favorable in the sense that up to times of order ε^{-2} , we have $\|X_n\|_{H^s} \sim L^{d/2} \delta^n$ which is less than $L^{d/2+\ln \delta}$ for $n \gtrsim \ln L$.

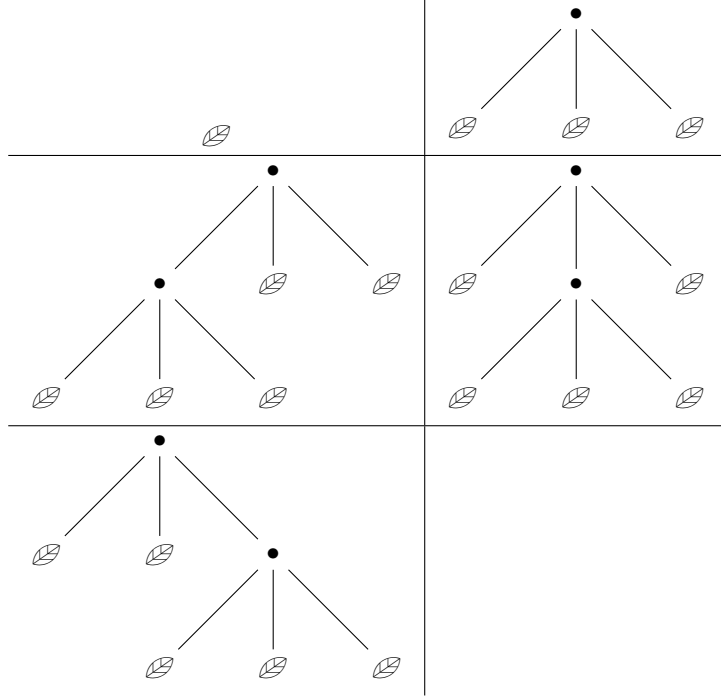
5.2 Diagrammatic

We have now reduced our problem to proving that $\|X_n\|_{H^s} \sim L^{d/2} \delta^n$. We do so by remarking that the recursive structure of our sequence $(X_n)_n$ is that of ternary trees. Instead of giving proper definitions we provide some examples and refer the curious reader to [dSST25], Subsection 2.1.

The first terms of the sequence $(X_n)_n$ write

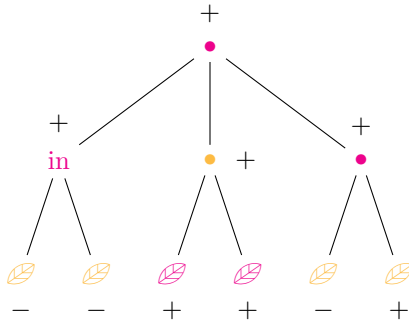
$$\begin{aligned}
 X_0 &= X_{\text{in}} \\
 X_1 &= \varepsilon^2 C(X_{\text{in}}, X_{\text{in}}, X_{\text{in}}) \\
 X_2 &= \varepsilon^2 C(X_1, X_{\text{in}}, X_{\text{in}}) + \varepsilon^2 C(X_{\text{in}}, X_1, X_{\text{in}}) + \varepsilon^2 C(X_{\text{in}}, X_{\text{in}}, X_1) \\
 &= \varepsilon^4 C(C(X_{\text{in}}, X_{\text{in}}, X_{\text{in}}), X_{\text{in}}, X_{\text{in}}) + \varepsilon^4 C(X_{\text{in}}, C(X_{\text{in}}, X_{\text{in}}, X_{\text{in}}), X_{\text{in}}) \\
 &\quad + \varepsilon^4 C(X_{\text{in}}, X_{\text{in}}, C(X_{\text{in}}, X_{\text{in}}, X_{\text{in}}))
 \end{aligned}$$

We represent them as trees as in the following table



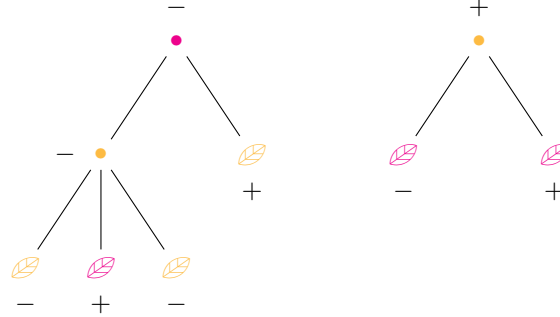
The leaves leaf represent the initial datum X_{in} ; the bullets $\bullet$ represent the interaction C . The upper left case represent X_0 ; upper right is X_1 ; middle left is $\varepsilon^2 C(X_1, X_{\text{in}}, X_{\text{in}})$; middle right is $\varepsilon^2 C(X_{\text{in}}, X_1, X_{\text{in}})$; bottom left is $\varepsilon^2 C(X_{\text{in}}, X_{\text{in}}, X_1)$; the sum of middle left, middle right and bottom left is X_2 .

Of course, this is schematic, we give an example of tree that appears in [dSST25].

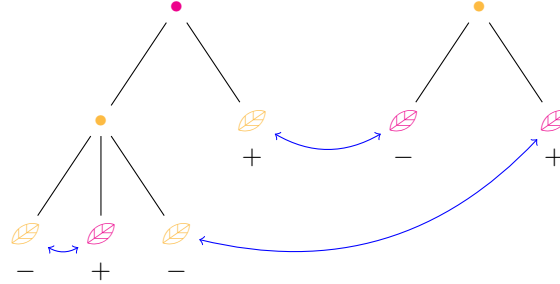


The colors as in yellow leaf or pink leaf are here to remember that we work on a system, they play the same role as parameter $\eta \in \{0, 1\}$ in (4). The colors of the leaves and nodes of a tree are completely determined by the color of its root. The signs play the same role as the signs in (4); they may be chosen arbitrarily, this echos with the fact that solutions to (2) are real-valued. The binary and ternary nodes correspond to respectively quadratic and cubic terms in (5); there are two quadratic terms in (5), hence there are two type of binary nodes: in and $\bullet$.

This allows to expand X_n in terms of trees, or of *Feynman diagrams*. But we wish to exploit the probabilistic structure of the initial datum and thus to estimate the average norm of X_n . In other words, what we are really interested in is estimating $\mathbb{E}(|X_n|^2)$ or more generally $\mathbb{E}(\overline{X_n}X_m)$. To do so, we expand $\mathbb{E}(\overline{X_n}X_m)$ thanks to what we call *couplings*. Couplings start with a couple of trees, which is inherited from the expansion of X_n in trees as in the following picture.



This picture represent a couple of trees involved in the expansion of $\overline{X_2}X_1$. We take its expectation: since all the probability lies within the initial datum, this amounts to taking the expectation of the product of the leaves $\text{orange leaf pink leaf orange leaf pink leaf}$. The initial datum is a Gaussian, it satisfies the Wick formula ([Jan97], Chapter 1), which allows to decompose the expectation of a product of Gaussian variables as the sum of products of expectations of pairs of Gaussians. Therefore, to expand $\mathbb{E}(\overline{X_n}X_m)$, we need to consider all couples of trees equipped with a pairing of their leaves, as in the following picture. The curved blue double arrows represent a possible pairing.



We now estimate the contribution of each coupling. Roughly, a coupling with n nodes contributes less than δ^n . The issue is that there are a lot of couplings, something like $n!$ couplings with n nodes. The factor δ^n cannot compensate this loss (remember that n is taken up to $\ln L$ that is, as $L \rightarrow \infty$, arbitrarily large). Fortunately, nodes that represent a non-resonant interaction get a better estimate: each one will yield a power of ε say ε^α with $\alpha > 0$. Couplings with only non-resonant interactions contribute therefore as $n!\varepsilon^{\alpha n}$, which, since $n \leq |\ln \varepsilon|$ is less than $(|\ln \varepsilon|\varepsilon^\alpha)^n$.

Resonant interactions imply a lot of structure: for instance the number of couplings with n nodes that all contribute in a resonant way is controlled geometrically, such that factor δ^n can compensate this loss.

To summarize, the core of the proof consists in expanding X_n thanks to trees or Feynman diagrams, expanding $\mathbb{E}(\overline{X_m}X_n)$ thanks to couplings, estimating $\mathbb{E}(\overline{X_m}X_n)$ by identifying the resonant structures in the couplings.

6 Conclusion

To conclude, let us mention that we provided an example of microscopic dynamics such that the mesoscopic dynamics is non trivial but driven by trivial resonances, that we adapted our diagrammatic to real-valued solutions, normal forms and systems of equations. We also provided a proof that is relatively robust with regard to the dispersion relation. To do so, we had to develop new (relatively simple) combinatorial tools.

Finally, for readers that are interested in the original text [dSST25], let us mention that I provided a companion program that randomly creates illustrations [dS]. And for readers who are interested in an introduction of wave turbulence following the paper of Deng and Hani, let us mention [dS24].

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