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ON TRAVELLING WATER WAVES WITH GENERAL VORTICITY

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ABSTRACT. We consider the classical problem of constructing two-dimensional travelling periodic gravity waves on water of finite depth. In the last two decades much progress has been made for waves with nonzero vorticity, also in the presence of critical layers and allowing the surface to overhang. We review recent progress of the theory, in particular with regard to finding a reformulation in the most general case which is amenable to tools from nonlinear functional analysis such as global bifurcation.

These notes originate in a talk held in the Séminaire Laurent Schwartz in February 2026 and are mainly based on joint work with Erik Wahlén [19].

1. INTRODUCTION

The water wave problem is one of the most classical problems in mathematics, having been studied since Euler, Lagrange and Legendre in the 18th century. It has immensely influenced other mathematical fields (e.g., it was the main motivation for Cauchy to develop his theory of complex analysis) and can be viewed as a benchmark for the progress of mathematics in general.

We are concerned with two-dimensional waves. These are wave trains that propagate in one horizontal direction and do not change their shape with respect to the other horizontal coordinate. These are quite common in nature; for example, if one observes waves near a beach looking towards the ocean, the picture typically remains (almost) the same when the observer moves along the coastline. From a mathematical point of view, two-dimensional waves are more tractable than fully three-dimensional waves because of three main reasons:

- By incompressibility, the velocity field can be recovered from just one scalar function (see (1.2)).
- Vorticity is conserved along particle trajectories (see (1.3)).
- Complex analysis is applicable.

A major challenge is that the water wave problem is a free boundary problem in that the determination of the free surface (the interface between water and air) is part of the problem. Thus, in general one has to determine the fluid domain Ω , bounded above by the free surface S and below by a rigid bed B , and the fluid quantities $\mathbf{u} = (u_1, u_2)$ (the velocity field) and p (the pressure). Both the domain and the fluid quantities depend on time in general. We assume the rigid bed to be a perfectly horizontal line (at height $-h$, say, where h refers to some mean depth), that is,

$$B = \mathbb{R} \times \{-h\}.$$

The equations describing the evolution of the water are:

- The *incompressible Euler equations* in Ω :

$$\frac{D\mathbf{u}}{Dt} = -\nabla(p + gY), \quad \nabla \cdot \mathbf{u} = 0.$$

Here we introduced the material derivative

$$\frac{D}{Dt} := \partial_t + (\mathbf{u} \cdot \nabla),$$

describing the rate of change when following particle trajectories. Moreover, $g > 0$ is the gravitational constant, and we denote by (X, Y) the spatial coordinates in $\mathbb{R}^2$.

- The *kinematic boundary condition* on S and B : It simply says that fluid particles never leave the fluid domain. If a part of the boundary is given by some equation $F(t, X, Y) = 0$, the boundary condition reads

$$\frac{DF}{Dt} = 0.$$

At the bed B , this equation simplifies to

$$u_2 = 0.$$

- Since we are dealing with a free boundary problem, there needs to be one more boundary condition on S . It is a pressure balance, the *dynamic boundary condition*:

$$p = p_{\text{atm}},$$

that is, the pressure is continuous across the water-air interface. Here, the atmospheric pressure p_{atm} is assumed to be a constant. Importantly, we have neglected surface tension here. The physical motivation for this is that we are interested in waves on the ocean with wavelengths of up to several hundreds of meters, while surface tension only becomes relevant on the much smaller scale of a few centimetres; thus, ocean waves are driven by gravity, not by surface tension. The mathematical motivation is that the pure gravity problem is more challenging (as surface tension has a regularising effect) and interesting (since one can prove more structural properties; see Section 2.4).

We do not deal with the fully time-dependent problem, but rather restrict ourselves to *travelling* waves. These propagate in the horizontal direction with a fixed wave speed c and without change of shape. In a frame moving with the wave at speed c , the wave and the equations therefore become stationary. This is Galilean invariance and why travelling waves are also called *steady* waves. More precisely, all quantities depend on (t, X) only through the combination $X - ct$. With some abuse of notation, writing X for $X - ct$ and $\mathbf{u}$ for $\mathbf{u} - (c, 0)$, the steady equations are

$$(1.1a) \quad \frac{D\mathbf{u}}{Dt} = -\nabla(p + gY) \quad \text{in } \Omega,$$

$$(1.1b) \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega,$$

$$(1.1c) \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } S,$$

$$(1.1d) \quad u_2 = 0 \quad \text{on } B,$$

$$(1.1e) \quad p = p_{\text{atm}} \quad \text{on } S,$$

where $\mathbf{n}$ is the outward unit normal of S and the material derivative is now simply

$$\frac{D}{Dt} = \mathbf{u} \cdot \nabla.$$

At this level, we have to solve for the fluid domain Ω and the three scalar functions u_1 , u_2 and p .

Much nicer to study is the *stream formulation*, which reduces the number of scalar fluid unknowns from three to one: By incompressibility (1.1b), the velocity field can be written as

$$(1.2) \quad \mathbf{u} = \nabla^\perp \psi := (\psi_Y, -\psi_X),$$

where ψ (defined up to an irrelevant constant) is called *stream function*. Clearly, ψ is conserved along particle trajectories:

$$\frac{D\psi}{Dt} = \mathbf{u} \cdot \nabla \psi = \nabla^\perp \psi \cdot \nabla \psi = 0.$$

(This is only true in the steady context.) Another conserved quantity is the *vorticity*

$$\omega := \nabla \times \mathbf{u} := \partial_X u_2 - \partial_Y u_1.$$

Indeed, taking the curl of the Euler equation (1.1a) yields

$$(1.3) \quad \frac{D\omega}{Dt} = 0.$$

(This is also true in the fully time-dependent context.) For this property the two-dimensional nature of the flow is absolutely crucial—for fully three-dimensional flows there would appear a vortex stretching term on the right-hand side. In other words, both ψ and ω have the same level sets (the streamlines of the flow), which is why it makes sense (and can be fully justified in many cases) to impose a global, single-valued functional relation

$$(1.4) \quad \omega = \gamma(\psi),$$

where in the following we consider γ to be arbitrary, but given. With this ansatz, the vorticity (at first being an expression in first derivatives of $\mathbf{u}$) magically becomes perturbative in some sense. More precisely, we have

$$(1.5a) \quad \Delta \psi = -\gamma(\psi) \quad \text{in } \Omega$$

since

$$\Delta \psi = -\nabla \times \nabla^\perp \psi = -\nabla \times \mathbf{u} = -\omega = -\gamma(\psi),$$

that is, vorticity now enters as a zeroth order term in a semilinear elliptic equation for ψ .

As for boundary conditions, the kinematic boundary condition says that the boundary parts are streamlines, so it comes at no surprise that ψ has to be constant on both S and B . We normalise ψ such that

$$(1.5b) \quad \psi = 0 \quad \text{on } S,$$

$$(1.5c) \quad \psi = -m \quad \text{on } B.$$

Here, the constant m has the physical interpretation of a *mass flux*.

Finally, in order to rewrite the dynamic boundary condition, we need *Bernoulli's law*. It states that

$$E := \frac{1}{2}|\mathbf{u}|^2 + p + gY + \int_0^\psi \gamma(s) ds$$

is constant throughout the fluid domain. (Again, this is only true in the steady context.) Indeed,

$$\nabla E = (\mathbf{u} \cdot \nabla) \mathbf{u} + \omega \mathbf{u}^\perp + \nabla(p + gY) + \gamma(\psi) \nabla \psi = 0,$$

using (1.1a) and (1.2) in the last step. Thus,

$$(1.5d) \quad \frac{1}{2}|\nabla \psi|^2 + gY = Q \quad \text{on } S$$

in view of (1.1e), (1.2), (1.4) and (1.5b), where

$$Q := E - p_{\text{atm}}$$

is the *Bernoulli constant*.

The stream formulation of (1.1) is therefore the system (1.5); see Figure 1 for an illustration of the setup and the equations. Conversely, a solution of (1.5) gives rise to a solution of (1.1) via defining $\mathbf{u}$ by (1.2) and setting

$$p := Q + p_{\text{atm}} - \frac{1}{2}|\mathbf{u}|^2 - gY - \int_0^\psi \gamma(s) ds.$$

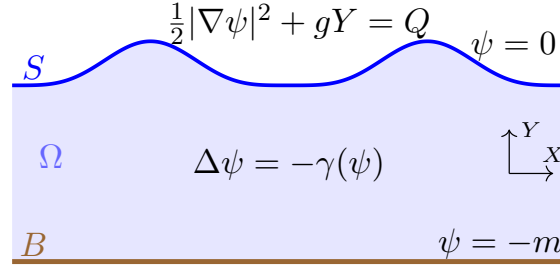


FIGURE 1. The setup and the equations.

2. THE MATHEMATICAL APPROACH

We are solely interested in *periodic waves*, while there is also much literature on *solitary waves*. Throughout, we do not aim to provide a complete overview of the immense literature and limit ourselves to only a few main references. Also, the water wave problem can be seen and studied from many different angles and has a huge number of different features, of which we can only consider very little. For a recent, excellent review paper we refer to [9].

In general, one cannot write down interesting solutions to (1.5) explicitly. The typical approach to rigorously construct solutions is hence to apply bifurcation theory. The scheme is the following; see Figure 2 for a visualisation:

- 1) Find *trivial solutions* depending on one parameter. Here, these are shear flows (independent of X) with a flat surface, parametrised (often) by a speed c . More precisely, $\Omega = \mathbb{R} \times (-h, 0)$ and ψ^c is determined by

$$\begin{aligned} \psi_{YY}^c &= -\gamma(\psi^c) \quad \text{in } (-h, 0), \\ \psi^c(0) &= 0, \quad \psi_Y^c(0) = c. \end{aligned}$$

(This set of equations has a unique solution if, say, γ is globally Lipschitz.)

- 2) Find *small-amplitude solutions* by applying a local bifurcation theorem, typically Crandall–Rabinowitz, identifying a critical parameter c_* .
- 3) Find *large-amplitude solutions* by applying a global bifurcation theorem.
- 4) Discuss at which configurations the solution set can terminate. This requires a structural analysis of solutions, applying strong maximum principles and a priori estimates.

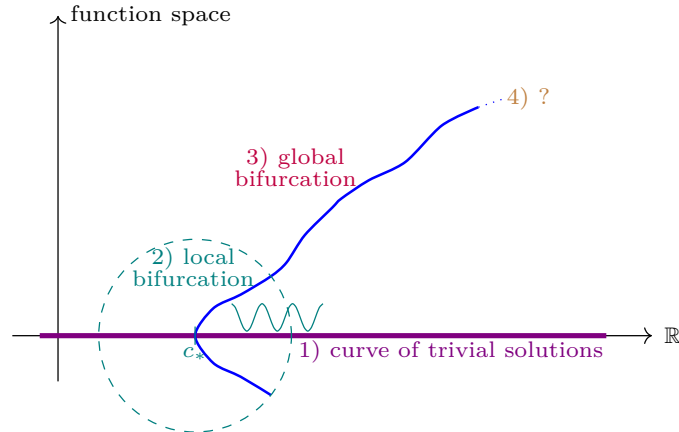
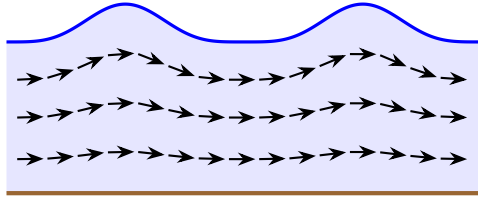


FIGURE 2. Typical steps in a bifurcation analysis.

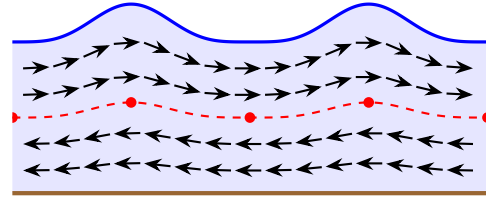
Bifurcation theorems need to be applied in a fixed Banach space. However, in our problem the fluid domain is not fixed and will vary for different solutions. Thus, we need to introduce an a priori unknown change of variables, mapping the fluid domain to a fixed reference domain, typically $\mathbb{R} \times (-h, 0)$, and then study the problem on this reference domain. Typical changes of variables are:

- A) For irrotational flows, use $X + iY \mapsto \varphi + i\psi$, where φ is the velocity potential (in that $\mathbf{u} = \nabla\varphi$) and $\varphi + i\psi$ the *complex velocity potential*. This is the most classical change of variables, primarily used in the last century. Note that in the presence of vorticity, however, φ does no longer exist.
- B) The *semi-hodograph transform* $(X, Y) \mapsto (X, \psi)$ (recall that ψ is constant on S and B). The drawback here is that *critical layers*, that is, sets for which $u_1 = \psi_Y = 0$, are a priori not allowed since at such points the change of variables would degenerate. See Figure 3 for a comparison of unidirectional waves (for which the horizontal velocity has a strict sign) and waves with critical layers (on which $u_1 = 0$) and *stagnation points* (at which $\mathbf{u} = 0$).
- C) The *naive flattening* $(X, Y) \mapsto (X, h(Y + h)/(\eta(X) + h) - h)$, assuming the surface is given by a graph $Y = \eta(X)$. The drawback here is that *overhanging waves* are a priori not allowed.
- D) The *conformal change of variables*, see Section 3.1. It does not involve any fluid quantities, and it is the only change of variables in this list which can allow for general vorticity, critical layers and overhanging profiles.

Our main motivation here is to establish a global bifurcation result for periodic waves that allows for all possible features, therefore utilising a conformal change of variables.



(A) A unidirectional wave.



(B) A wave with one critical layer (in red, with stagnation points indicated by dots).

FIGURE 3. Two types of internal flow structure.

2.1. Local bifurcation. Local bifurcation à la Crandall–Rabinowitz is purely perturbative, in the end reducing essentially to an implicit function theorem. Any such construction near trivial solutions can therefore only establish small-amplitude waves. Crandall–Rabinowitz amounts to finding parameters c such that the linearised problem (with respect to the state, not the parameter) has one kernel element (and, additionally, checking a nondegeneracy condition, the transversality condition).

In physical language, such a kernel equation is the *dispersion relation*, which relates the speed to the frequency of the wave. In the easiest setting of irrotational flows (for the dispersion relation for general γ , see [19, Section 4]), it reads

$$\frac{c^2}{gh} = \frac{\tanh \xi}{\xi},$$

where $\xi = 2\pi h/L$ and L is the wavelength. Here, one observes that bifurcation of periodic waves is only possible if $0 < c^2/gh < 1$ (while one has solitary waves for $0 < c^2/gh - 1 \ll 1$), and that in this case there is exactly one solution $\xi > 0$. Moreover, the trivial solution is unidirectional since $\psi_Y^c \equiv c$.

This can change drastically in the rotational case. Even in the case of constant γ , ψ_Y^c is affine and can change sign, therefore giving rise to critical layers which persist for small-amplitude waves nearby [18]. For affine γ , it is even possible that more than one positive ξ solve the dispersion relation with the same c , such that more than one sinusoidal mode contributes to small-amplitude waves (which are then constructed as bifurcations from higher-dimensional kernels, not from one-dimensional kernels as Crandall–Rabinowitz) at leading order [7, 8].

2.2. Global bifurcation. For global bifurcation, one tries to leave the perturbative regime of local bifurcation, thereby trying to establish existence of solutions which have no longer an almost flat surface.

As much as any perturbative theory relies on the implicit function theorem, a global theory usually relies on a *topological degree*. Like many other tools from nonlinear functional analysis, degree methods are essentially tailored to problems which are ‘almost finite-dimensional’. More precisely, they typically require Fredholm properties and certain compactness, that is, certain ellipticity. While ellipticity in the stream function is maybe not so surprising in view of (1.5a), the key is to uncover ellipticity in a parametrisation of the free surface.

The details differ for different changes of variables:

- A) In the irrotational case, one can write the problem as just an equation on the free surface, the *Nekrasov equation*. The first construction of large waves [13] used positive operator theory, while later abstract bifurcation theorems became available and were applied to the water wave problem [11].
- B) For the case with vorticity, the *Healey–Simpson degree* (originally introduced in the context of nonlinear elasticity) can be applied in conjunction with the semi-hodograph transform. This was first done in the groundbreaking work [3].
- C) In the framework of the naive flattening transform, a global result can be established at least using analytic global bifurcation, but allowing for critical layers [17].
- D) For constant vorticity and utilising a conformal change of variables (thereby allowing for overhanging waves), a pseudo-differential structure together with commutator estimates can pave the way to uncover ellipticity [4].

Our approach, which we shall present in Section 3, is more general and powerful in that it

- allows for general vorticity, critical layers and overhanging profiles simultaneously;
- does neither rely on an analytic theory nor a fancy degree, but rewrites the problem instead in the form ‘identity plus compact’ which is amenable to the most basic *Leray–Schauder degree*.

2.3. Why overhanging waves? We still have to motivate why it makes sense at all to allow for overhanging waves, which look somehow exotic; see Figure 4 for a sketch. Based on ideas of [16], one can show that no overhanging waves can exist if the vorticity has a certain fixed sign (with our convention, if $c\gamma \geq 0$). However, if γ has the other sign, even if it is constant, numerics have long indicated the formation of overhanging profiles [15].

For a rigorous verification of this observation, only fairly recently progress has been made, but there is more to be studied. Many constructions perturb an explicit zero gravity solution, thereby having to restrict themselves to a parameter regime of small gravity [6, 10].

Very recently, the existence of overhanging profiles with non-small gravity was established with computer assistance [2].

All of this strongly motivates to allow for overhanging waves and therefore work with a conformal change of variables.

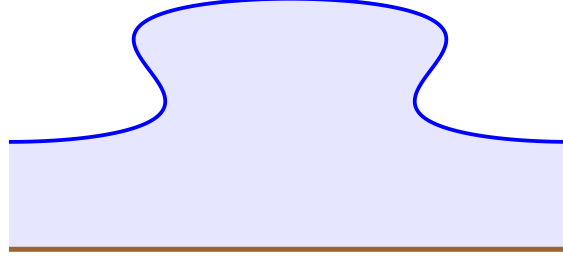


FIGURE 4. An overhanging wave.

2.4. Nodal properties. How do (large) solutions look like? Let us assume that, as the starting point, we have a Crandall–Rabinowitz bifurcation, that is, a one-dimensional kernel of the linearised problem. The surface of corresponding linear waves is then simply a sine, and small nonlinear waves are nearly sinusoidal.

Additionally, at least under some spectral assumption (e.g., $\sup \gamma' < \pi^2/h^2$, which ensures that the first Dirichlet eigenvalue of the linearised ODE operator $-\partial_Y^2 - \gamma'(\psi^c)$ on $(-h, 0)$ is positive), for small solutions the vertical velocity $-\psi_X$ has a strict sign (with the exception of the bed, at which it vanishes) in each open half period.

This provides the basis for a continuation argument, which is well known for pure gravity waves. Here, the abstract motivation to look at ψ_X is that ∂_X generates the group of translations in X , under which the water wave problem is invariant. In particular, one can apply strong maximum principle arguments (including Hopf’s lemma and Serrin’s corner-point lemma) to ψ_X in view of

$$(\Delta + \gamma'(\psi))\psi_X = 0.$$

With this at hand, one can then prove that the sign property of ψ_X persists on all of the global continuum of solutions, which more or less immediately yields that:

- The waves look sinusoidal in that there is exactly one crest and one trough in each minimal period and the surface elevation is strictly monotone from crest to trough (and vice versa). (However, a distinguishing feature of large solutions is that typically the wave has a narrow crest and a wide trough.)
- The surface can never intersect with the bed (also not in the limit of a sequence of solutions).
- Self-intersection of the surface can only happen exactly above a trough.

As the bullet point on self-intersection above suggests, an important motivation of studying nodal properties is to narrow down the number of alternatives which can occur ‘at the end’ of the global bifurcation set. A typical global bifurcation result asserts that the global continuum of solutions emanating from a trivial solution has to

- a) be unbounded, or
- b) approach a configuration at which the equations are no longer defined, or
- c) loops back to another trivial solution.

Of course, what a) and b) mean precisely, depends heavily on the norm and the formulation used. For many choices of γ (see [19, Theorem 5.6] for details), the nodal properties and a study of the dispersion relation imply that c) can never happen.

Numerics (and also some theoretical results) suggest that one should expect two possible end configurations:

- an extreme wave with a Stokes corner of 120° at the crest, as in the irrotational case (rigorously verified in [1], see also [12] for the rotational case), or
- an overhanging wave that self-intersects exactly above a trough.

However, a rigorous proof that these are the only two possibilities, for arbitrary γ , is not known.

3. THE NEW REFORMULATION

3.1. The conformal change of variables. Take any fluid domain Ω (with its surface neither intersecting itself nor the bed). Then [5], there exists a unique $h > 0$ and a (up to horizontal translations) unique conformal change of variables $H = U + iV$ from $\Omega_h := \mathbb{R} \times (-h, 0)$ to Ω which maps surface to surface, bottom to bottom, and respects periodicity. The uniqueness property is essentially due to the fact that, topologically, Ω is an annulus by periodicity. We denote points in Ω_h by (x, y) .

Our new unknowns (instead of ψ , Ω and Q) are (ϕ, w, q) , where

$$\psi \circ H = \psi^c + \phi, \quad w = V(\cdot, 0), \quad Q = \frac{c^2}{2} + q.$$

That is, ϕ is the deviation of the stream function from the trivial solution in conformal coordinates, w parametrises the height of the surface and Q is the deviation from its value at a trivial solution. Uniqueness of h demands that the average $\langle w \rangle$ over one period vanishes. We shall consider h arbitrary, but fixed throughout.

Thus, V can be recovered from w as the solution of

$$\begin{aligned} (3.1a) \quad & \Delta V = 0 && \text{in } \Omega_h, \\ (3.1b) \quad & V = w && \text{on } y = 0, \\ (3.1c) \quad & V = 0 && \text{on } y = -h, \end{aligned}$$

while (1.5) now reads

$$\begin{aligned} (3.2a) \quad & \Delta \phi = -\gamma(\psi^c + \phi)|\nabla V|^2 + \gamma(\psi^c) && \text{in } \Omega_h, \\ (3.2b) \quad & \phi = 0 && \text{on } y = 0, \\ (3.2c) \quad & \phi = 0 && \text{on } y = -h, \\ (3.2d) \quad & \frac{(\phi_y + c)^2}{2((1 + \mathcal{C}w')^2 + w'^2)} + gw = \frac{c^2}{2} + q && \text{on } y = 0. \end{aligned}$$

Here, the mass flux in (1.5c) is given by

$$m = m(c) := -\psi^c(-h).$$

Moreover,

$$(1 + \mathcal{C}w')^2 + w'^2 = |\nabla V|^2|_{y=0}$$

is the conformal factor at the surface, where naturally the periodic Hilbert transform on a strip, given on the Fourier side by

$$\mathcal{C} \equiv -i \coth\left(\frac{2\pi kh}{L}\right), \quad k \in \mathbb{Z},$$

appears. It is clear that, for any $c \in \mathbb{R}$, $(\phi, w, q) = (0, 0, 0)$ is a solution (the trivial one) of (3.2), from which our bifurcation shall occur at a certain c .

3.2. Uncovering ellipticity. So how can global bifurcation theory be applied in this framework (assuming the local theory has already been set up)? Our goal is to write (3.2) as $F = 0$, where F admits the structure ‘identity plus compact’, to which the classical Leray–Schauder degree is tailored.

Of course, (3.2) is not yet of this form. We therefore somehow have to single out and then invert the principal operator.

Looking first at (3.2a)–(3.2c), it is quite obvious that we should invert this Dirichlet problem by writing

$$(3.3a) \quad \phi = \Delta_0^{-1} \left(-\gamma(\psi^c + \phi)|\nabla V|^2 + \gamma(\psi^c) \right),$$

where Δ_0^{-1} denotes the inverse Laplacian with Dirichlet boundary conditions. Now, the right-hand side of (3.3a) is a compact operator in (ϕ, w, q) . Indeed, in view of (3.1) and by classical Schauder theory, V lives at the same level of regularity as w , so that one derivative is lost in the right-hand side of (3.2a). Applying the inverse Laplacian gains two derivatives, that is, in total we gain one derivative, which yields compactness. Here and throughout, we work in Hölder spaces, and the compactness in the end is simply a compact embedding on the Hölder scale; note that our base domain $(0, L) \times (-h, 0)$ is bounded since we deal with periodic waves.

Although the above observation seems quite trivial, it is important to point out that a conformal change of variables is crucial. Indeed, when writing the Laplacian in new coordinates, one gets as coefficients second order derivatives of the change of variables. So for a general change of variable the above reasoning would no longer work (first losing two and then gaining two derivatives does not result in compactness anymore). Crucially, for a conformal change of variables only first order derivatives of the change of variables appear as coefficients in the transformed Laplacian (which is simply scaled by the conformal factor $|\nabla V|^2$), which is because both components of the change of variables are harmonic.

On the other hand, uncovering ellipticity in w with the help of (3.2d) is the really interesting part. Indeed, for example, $\phi \equiv 0$ for any irrotational wave. Also, the conformal factor $(1 + \mathcal{C}w')^2 + w'^2$ seems to be the leading order term, but one cannot simply algebraically solve (3.2d) for w' . However, a nice complex analysis argument does the job.

Assuming (3.3a), we can write (3.2d) as

$$(3.4) \quad \sqrt{(1 + \mathcal{C}w')^2 + w'^2} = R,$$

where

$$R := \left. \frac{\left| \partial_y \Delta_0^{-1} (-\gamma(\psi^c + \phi)|\nabla V|^2 + \gamma(\psi^c)) + c \right|}{\sqrt{c^2 + 2q - 2gw}} \right|_{y=0}.$$

Note that R can be regarded as zero order in terms of number of derivatives. The main idea now is that a logarithm of H' can be defined; this is in fact quite reminiscent of Levi-Civita's formulation [14], where he uses the logarithm of the complex derivative of the complex velocity potential. We let

$$\log H'(z) := \ln H'(-ih) + \int_{-ih}^z \frac{H''(\zeta)}{H'(\zeta)} d\zeta.$$

Here, note that $H' \neq 0$ since H is conformal and that $H'(-ih) \in (0, \infty)$. The latter follows from the facts that the flat bottom $\{y = -h\}$ is mapped to the flat bottom $\{Y = -h\}$, implying that $\text{Im } H'(\cdot, -h) \equiv 0$, and that V attains its minimum everywhere on the bottom, so that $V_y(\cdot, -h) > 0$ by Hopf's lemma. Now, one can easily verify that indeed the above definition provides a logarithm of H' in that

$$H' = e^{\log H'}.$$

Let us introduce

$$\rho := \text{Re } \log H', \quad \vartheta := \text{Im } \log H'.$$

Note that ϑ vanishes on the bottom. As usual in complex analysis, since $\log H'$ is holomorphic, ρ and ϑ have to be related on the surface by means of a Hilbert transform. More precisely,

$$\mathcal{P}\rho = \mathcal{C}\vartheta \quad \text{on } y = 0,$$

where $\mathcal{P}$ denotes the projection onto the space of functions with zero average.

As a matter of fact,

$$H' = (1 + \mathcal{C}w') + iw' \quad \text{on } y = 0,$$

while by (3.4)

$$R = |H'| = \left| e^{\rho + i\vartheta} \right| = e^{\rho} \quad \text{on } y = 0.$$

Thus,

$$(1 + \mathcal{C}w') + iw' = e^{\rho} e^{i\vartheta} = e^{\rho} e^{i\mathcal{C}^{-1}\mathcal{P}\rho} = R e^{i\mathcal{C}^{-1}\mathcal{P} \ln R} \quad \text{on } y = 0.$$

Taking the imaginary part yields

$$(3.3b) \quad w' = R \sin \left(\mathcal{C}^{-1}\mathcal{P} \ln R \right),$$

and averaging the real part over one period implies

$$(3.3c) \quad 1 = \left\langle R \cos \left(\mathcal{C}^{-1}\mathcal{P} \ln R \right) \right\rangle.$$

This new system of equations (3.3) is our reformulation. We have seen that (3.2) implies (3.3), but we can in fact also go in the opposite direction [19, Lemma 3.1].

While (3.3c) can be viewed as the equation for q , (3.3b) is the equation for w . Importantly, after integrating (3.3b) once, the right-hand side is compact since R is zero order and both $\mathcal{C}$ and $\mathcal{P}$ do not affect (Hölder) regularity.

Notice, however, the reformulation breaks down when certain terms become zero, which is entirely expectable. Indeed, the vanishing of $(1 + \mathcal{C}w') + iw'$ at some point would mean that the conformal map degenerates, so that we expect a non-smooth surface there. Moreover, if the denominator in R , that is, $Q - gw$, vanishes at some point, the wave is of greatest height featuring surface stagnation. This is the case for which a Stokes corner of 120° appears in the classical theory, and it is clear that at such a point we can never expect any ellipticity, from which regularity could be bootstrapped.

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REFERENCES

1. C. J. Amick, L. E. Fraenkel, and J. F. Toland, *On the Stokes conjecture for the wave of extreme form*, Acta Math. **148** (1982), 193–214.
2. M. Cadiot and S. Haziot, *Global bifurcation and the constructive existence of overhanging periodic steady water waves*, 2026, [arXiv:2607.13567](#).
3. A. Constantin and W. A. Strauss, *Exact steady periodic water waves with vorticity*, Comm. Pure Appl. Math. **57** (2004), no. 4, 481–527.
4. A. Constantin, W. A. Strauss, and E. Vărvăruță, *Global bifurcation of steady gravity water waves with critical layers*, Acta Math. **217** (2016), no. 2, 195–262.
5. A. Constantin and E. Vărvăruță, *Steady periodic water waves with constant vorticity: regularity and local bifurcation*, Arch. Ration. Mech. Anal. **199** (2011), no. 1, 33–67.
6. J. Dávila, M. del Pino, M. Musso, and M. H. Wheeler, *Overhanging solitary water waves*, Invent. Math. **244** (2026), no. 2, 643–767.
7. M. Ehrnström, J. Escher, and E. Wahlén, *Steady water waves with multiple critical layers*, SIAM J. Math. Anal. **43** (2011), no. 3, 1436–1456.
8. M. Ehrnström and E. Wahlén, *Trimodal steady water waves*, Arch. Ration. Mech. Anal. **216** (2015), no. 2, 449–471.
9. S. V. Haziot, V. M. Hur, W. A. Strauss, J. F. Toland, E. Wahlén, S. Walsh, and M. H. Wheeler, *Traveling water waves — the ebb and flow of two centuries*, Quart. Appl. Math. **80** (2022), 317–401.
10. V. M. Hur and M. H. Wheeler, *Overhanging and touching waves in constant vorticity flows*, J. Differential Equations **338** (2022), 572–590.
11. G. Keady and J. Norbury, *On the existence theory for irrotational water waves*, Math. Proc. Cambridge Philos. Soc. **83** (1978), no. 1, 137–157.

12. V. Kozlov and E. Lokharu, *Global bifurcation and highest waves on water of finite depth*, Arch. Ration. Mech. Anal. **247** (2023), no. 5, Paper No. 98.
13. Ju. P. Krasovskiĭ, *On the theory of steady-state waves of finite amplitude*, Ž. Vyčisl. Mat i Mat. Fiz. **1** (1961), 836–855.
14. T. Levi-Civita, *Determinazione rigorosa delle onde irrotazionali periodiche in acqua profonda*, Rend. Accad. Lincei **33** (1924), 141–150.
15. J. A. Simmen and P. G. Saffman, *Steady deep-water waves on a linear shear current*, Stud. Appl. Math. **73** (1985), no. 1, 35–57.
16. E. R. Spielvogel, *A variational principle for waves of infinite depth*, Arch. Ration. Mech. Anal. **39** (1970), 189–205.
17. K. Varholm, *Global bifurcation of waves with multiple critical layers*, SIAM J. Math. Anal. **52** (2020), no. 5, 5066–5089.
18. E. Wahlén, *Steady water waves with a critical layer*, J. Differential Equations **246** (2009), no. 6, 2468–2483.
19. E. Wahlén and J. Weber, *Large-amplitude steady gravity water waves with general vorticity and critical layers*, Duke Math. J. **173** (2024), no. 11, 2197–2258.

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