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The semiclassical limit of the Dirac equation

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Abstract

The Dirac equation was discovered by Paul Dirac in 1928 in order to find a first-order in time Lorentz-invariant Schrödinger equation for particles with half-integer spin. He discovered that the scalar wave function of the Schrödinger equation had to be replaced by a 4-component vector-valued wave function, which includes the two spin states of a charged fermion and predicted its antiparticle. The vector-valued nature implies that the Dirac operator is matrix-valued. In order to treat the semiclassical limit we transform the Dirac equation into the Wigner equation for the matrix-valued Wigner transform, which converges to the positive definite Wigner matrix measure. The latter obeys a matrix-valued Liouville equation where the oscillating part of the Wigner equation converges to a Lagrange multiplier that maintains the constraint that the Wigner matrix commutes with the Weyl symbol of the Dirac operator. The relativistic Vlasov equations for the electron and positron can then be recovered by projecting the Wigner matrix on the eigenspaces of the Dirac symbol and taking the trace over the spin components.

1 Introduction

The content of this report is the following (informal) theorem, proved in [6]:

Theorem 1. *Let $\hbar$ be a small parameter. Then the matrix-valued Wigner transform $W^\hbar$ of a mixed state density matrix $R^\hbar$ – whose eigenvalues $\lambda_j^\hbar$ satisfy*

$$\sum_{j=1}^{\infty} (\lambda_j^\hbar)^2 \leq C(2\pi\hbar)^3,$$

and which solves the von Neumann equation associated with the Dirac equation

$$i\hbar\partial_t\Psi(t,x) = (H\Psi)(t,x) := (\alpha \cdot (-i\hbar\nabla - A(t,x)) + mc\beta + V(t,x)) \Psi(t,x),$$

– converges weakly* in $L^\infty(\mathbb{R}_t, L^2(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$ to the Wigner matrix measure W , which is a solution of the matrix-valued Liouville equation

$$\begin{cases} \partial_t W = \frac{1}{i}[P, Y] + \frac{1}{2}(\{P, W\} - \{W, P\}), & P(t, x, \xi) = \alpha \cdot (\xi - A(t, x)) + \beta + V(t, x), \\ [P, W] = 0, \\ W|_{t=0} = W_+^{\text{in}} + W_-^{\text{in}}, \end{cases}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket and P is the matrix-valued symbol of the Dirac Hamilton operator H . The matrix-valued distribution Y is the Lagrange multiplier associated with the constraint $[P, W] = 0$.

The evolution of the electron and positron densities $f_\pm := \text{tr}_{\mathbb{C}^4}(W\Pi_\pm)$, where $\Pi_\pm$ are the projections on the electron and positron subspaces of the symbol P is governed by Liouville's equation and the classical relativistic Hamilton functions

$$\mathcal{H}_\pm(t, x, \xi - A(x)) = \pm\sqrt{1 + |\xi - A(x)|^2} + V(x).$$

The exact statements can be found in Theorems 2 and 3. In the following we will discuss the meaning and proof of this theorem in greater detail.

1.1 A brief history of spin

In 1926, G. E. Goudsmit and S. A. Uhlenbeck postulated the existence of an intrinsic angular momentum of the electron, i.e the *spin*, in order to explain the anomalous Zeeman effect [13]. In 1927, W. Pauli found a generalization of Schrödinger's equation which includes spin (the *Pauli equation*) [10], but this equation is not Lorentz-invariant (in fact, it is a vector-valued magnetic Schrödinger equation with an additional potential term coupling the two components of the Pauli spinor). Later still, in 1928, P. A. M. Dirac found a relativistic version of Schrödinger's equation where the spin occurs intrinsically, which we now know as the *Dirac equation* [3]. Due to the fact that the spin appears as a quantum mechanical phenomenon in the Dirac equation one might be tempted to expect it to disappear in the (semi-)classical limit. However, L. Thomas [12], J. Frenkel [5] and H. A. Kramers [8] and later V. Bargmann, L. Michel, and V. Telegdi [1], developed a classical theory of spin based on special relativity which does not refer to quantum mechanics.

Their *BMT equation* (Bargmann-Michel-Telegdi equation) for the spin $\mathbf{s}$ reads

$$\frac{d}{dt}\mathbf{s} = \frac{e}{mc}\mathbf{s} \times \left(\frac{1}{\gamma}B - \frac{1}{c(1+\gamma)}v \times E \right), \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}.$$

Spohn in [11] showed that the classical BMT equation for the classical spin precession can indeed be derived from the Dirac equation and the quantum mechanical spin precession. We will return to this derivation below.

1.2 A classical description of relativistic electrodynamics

In the kinetic description of classical mechanics, a particle is described by a phase space distribution function $f(t, x, \xi) \geq 0$ obeying the *Liouville equation*

$$\partial_t f(t, x, \xi) = \{\mathcal{H}, f\}(t, x, \xi), \quad \{a, b\} := \nabla_x a \cdot \nabla_\xi b - \nabla_\xi a \cdot \nabla_x b,$$

where $\mathcal{H}$ is the classical Hamilton function, $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ is a point in phase space and $\{a, b\}$ is the Poisson bracket. The relativistic dynamics of an electron or positron - represented by a phase space distribution function $f_\pm$ - which subjected to an electromagnetic field $E = -\partial_t A - \nabla V$, $B = \nabla \times A$, where $A: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $V: \mathbb{R}^d \rightarrow \mathbb{R}$ are the electromagnetic potentials, are governed by the Hamilton functions

$$\mathcal{H}_\pm(t, x, \xi - A(x)) = \pm \sqrt{1 + |\xi - A(x)|^2} + V(x).$$

Moreover, in a self-consistent model where the field is generated by the one-particle densities (ρ, J) we obtain the *Vlasov-Maxwell equation*

$$\begin{cases} \partial_t f_\pm \pm (v/\langle v \rangle) \cdot \nabla_x f_\pm + (E \pm (v/\langle v \rangle) \times B) \cdot \nabla_v f_\pm = 0, \\ \partial_t E - \nabla \times B = -4\pi J := -4\pi \int (v/\langle v \rangle)(f_+ - f_-)dv, & \partial_t B + \nabla \times E = 0, \\ \nabla \cdot E = 4\pi \rho := 4\pi \int (f_+ + f_-)dv, & \nabla \cdot B = 0, \\ (f, E, B)|_{t=0} = (f^{\text{in}}, E^{\text{in}}, B^{\text{in}}). \end{cases} \quad (1.1)$$

Here,

$$\frac{v}{\langle v \rangle} := \frac{v}{\sqrt{1 - |v|^2}} \quad (1.2)$$

is the relativistic velocity. Note that there is no mention of spin in the relativistic Vlasov equation, although the BMT equation describes the classical evolution of the spin.

Our goal is then twofold:

- a) Derive the relativistic Vlasov equation for external electromagnetic fields (the derivation of the relativistic Vlasov-Maxwell equation (1.1) remains an open problem)
- b) Derive the BMT equation and the classical description of spin

1.3 Dirac's equation: A quick derivation

In the following we briefly sketch Dirac's original motivation for the Dirac equation. Recall the relativistic energy-momentum relation

$$E^2 = p^2 + m^2 c^2,$$

where $p = (p_1, p_2, p_3)^\top$ is the momentum 3-vector, m is the mass and c is the speed of light. The usual quantization rule replaces

$$E \rightarrow i\hbar \partial_t \quad \text{and} \quad p \rightarrow -i\hbar \nabla. \quad (1.3)$$

This yields the *Klein-Gordon equation* for the scalar wave function u

$$(-\frac{\hbar^2}{c^2}\partial_t^2 + \hbar^2\Delta)u - m^2c^2u = 0. \quad (1.4)$$

Dirac's goal was to find a linear first order equation in time. A first naive approach is to take the square root of the energy-momentum relation and linearize the square root by introducing the quantities α_j , $j = 1, \dots, 4$.

$$E = \sqrt{p^2 + m^2c^2} = \alpha_1p_1 + \alpha_2p_2 + \alpha_3p_3 + \alpha_4m^2c^2.$$

Now we quantize according to (1.3):

$$i\frac{\hbar}{c}\partial_t\Psi = [\alpha_1(-i\hbar\partial_{x_1}) + \alpha_2(-i\hbar\partial_{x_2}) + \alpha_3(-i\hbar\partial_{x_3}) + \alpha_4m^2c^2]\Psi, \quad (1.5)$$

for some unknown Ψ . The square of this operator should of course satisfy the Klein-Gordon equation (1.4). This forces the following constraints on the quantities α_j :

$$\alpha_j\alpha_i + \alpha_i\alpha_j = 0 \text{ for } i \neq j, \quad \alpha_i^2 = 1.$$

These are the defining relations for the Clifford algebra $\text{Cl}_{1,3}(\mathbb{C})$, given by

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3, \quad \alpha_4 \equiv \beta = \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{pmatrix}.$$

It is customary to denote $\alpha_4 \equiv \beta$. The 2×2 matrices σ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, in (1.5) we note that Ψ is a vector-valued function with values in $\mathbb{C}^4$. By minimal coupling (i.e. by replacing $-i\hbar\nabla$ by $-i\hbar\nabla - \frac{1}{c}A$) and addition of an electric potential V we finally obtain *Dirac's equation*

$$i\frac{\hbar}{c}\partial_t\Psi(t, x) = (\alpha \cdot (-i\hbar\nabla - \frac{1}{c}A(t, x)) + mc\beta + V(t, x)I_4)\Psi(t, x),$$

where $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^\top \in L^2(\mathbb{R}^d, \mathbb{C}^4)$ is the Dirac 4-spinor. We use the notation

$$\alpha \cdot (-i\hbar\nabla - \frac{1}{c}A(t, x)) := \sum_{k=1}^3 \alpha_k(-i\hbar\partial_{x_k} - \frac{1}{c}A_k(t, x)).$$

The first two components of Ψ correspond to the two spin states of a spin- $\frac{1}{2}$ -particle (e.g. the electron). Indeed, after a Foldy-Wouthuysen transformation $\psi = e^{itc^2/\hbar}\Psi$ one recovers the *Pauli equation* for the 2-spinor $\psi^+ = (\psi_1^+, \psi_2^+)^\top \in L^2(\mathbb{R}^d, \mathbb{C}^2)$

$$i\hbar\partial_t\psi^+(t, x) = \left(\frac{1}{2m}(-i\hbar\nabla - \frac{1}{c}A(t, x))^2 + V(t, x)I_2 + \frac{\hbar}{2mc}(\sigma \cdot B(t, x))\right)\psi^+(t, x)$$

The lower two components ψ_3, ψ_4 were later interpreted to correspond to the *antiparticle* with opposite charge (e.g. the positron).

2 Quantum Mechanics on Phase Space

2.1 Density matrices and Wigner transform

A *quantum state* is represented by a density operator $R = R^* \geq 0$, which is a *positive trace class* (i.e. $R \in \mathcal{L}^1(\mathfrak{H})$) operator on a Hilbert space $\mathfrak{H}$ with

$$\mathrm{tr}_{\mathfrak{H}}(R) = 1.$$

A *pure state* is defined by $\mathrm{tr}_{\mathfrak{H}}(R^2) = 1$ and corresponds to a rank one projection on $\mathbb{C}\psi$ where $\psi \in \mathfrak{H}$, i.e.

$$R = |\psi\rangle\langle\psi|.$$

On the other hand, a *mixed state* corresponds to a convex combination of rank one projections such that

$$\begin{cases} R = \sum_{j \geq 1} \lambda_j |\psi_j\rangle\langle\psi_j|, \\ \{\psi_j\} \text{ orthonormal basis of } \mathfrak{H}, \\ \lambda_j \geq 0, \\ \sum_{j \geq 1} \lambda_j = 1. \end{cases} \quad (2.1)$$

Note that R can be represented as an integral operator with kernel

$$R(t, X, Y) = \sum_{j \geq 1} \lambda_j \psi_j(X) \psi_j(Y)^*, \quad \psi_j^* = \overline{\psi_j}^\top.$$

We identify R with its kernel and denote them by the same letter, which is a very common abuse of notation in the literature.

The dynamics of R are governed by *von Neumann's equation*

$$i\hbar \partial_t R(t) = [H, R(t)], \quad R(0) = R^{\mathrm{in}}.$$

Since H and R are matrix-valued it is important to recall that this is an equation for operators where the order is important. Choose $f \in L^2$. Then the von Neumann equation for the operator R applied to f becomes

$$i\hbar \int dY \partial_t R(t, X, Y) f(Y) = \int dY (H_X R(t, X, Y) f(Y) - R(t, X, Y) H_Y f(Y)).$$

Now given a Hamilton function $\mathcal{H}(t, x, \xi)$ on phase space $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$ one may ask what the corresponding operator is going to be. One commonly defines the Weyl operator H (or Weyl quantization) as

$$(Hf)(t, x) = \mathrm{Op}^W[\mathcal{H}](f)(t, x) := \iint \mathcal{H}(t, \frac{x+y}{2}, \xi) f(t, y) e^{i(x-y) \cdot \xi} dy d\xi.$$

Note that there exist other quantization schemes, such as Toeplitz quantization.

The inverse mapping of the Weyl quantization is given by the *Wigner transform* [14]:

$$W^h[R](t, x, \xi) := \frac{1}{(2\pi)^d} \int e^{-i\xi \cdot y} R(t, x + \hbar y/2, x - \hbar y/2) dy.$$

The Wigner transform is Hermitian ($W^h = (W^h)^*$) but *not* positive definite (this is a consequence of the uncertainty principle). On the other hand, its moments correspond to the macroscopic densities. In particular, the diagonal of R corresponds to the integral of W_h over the momentum variable, i.e.

$$R(t, x, x) = \int W^h(t, x, \xi) d\xi, \quad \iint \text{tr}_{\mathbb{C}^d}(W^h(t, x, \xi)) dx d\xi = 1.$$

For the Dirac equation, the charge and current densities ρ^h, J^h are given by

$$\rho^h(t, x) = \int \text{tr}(W^h(t, x, \xi)) d\xi, \quad J_k^h(t, x) = \int \text{tr}(\alpha_k W^h(t, x, \xi)) d\xi.$$

In the limit $\hbar \rightarrow 0$, W^h converges in $\mathcal{S}'$ to the positive definite *Wigner matrix measure* [7, 9]

$$W = W^* \geq 0.$$

The matrix-valued Wigner transform $W_h[R]$ of R solves the *Wigner equation*

$$\partial_t W_h = \frac{1}{i\hbar} [P, W_h] + \frac{1}{2} (\{P, W_h\} - \{W_h, P\}) + r_h, \quad W_h(0) = W_h[R^{\text{in}}] =: W_h^{\text{in}}. \quad (2.2)$$

Here, r_h is an $O(\hbar)$ remainder term. In the scalar case the oscillating term $\frac{1}{i\hbar} [P, W_h]$ vanishes. We will treat the limit of this term below.

2.2 Dirac's equation in the Wigner formalism

After this general discussion, let us now discuss the Dirac equation and its formulation on phase space in greater detail. The Hilbert space of 4-spinors is given by

$$\mathfrak{H} = L^2(\mathbb{R}^3, \mathbb{C}^4).$$

Recall the Dirac Hamiltonian and the von Neumann equation

$$H = \alpha \cdot (-i\hbar \nabla - A) + \beta + V, \quad i\hbar \partial_t R(t) = [H, R(t)], \quad R(0) = R^{\text{in}}. \quad (2.3)$$

Here, A and V are understood as multiplication operators on $\mathfrak{H}$. The Hamilton operator is realized as the Weyl operator $\text{Op}_h^W[P]$ with Weyl symbol

$$P(t, x, \xi) = \alpha \cdot (\xi - A(t, x)) + \beta + V(t, x) I_4.$$

The matrix-valued symbol P has two eigenvalues $\lambda_{\pm}$ of *constant* multiplicity 2

$$\lambda_{\pm}(t, x, \xi) = \pm \sqrt{1 + |\xi - A(t, x)|^2} + V(t, x), \quad (2.4)$$

where the corresponding projections $\Pi_{\pm}$ on the eigenspaces associated with $\lambda_{\pm}$ are given by

$$\Pi_{\pm}(t, x, \xi) = \frac{1}{2} \left(I_4 \pm \frac{\alpha \cdot (\xi - A(t, x))}{\sqrt{1 + |\xi - A(t, x)|^2}} \right) \in C^{\infty}(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_{\xi}^d, \mathbb{M}_4(\mathbb{C})). \quad (2.5)$$

Note that since the eigenvalues $\lambda_{\pm}$ are separated, no mode crossing occurs and the projections are smooth everywhere. Problems involving mode crossing have been treated e.g. in [4, 2].

By taking the Wigner transform of the von Neumann equation with the Dirac Hamilton operator H we obtain the Wigner equation

$$\partial_t W^{\hbar} = \frac{1}{i\hbar} [P, W^{\hbar}] + \frac{1}{2} (\{P, W^{\hbar}\} - \{W^{\hbar}, P\}) + r_{\hbar}, \quad (2.6)$$

where $r_{\hbar}$ is a remainder term that is explicitly given by

$$r_{\hbar} = \hbar \Delta[A_k][\alpha_k, W^{\hbar}] + \frac{1}{2} (-\nabla_x A_k \cdot \nabla_{\xi} - \theta[A_k])[\alpha_k, W^{\hbar}]_+ + (-\nabla V \cdot \nabla_{\xi} - \theta[V])W^{\hbar}, \quad (2.7)$$

where $[\cdot, \cdot]_+$ denotes the anticommutator. Here, θ is the pseudo-differential operator defined by

$$(\theta[g]f)(x, \xi) := \int_{\mathbb{R}^6} \frac{1}{i\hbar} (g(x + \hbar y/2) - g(x - \hbar y/2)) f(x, \eta) e^{-i(\xi - \eta) \cdot y} d\eta dy, \quad (2.8)$$

while Δ is the pseudo-differential operator defined by

$$(\Delta[g]f)(x, \xi) := \frac{1}{2i} \int_{\mathbb{R}^6} \frac{g(x + \hbar y/2) - 2g(x) + g(x - \hbar y/2)}{\hbar^2} f(x, \eta) e^{-i(\xi - \eta) \cdot y} d\eta dy. \quad (2.9)$$

Assuming sufficient regularity for the potentials A, A_0 one can then show that the remainder $r_{\hbar}$ converges to zero in $\mathcal{S}'$ (or H_{loc}^{-1} , respectively).

3 The semiclassical limit

In the following we shall discuss the semiclassical limit of the Wigner equation (2.6). First we formulate the “global” limit of the Wigner transform $W^{\hbar}$, where we pass to the limit in the Wigner equation (2.6) without projecting $W_{\hbar}$ on the electron or positron subspace, i.e. without considering $\Pi_{\pm} W_{\hbar} \Pi_{\pm}$. In that case, one would have (supposing for the moment that P does not depend on time for simplicity)

$$\partial_t \Pi_{\pm} W_{\hbar} \Pi_{\pm} = \frac{1}{i\hbar} \Pi_{\pm} [P, W^{\hbar}] \Pi_{\pm} + \frac{1}{2} \Pi_{\pm} (\{P, W^{\hbar}\} - \{W^{\hbar}, P\}) \Pi_{\pm} + \Pi_{\pm} r_{\hbar} \Pi_{\pm}.$$

Now note that since

$$P = \lambda_+ \Pi_+ + \lambda_- \Pi_-,$$

it follows that

$$\frac{1}{i\hbar} \Pi_\pm [P, W^h] \Pi_\pm = 0,$$

and so

$$\partial_t \Pi_\pm W_h \Pi_\pm = \frac{1}{2} \Pi_\pm (\{P, W^h\} - \{W^h, P\}) \Pi_\pm + \Pi_\pm r_h \Pi_\pm.$$

Our approach is to not project on the energy eigenspaces but consider the limit of W_h instead of $\Pi_\pm W_h \Pi_\pm$. To this end we will introduce a Lagrange multiplier that maintains the constraint that $[P, W] = 0$. The Lagrange multiplier is given by the limit of the oscillating term $\frac{1}{i\hbar} [P, W_h]$.

For simplicity we will present the theory of the Lagrange multiplier in the sense of Schwartz distributions and smooth external potentials A, V . By careful analysis of the remainder term (2.7) one can lower the regularity needed for the potentials. The details can be found in [6].

3.1 The global limit

3.1.1 The Matrix Liouville equation

First we make some assumptions on the density matrix

$$\|R^{\text{in}}\|_2 = \left(\sum_{j \geq 1} \lambda_j^2\right)^{1/2} \leq C(2\pi\hbar)^{d/2} \quad \Leftrightarrow \quad \|W_h^{\text{in}}\|_2 = \frac{1}{(2\pi\hbar)^{d/2}} \|R^{\text{in}}\|_2 \leq C.$$

The L^2 norm is conserved by the flow of the Wigner equation:

$$\|W^h\|_{L^\infty(\mathbb{R}_t, L^2(\mathbb{R}_x^d \times \mathbb{R}_\xi^d))} = \text{const.}$$

Therefore the Wigner matrix measure W is uniformly bounded in L^2 , i.e.

$$W \in L^\infty(\mathbb{R}_t, L^2(\mathbb{R}_x^d \times \mathbb{R}_\xi^d, \mathbb{M}_4(\mathbb{C}))). \quad (3.1)$$

Multiplying the Wigner equation (2.6) by $\hbar$ and passing to the limit in $\mathcal{S}'$ yields

$$[P, W^h] = i\hbar \left(\partial_t W_h - \frac{1}{2} (\{P, W_h\} - \{W_h, P\}) - r_h \right) \quad \Rightarrow \quad [P, W] = 0.$$

Since

$$\partial_t W^h \rightarrow \partial_t W \quad \text{and} \quad \frac{1}{2} (\{P, W_h\} - \{W_h, P\}) \rightarrow \frac{1}{2} (\{P, W\} - \{W, P\})$$

in $\mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d, \mathbb{M}_4(\mathbb{C}))$, while

$$r_h \rightarrow 0 \quad \text{in } \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d, \mathbb{M}_4(\mathbb{C})),$$

we deduce that

$$\frac{1}{i\hbar}[P, W^h] \rightarrow X \quad \text{in } \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d, \mathbb{M}_4(\mathbb{C})).$$

Thus the Wigner measure W solves the *Matrix Liouville equation*

$$\begin{cases} \partial_t W = X + \frac{1}{2}(\{P, W\} - \{W, P\}), \\ [P, W] = 0. \end{cases} \quad (3.2)$$

in $\mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d, \mathbb{M}_4(\mathbb{C}))$. Note that up to now we are not able to pose the Cauchy problem. To this end we need to study the Lagrange multiplier X in greater detail.

3.1.2 Properties of the Lagrange multiplier

In general, the Lagrange multiplier X has the following properties.

Lemma 1. *The matrix-valued distribution X satisfies the following properties:*

- (a) $X = X^*$,
- (b) for all $\Phi \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d, \mathbb{M}_m(\mathbb{C}))$.

$$[P, \Phi] = 0 \text{ on } \mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d \implies \text{tr}(\Phi X) = 0 \text{ in } \mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d).$$

In particular

$$\text{tr}(X) = \text{tr}(PX) = \dots = \text{tr}(P^n X) = 0 \quad \text{for all } n \geq 0.$$

Notice however that $X \in \mathcal{S}'(\mathbb{R}_t^d \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d, \mathbb{M}_4(\mathbb{C}))$, so that Lemma 1 does not imply in general that $W|_{t=0}$ is well-defined. More information is needed in order to formulate the Cauchy problem. In fact, at least for the Dirac symbol P , we can say more about the structure of the Lagrange multiplier:

Lemma 2. *Let P be the matrix-valued symbol of the Dirac operator and let $\Pi_\pm$ be its projections on the eigenspaces associated with $\lambda_\pm$. Then*

- (a) $\Pi_\pm X \Pi_\pm = 0 \implies$ there is a $Y = Y^* \in \mathcal{S}'$ such that

$$X = \frac{1}{i}[P, Y] \quad \text{and} \quad Y = \frac{1}{4\langle \xi - A \rangle^2} [P, [P, Y]].$$

- (b) Using $[P, W] = 0$, we have the following formula for Y

$$Y = \frac{i \left([\partial_t P, W] - \frac{1}{2} [P, \{P, W\} - \{W, P\}] \right)}{4\langle \xi - A \rangle^2}.$$

Now the limiting Matrix Liouville equation (3.2) becomes

$$\begin{cases} \partial_t W = \frac{1}{i}[P, Y] + \frac{1}{2}(\{P, W\} - \{W, P\}), \\ Y = \frac{i([\partial_t P, W] - \frac{1}{2}[P, \{P, W\} - \{W, P\}])}{4\langle \xi - A \rangle^2}, \\ [P, W] = 0. \end{cases} \quad (3.3)$$

in $\mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_\xi^d, \mathbb{M}_4(\mathbb{C}))$.

3.1.3 Cauchy problem for the Matrix Liouville equation

Due to the additional structure of the Lagrange multiplier of the Dirac equation we can now formulate the Cauchy problem for the Matrix Liouville equation. Due to Lemma 2, the smoothness of P and the fact that $W \in L^\infty(\mathbb{R}_t, L^2)$, one has

$$Y = \frac{i([\partial_t P, W] - \frac{1}{2}[P, \{P, W\} - \{W, P\}])}{4\langle \xi - A \rangle^2} \in L^\infty(\mathbb{R}_t, \mathcal{S}'(\times \mathbb{R}_x^d \times \mathbb{R}_\xi^d, \mathbb{M}_m(\mathbb{C}))).$$

Thus, using the Wigner equation we obtain

$$\partial_t W = \frac{1}{i}[P, Y] + \frac{1}{2}(\{P, W\} - \{W, P\}) \in L^\infty(\mathbb{R}_t, \mathcal{S}'(\times \mathbb{R}_x^d \times \mathbb{R}_\xi^d, \mathbb{M}_m(\mathbb{C}))). \quad (3.4)$$

The fact that $\partial_t W \in L^\infty$ implies that W is Lipschitz-continuous in t and so $W|_{t=0} = W^{\text{in}}$ is well-defined and the Cauchy problem can be posed:

$$\begin{cases} \partial_t W = \frac{1}{i}[P, Y] + \frac{1}{2}(\{P, W\} - \{W, P\}), & W|_{t=0} = W^{\text{in}}, \\ Y = \frac{i([\partial_t P, W] - \frac{1}{2}[P, \{P, W\} - \{W, P\}])}{4\langle \xi - A \rangle^2}, \\ [P, W] = 0. \end{cases} \quad (3.5)$$

3.1.4 Main result

Thus we are finally able to formulate the rigorous version of the informal theorem in the introduction:

Theorem 2. *Let $A \in C^1(\mathbb{R}_t, H^s(\mathbb{R}_x^3))$, where $s > 5/2$ is a real number, and let $V \in C^1(\mathbb{R}_t, H^1(\mathbb{R}^3))$. Let $R_{\text{in}} \in \mathcal{L}(\mathfrak{H})$ be a density operator such that*

$$\|R_{\text{in}}\|_{L^2}^2 = \sum_{j=1}^{\infty} (\lambda_j^h)^2 \leq C(2\pi\hbar)^3, \quad (3.6)$$

$$\text{tr}_{\mathfrak{H}} \left(\sqrt{1 - \Delta} R_{\text{in}} \sqrt{1 - \Delta} \right) < +\infty. \quad (3.7)$$

Let R be the unique solution to the Dirac-von Neumann equation (2.3) with initial datum R_{in} and let $W^h = W^h[R]$ be its Wigner transform. Assume that

$$W_{\pm}^{\text{in},h} := \Pi_{\pm} W^{\text{in},h} \Pi_{\pm} \xrightarrow{h \rightarrow 0} \Pi_{\pm} W^{\text{in}} \Pi_{\pm} =: W_{\pm}^{\text{in}} \quad \text{weakly in } L^2(\mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3, \mathbb{M}_4(\mathbb{C})), \quad (3.8)$$

where $\Pi_{\pm}$, defined in (2.5), are the projections onto the electron and positron subspaces of P . Then for any $T > 0$, W^h converges weakly* in $L^{\infty}(\mathbb{R}_t, L^2(\mathbb{R}_{x,\xi}^6, \mathbb{M}_4(\mathbb{C})))$ to a non-negative matrix-valued Radon measure $W \in L^{\infty}(\mathbb{R}_t, L^2(\mathbb{R}_{x,\xi}^6, \mathbb{M}_4(\mathbb{C})))$, called the Wigner matrix measure. Moreover, there exists

$$Y \in H_{\text{loc}}^{-1}(\mathbb{R}_t \times \mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3, \mathbb{M}_4(\mathbb{C})), \quad (3.9)$$

such that W solves the matrix-valued Liouville equation

$$\begin{cases} \partial_t W = \frac{1}{i}[P, Y] + \frac{1}{2}(\{P, W\} - \{W, P\}), & P = \alpha \cdot (\xi - A) + \beta - V, \\ [P, W] = 0, \\ W|_{t=0} = W_+^{\text{in}} + W_-^{\text{in}}, \end{cases} \quad (3.10)$$

in $H_{\text{loc}}^{-1}(\mathbb{R}_t \times \mathbb{R}_x^3 \times \mathbb{R}_{\xi}^3)$. The matrix-valued distribution Y satisfies $Y = Y^*$, $\Pi_{\pm} Y \Pi_{\pm} = 0$ and is given by

$$Y = \frac{i}{4(\xi - A)^2} \left([\partial_t P, W] - \frac{1}{2}[P, \{P, W\} - \{W, P\}] \right). \quad (3.11)$$

Finally, we have that

$$\iint \text{tr}(W(t, x, \xi)) dx d\xi = 1 \quad \text{for almost every } t, \quad (3.12)$$

and

$$\rho^h(t, \cdot) = \int \text{tr}(W^h(t, \cdot, \xi)) d\xi \xrightarrow{h \rightarrow 0} \int \text{tr}(W(t, \cdot, \xi)) d\xi =: \rho(t, \cdot), \quad (3.13)$$

$$J_k^h(t, \cdot) = \int \text{tr}(\alpha_k W^h(t, \cdot, \xi)) d\xi \xrightarrow{h \rightarrow 0} \int \text{tr}(\alpha_k W(t, \cdot, \xi)) d\xi =: J_k(t, \cdot), \quad (3.14)$$

narrowly on $\mathbb{R}_x^3$ for almost every t .

3.1.5 Analogy with Navier-Stokes

The matrix-valued distribution Y should be thought of as the Lagrange multiplier associated with the constraint $[P, W] = 0$. There is an interesting analogy with the pressure term $p \equiv p(t, x) \in \mathbb{R}$ in the Navier-Stokes equation for incompressible fluids

$$\begin{aligned} \partial_t u + u \cdot \nabla_x u + \nabla_x p &= \Delta_x u, & x \in \mathbb{R}^3 \\ \nabla_x \cdot u &= 0, \end{aligned}$$

Notice that Y is analogous to p and $-i[P, \cdot]$ is analogous to ∇_x .

There is also an analogy with Lemma 2. Indeed, one can express the pressure p solely in terms of u and its derivatives:

$$-\Delta_x p = \nabla_x^2 : (u \otimes u).$$

Therefore

$$p = (-\Delta_x)^{-1} \nabla_x^2 : (u \otimes u).$$

This is completely analogous to the formula

$$Y = \frac{1}{4\langle \xi - A \rangle^2} ([\partial_t P, W] - \frac{1}{2}[P, \{P, W\} - \{W, P\}]).$$

3.2 The “Projected” Limit

Recall the matrix-valued Liouville equation:

$$\begin{cases} \partial_t W = \frac{1}{i}[P, Y] + \frac{1}{2}(\{P, W^h\} - \{W^h, P\}), \\ [P, W] = 0. \end{cases}$$

Given that $P = \lambda_+ \Pi_+ + \lambda_- \Pi_-$, it follows that $\Pi_\pm [P, Y] \Pi_\pm = 0$. Consequently, the time evolution of W projected onto the subspaces defined by $\Pi_\pm$ simplifies to:

$$\Pi_\pm \partial_t W \Pi_\pm = \frac{1}{2} \Pi_\pm \left(\{P, W^h\} - \{W^h, P\} \right) \Pi_\pm, \quad [\Pi_\pm, W] = 0.$$

Again, an analogy can be drawn with the Navier-Stokes equation, where the Leray-Helmholtz projection $\mathbb{P} = I + \nabla(-\Delta)^{-1} \nabla \cdot$ eliminates the pressure gradient ∇p in the equation for divergence-free vector fields:

$$\partial_t u + \mathbb{P}(u \cdot \nabla_x u) = \Delta_x u.$$

Theorem 3. *Let $W_\pm^{\text{in}} = \Pi_\pm W_{\text{in}} \Pi_\pm$, where $\Pi_\pm$ is given by (2.5). Under the assumptions of Theorem 2 it follows that $W_\pm := \Pi_\pm W \Pi_\pm$ solves*

$$\partial_t W_\pm = \{\lambda_\pm, W_\pm\} + [H_{\text{be}} + H_{\text{pc}}, W_\pm], \quad W_\pm|_{t=0} = W_\pm^{\text{in}} \quad \text{in } H_{\text{loc}}^{-1}, \quad (3.15)$$

where $\lambda_\pm$ is given by (2.4) and where the Berry term H_{be} and the Poissonian curvature H_{pc} are given by

$$H_{\text{be}} := [\Pi_\pm, \{\lambda_\pm, \Pi_\pm\} - \partial_t \Pi_\pm], \quad (3.16)$$

$$H_{\text{pc}}^\pm := \pm \langle \xi - A \rangle \Pi_\pm \{\Pi_\mp, \Pi_\mp\} \Pi_\pm, \quad (3.17)$$

In particular, the scalar Wigner measures $f_\pm(t, x, v) := \text{tr}(W_\pm)(t, x, \xi - A(t, x))$ solve the relativistic Vlasov equations with Lorentz force and external magnetic and electric fields A, V for the electron and positron phase space densities f_+ and f_- , respectively, i.e. $f_\pm$ solves

$$\partial_t f_\pm \pm \frac{v}{\langle v \rangle} \cdot \nabla_x f_\pm + (E \pm \frac{v}{\langle v \rangle} \times B) \cdot \nabla_v f_\pm = 0, \quad f_\pm|_{t=0} = f_\pm^{\text{in}} := \text{tr}(W_\pm^{\text{in}}), \quad (3.18)$$

$$E = -\nabla V - \partial_t A, \quad B = \nabla \times A.$$

in H_{loc}^{-1} with the charge and current densities

$$\rho(t, x) = \int_{\mathbb{R}^3} (f_+ + f_-)(t, x, v) dv, \quad J(t, x) = \int_{\mathbb{R}^3} \frac{v}{\langle v \rangle} (f_+ - f_-)(t, x, v) dv. \quad (3.19)$$

For completeness we give explicit expressions for the Berry term H_{be} and the Poissonian curvature H_{pc} . Recall that $v(t, x, \xi) = \xi - A(t, x)$. The Berry term H_{be} is given by

$$H_{\text{be}} = -\frac{1}{2\langle v \rangle^2} (\beta \alpha \cdot F_{\pm}^{\text{L}} - i\gamma^5 \alpha \cdot (v \times F_{\pm}^{\text{L}})), \quad (3.20)$$

$$F_{\pm}^{\text{L}} := E \pm \frac{v}{\langle v \rangle} \times B = -\nabla V - \partial_t A \pm \frac{v}{\langle v \rangle} \times (\nabla \times A), \quad (3.21)$$

where $F_{\pm}^{\text{L}}$ denotes the Lorentz force for electrons and positrons. Note that the electric field E in the definition of $F_{\pm}^{\text{L}}$ contains the magnetic contribution $\partial_t A$ which comes from the term $\partial_t \Pi_j$ and which was omitted in [7, 11] since these references assume that the electromagnetic potentials do not depend on t .

The Poissonian curvature H_{pc} is given by

$$H_{\text{pc}} = \pm \frac{i}{2\langle v \rangle} \Pi_{\pm} (\gamma^5 (\alpha \cdot B)) \Pi_{\pm}. \quad (3.22)$$

The proof of Theorem 3 essentially relies on the following observation which connects derivations of two commuting elements of an algebra of which one is a projection.

Lemma 3. *Let Π be a projection ($\Pi^2 = \Pi$), let W satisfy $[\Pi, W] = 0$, and let D be a derivation, i.e., $D(AB) = (DA)B + A(DB)$. Then:*

$$\Pi(DW)\Pi = D(\Pi W \Pi) - [\Pi W, [\Pi, D\Pi]].$$

Note that this lemma holds independently of the specific structure of W , except the condition that it should commute with the projection Π . This lemma is now applied to

$$\Pi_{\pm} \partial_t W \Pi_{\pm} = \frac{1}{2} \Pi_{\pm} \left(\{P, W^{\hbar}\} - \{W^{\hbar}, P\} \right) \Pi_{\pm},$$

by observing that

$$\partial_t \quad \text{and} \quad \{P, \cdot\}$$

are derivations on $\mathfrak{H}$. One immediately notices that

$$\frac{1}{2} \Pi_{\pm} (\{P, W^{\hbar}\} - \{W^{\hbar}, P\}) \Pi_{\pm} = \Pi_{\pm} \{\lambda_{\pm}, W\} \Pi_{\pm} + \frac{\lambda_+ - \lambda_-}{2} \Pi_{\pm} (\{\Pi_{\mp}, W^{\hbar}\} - \{W^{\hbar}, \Pi_{\mp}\}) \Pi_{\pm}. \quad (3.23)$$

From the first term on the right hand side we extract, using Lemma 3,

$$\begin{aligned} \Pi_{\pm} \{\lambda_{\pm}, W\} \Pi_{\pm} &= \{\lambda_{\pm}, \Pi_{\pm} W \Pi_{\pm}\} - [\Pi_{\pm} W \Pi_{\pm}, [\Pi_{\pm}, \{\lambda_j, \Pi_{\pm}\}]] \\ &= \{\lambda_{\pm}, W_{\pm}\} - [W_{\pm}, [\Pi_{\pm}, \{\lambda_j, \Pi_{\pm}\}]]. \end{aligned}$$

The second term is the Poisson bracket part of the Berry term. The second term on the right hand side of (3.23) becomes, after suitable manipulation, the Poissonian curvature term.

3.3 The BMT equation

Finally we explain how the BMT equation (3.27) can be obtained from the semiclassical limit of the Dirac equation. Suppose that $\hat{A}$ is an observable (i.e. a self-adjoint operator on $\mathfrak{H}$). Let a be its (matrix-valued) Weyl symbol such that

$$[P, a] = 0. \quad (3.24)$$

The expectation value of $\hat{A}$ is given by

$$\text{Tr}(\hat{A}R(t)) = \iint \text{tr}(a(x, \xi)W^h(t, x, \xi))dx d\xi.$$

Supposing that a is sufficiently regular, we obtain in the limit $\hbar \rightarrow 0$

$$\begin{aligned} \iint \text{tr}(a(x, \xi)W(t, x, \xi))dx d\xi &\Rightarrow \frac{d}{dt} \int \text{tr}(aW) = \int \underbrace{\text{tr}(a \frac{1}{i}[P, Y])}_{=0} + \int \text{tr}(a \frac{1}{2}(\{P, W\} - \{W, P\})) \\ &= - \int \text{tr}(\frac{1}{2}(\{P, a\} - \{a, P\})W). \end{aligned}$$

The first term on the right hand side of the first line vanishes on account of (3.24) and the cyclicity of the trace. Otherwise put, the time evolution of $\hat{A}$ in the semiclassical limit is given by

$$\frac{d}{dt} \int \text{tr}(aW) + \int \text{tr}(\frac{1}{2}(\{P, a\} - \{a, P\})W) = 0. \quad (3.25)$$

In particular, choosing $a \rightarrow \Pi_{\pm} a \Pi_{\pm}$ gives the time evolution for the eigenmodes $W_{\pm} = \Pi_{\pm} W \Pi_{\pm}$ of the Dirac Hamiltonian

$$\frac{d}{dt} \int \text{tr}(aW_{\pm}) + \int \text{tr}(\Pi_{\pm} \frac{1}{2}(\{P, a\} - \{a, P\})\Pi_{\pm} W) = 0. \quad (3.26)$$

Now the BMT equation for the spin $\mathbf{s}$ reads

$$\frac{d}{dt} \mathbf{s} = \frac{e}{mc} \mathbf{s} \times \left(\frac{1}{\gamma} B - \frac{1}{c(1+\gamma)} v \times E \right), \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (3.27)$$

In [11], Spohn derived the BMT equation (3.27) for the electron (represented by W_+) by choosing

$$\hat{A} = \mathbf{a} \cdot \mathbf{S},$$

where $\mathbf{a} \cdot \mathbf{S}$ is the polarization matrix of the spin along a direction $\mathbf{a} \in \mathbb{R}^3$, $|\mathbf{a}|^2 = 1$. More explicitly, the (matrix-valued) symbol $a_{\mathbf{S}}$ of $\mathbf{a} \cdot \mathbf{S}$ is given by

$$a_{\mathbf{S}}(x, \xi) = \left(\beta \alpha \cdot \mathbf{g} + \frac{1}{\langle \xi - A \rangle} \mathbf{a} \cdot (\xi - A) \right) \gamma^5, \quad \gamma^5 = -i\alpha_1 \alpha_2 \alpha_3,$$

where

$$\mathbf{g}(x, \xi) = \mathbf{a} - \frac{(\mathbf{a} \cdot (\xi - A))(\xi - A)}{\langle \xi - A \rangle (\langle \xi - A \rangle + 1)}.$$

Then, at the classical level, the spin $\mathbf{s}$ is represented by

$$\mathbf{s}(t) := \iint \mathrm{tr}(a_{\mathbf{S}}(x, \xi) W(t, x, \xi)) dx d\xi.$$

Thus, (3.26) with $a = a_{\mathbf{S}}$ yields the BMT equation (3.27).

This shows why the spin does not show up in the relativistic Vlasov-Maxwell equation (1.1): The time evolution of $\mathrm{tr}(W_{\pm})$ does not contain any information about the observable a . One has to evolve $\mathrm{tr}(aW_{\pm})$ in time.

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