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UNBOUNDED SOLUTIONS TO THE EULER EQUATION

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ABSTRACT. In this article, we will study unbounded solutions of the 2D incompressible Euler equations. One of the motivating factors for this is that the usual functional framework for the Euler equations (e.g. based on finite energy conditions, such as L^2) does not respect some of the symmetries of the problem, such as Galileo invariance.

Our first result, global existence and uniqueness of solutions for initial data with square-root growth $O(|x|^{1/2-\varepsilon})$ and bounded vorticity, is based on two key ingredients. Firstly an integral decomposition of the pressure, and secondly examining local energy balance leading to solution estimates in weighted L^2 spaces. We also prove continuity of the initial data to solution map by a substantial adaptation of Yudovich's uniqueness argument.

The second result is global existence and uniqueness of solutions for initial data with sublinear growth $O(|x|^{1-\varepsilon})$. This is work in progress.

1. INTRODUCTION

The Euler equations

$$u_t + u \cdot \nabla u + \nabla \pi = 0, \quad \nabla \cdot u = 0, \quad \text{in } \mathbb{R} \times \mathbb{R}^2$$

are fundamental in fluid dynamics. Global existence and uniqueness under natural weak conditions is well understood. A qualitative understanding of the dynamics is important and challenging.

Here $u(t, x) \in \mathbb{R}^2$ is the velocity at time t and the point x , so that the trajectory of a particle satisfies

$$\dot{x} = u(t, x)$$

and it defines a flow map $x(t) = \Phi(t, x(0))$. The incompressibility condition $\nabla \cdot u = 0$ ensures that the flow map Φ is area preserving,

$$|\Phi(t, A)| = |A|.$$

The term $u_t + u \cdot \nabla u$ is the time derivative of the momentum of the particle, hence the acceleration. The acceleration is driven by the negative gradient of the pressure, which is the only force field in the equation. Taking divergence of the equation yields

$$(1) \quad -\Delta \pi = \partial_{ij}^2 (u^i u^j)$$

with the summation convention that there is an implicit sum on repeated exponents/indices.

The Navier-Stokes equations are the basic equation for the flow of incompressible fluids. The viscosity becomes negligible in many physically relevant situations whenever one considers large spatial scales like in oceans and the atmosphere, which is compressible and has a vertical scale much smaller than the horizontal one. Likewise, compressibility seems not relevant on large scales. This is especially true in

the presence of a strong Coriolis force, where the Talor-Proudman theorem also implies that the vertical velocity component loses its importance on large scale, which leads with some simplifications to the 2d Euler equations.

Taking the curl of the equation we arrive at the equation for the vorticity $\xi = \nabla \times u$,

$$\xi_t + u \cdot \nabla \xi = 0$$

which is a transport equation with the divergence free vector field u . As a consequence, $\|\xi(t)\|_{L^p}$ is constant under weak assumptions on the solutions or the initial data. The most interesting case is $p = \infty$, which gives a log-Lipschitz continuity for the velocity u with respect to x , just enough to get uniqueness for the flow map and the solutions to the Euler equation itself.

This is a fundamental result of Yudovich [Yud95; Jud63]: Weak solutions are unique as soon as the vorticity ξ is in $L^\infty(\mathbb{R}^2)$ and the velocity is in $L^2(\mathbb{R}^2)$. Let us sketch the argument, which we will modify later. Let P be defined by the Leray projection to solenoidal vector fields in L^2 and consider

$$(2) \quad P(\partial_t u + u \cdot \nabla u) = 0.$$

Let u_1 and u_2 be solutions and $v = u_2 - u_1$ and $u = u_2 + u_1$. We recall that the kinetic energy is conserved and also the L^p norms of $\nabla \times u$. Let $1 < r < \infty$ and

$$K = \|u_1(0)\|_{L^2} + \|\nabla \times u_1(0)\|_{L^\infty} + \|\nabla \times u_1(0)\|_{L^r} \\ + \|u_2(0)\|_{L^2} + \|\nabla \times u_2(0)\|_{L^\infty} + \|\nabla \times u_2(0)\|_{L^r}.$$

The equation for the difference v is

$$v_t + u \cdot \nabla v + v \cdot \nabla u + \nabla \pi = 0, \quad -\Delta \pi = 2\partial_{ij}^2(u^i v^j)$$

where we choose the solution to the pressure equation given by the convolution with the fundamental solution. This is equivalent to the formulation (2). The key for uniqueness is the estimate

$$(3) \quad \left| \frac{d}{dt} \|v(t)\|_{L^2}^2 \right| \leq 8e^2 K \|v(t)\|_{L^2}^2 \log \left(\frac{2K}{\|v(t)\|_{L^2}} \right),$$

which we will derive in a moment. We recall that $\|v(t)\|_{L^2} = \|v_0\|_{L^2} \leq K$. It implies by Grönwall's inequality (and solving

$$\dot{z} = 8e^2 K z \log(2K/z)$$

that

$$\|v(t)\|_{L^2} \leq \|v(0)\|_{L^2}^{e^{-8e^2 K t}} (2K)^{1-e^{-8e^2 K t}}.$$

The estimate (3) is proven as follows:

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2}^2 = - \int v^j u^i \partial_i v^j + v^j v^i \partial_i u^j dx = - \int v^j v^i \partial_i u^j dx$$

Hence, if $1/q + 1/p = 1$, by Hölder's inequality, and, with $p > \max\{2, r\}$

$$\left| \frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 \right| \leq \|u - v\|_{L^{2p/(p-1)}}^2 \|\nabla(u + v)\|_{L^p} \\ \leq \|u - v\|_{L^2}^{2p/(p-1)} \left(\|u\|_{L^\infty} + \|v\|_{L^\infty} \right)^{2/(p-1)} \left(\|\nabla u\|_{L^p} + \|\nabla v\|_{L^p} \right) \\ \leq 8pK \|u - v\|_{L^2}^2 \left(\frac{K}{\|v\|_{L^2}} \right)^{2/p}$$

where we used the Calderón-Zygmund estimate (recall $\nabla \cdot u = 0$)

$$(4) \quad \|\nabla u\|_{L^p(\mathbb{R}^2)} \leq \frac{4p^2}{p-1} \|\nabla \times u\|_{L^p(\mathbb{R}^n)} \leq \frac{4p^2}{p-1} \|\nabla \times u\|_{L^\infty}^{(p-r)/p} \|\nabla \times u\|_{L^r}^{r/p} \leq 8pK.$$

Of course the arguments work without the explicit constant 4. We then optimize our choice of p . Define the exponent

$$p = \log \left(\frac{2e^r K}{\|v(t)\|_{L^2}} \right) \geq r.$$

Then, the last factor in (4) becomes

$$\left(\frac{K}{\|v\|_{L^2}} \right)^{2/\log(K/\|v\|_{L^2})} = e^2$$

and we obtain (3), and hence an estimate of the modulus of continuity.

In this text, we will consider solutions with bounded vorticity. This is consistent with the scaling of the equations, which possess a two-parameter family of scaling invariances. For any $\lambda, \mu > 0$ and any solution $u(t, x)$, the function

$$u_{\lambda\mu}(t, x) := \lambda^{-1} \mu u(\mu t, \lambda x)$$

also is a solution. This means that function spaces with the same scaling for u are given by the norms

$$\|\nabla u\|_{L^\infty}, \quad \| |x|^{-2} u_0 \|_{L^2}, \quad \sup_{R>0} R^{-2} \|u_0\|_{L^2(B_R(0))}.$$

This observation about scaling has an interpretation in terms of energy balance in the solution. Consider a solution which behaves roughly as $u(t, x) \propto |x|^\alpha$, so that $\alpha < 1$ corresponds to a subcritical situation, $\alpha = 1$ a critical one, and $\alpha > 1$ is a supercritical situation. Then, the kinetic energy in a ball $B_R(0)$ of radius $R > 0$ and the energy flux through the sphere $\partial B_R(0)$ due to convection are, respectively

$$E_R(t) \propto R^{2+2\alpha} \quad \text{and} \quad \Sigma_R(t) \propto R^{1+3\alpha}.$$

Energy conservation implies that $E'_R(t) \approx \Sigma_R(t)$ (neglecting the contribution of the pressure, which we will see is of same order as $\Sigma_R(t)$). Therefore, we have

$$E'_R(t) \ll E_R(t), \quad E'_R(t) \approx E_R(t) \quad \text{or} \quad E'_R(t) \gg E_R(t)$$

depending on whether the exponent α is subcritical, critical or critical. From these formal differential inequalities, we expect solutions to exist globally in time for subcritical exponents $\alpha < 1$, and to blowup in finite time for supercritical exponents $\alpha > 1$, while the critical case $\alpha = 1$ is borderline. In fact, is it easy to show that Lipschitz solution may **blow up** and may **not be unique**. An explicit example is the following:

$$u(t, x) = \frac{1}{1-t/T} \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}, \quad \pi(t, x) = -\frac{1}{2T(1-t/T)^2} (x_1^2 - x_2^2 + T(x_1^2 + x_2^2)).$$

Since $u = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$ is a smooth stationary solution there is non-uniqueness, and since the solution blows up at time $t = T$ this solution cannot be extended to a global solution.

In addition to scaling, there is another source of non-uniqueness, the **Galilean symmetry**. Let $\gamma \in C^1(\mathbb{R}; \mathbb{R}^2)$ and

$$(5) \quad u_\gamma(t, x) = u(t, x - \gamma(t)) + \dot{\gamma}(t), \quad \pi_\gamma = \pi(t, x - \gamma(t)) - \ddot{\gamma} \cdot x.$$

This corresponds to a (non-inertial) change of reference frames, where the inertial force is absorbed in the pressure. A simple calculation shows that u_γ and π_γ satisfy the Euler equations. If $\gamma(0) = \dot{\gamma}(0) = 0$ then the initial data is $u_\gamma(0, x) = u_0(x)$. Any choice of γ with these initial conditions leads to a solution with the same initial data.

Despite causing non-uniqueness issues, there is an appealing feature of the Galilean symmetry: Let $\gamma(t)$ be the path of the particle which is at $x = 0$ at time zero. The Galilean transform leads to a solution for which the velocity is 0 at $x = 0$ and the particle stays at $x = 0$. We should note that infinite energy solutions (*i.e.*, non- L^2 solutions) are especially relevant to large scale fluids, such as oceans. There, bounded energy is inadequate and even boundedness seems too restrictive. Moreover the projection P to solenoidal vector fields is a global (and not a local) normalization and it becomes questionable.

There has been considerable interest in unbounded solutions. It is not possible to cover all contributions and the following list is incomplete.

- (1) Cozzi and Kelliher [CK19] consider local existence for mildly unbounded solutions with bounded vorticity.
- (2) Chae and Wolf [CW21] prove local existence to the Euler equation close to Lipschitz spaces in all space dimensions.
- (3) Elgindi and Jeong [EJ20] consider linear growth under symmetry assumptions.
- (4) McOwen and Topalov [MT17] proved well-posedness for functions growing slower than $\sqrt{|x|}$ at infinity, but with some expansion at ∞ .
- (5) Cobb [Cob24] proved uniqueness under a number of equivalent normalization conditions for infinite energy bounded solutions.

There are two main cases in our study. If the initial datum satisfies $u_0(x) = O(|x|^\alpha)$, then we distinguish between the case where $\alpha < 1/2$ and the case where $1/2 \leq \alpha < 1$. This is very much linked to the non-uniqueness issue posed by the Galilean symmetry (5), which acts as a gauge group for the problem. Solutions can only be unique up to the choice of a reference frame. We consider two different ways to fix this gauge, depending on the value of α .

We give a formal explanation of why these two cases naturally appear. Once again, consider a solution which behaves as $u(t, x) \propto |x|^\alpha$. Substituting this in (2) gives

$$\begin{aligned} \partial_t u &\propto -P(u \cdot \nabla u + \nabla(-\Delta)^{-1} \partial_{ij}^2(u^i u^j)) \\ &\propto |x|^{2\alpha-1}. \end{aligned}$$

Observe that if $\alpha < 1/2$, the derivative $\partial_t u$ naturally has some decay at $|x| \rightarrow \infty$, whereas it is not the case if $\alpha \geq 1/2$. On the one hand, if $\alpha < 1/2$, then a way to fix the gauge $\gamma(t)$ is to enforce some kind of decay for $u(t) - u(0)$ at infinity (*i.e.*, a primitive of $\partial_t u$). On the other hand, this strategy is no longer available for $\alpha \geq 1/2$.

This is linked to the fact that below $1/2$ one can define the Leray projection, but this is impossible under weaker growth conditions. The critical issue is equation (1). The gradient of the pressure enters the equation. For any growth below the square root we can express the gradient of the pressure in (1) by a convolution with the right hand side. The integral kernel has homogeneity -3 . Local regularity is easy,

since the equation is linear. However at infinity we have to be able to integrate

$$\int \langle x \rangle^{-3} |u|^2 dx$$

which is possible for any growth $|u| \leq c \langle x \rangle^\alpha$ with $\alpha < 1/2$. This is the situation studied in [CK26].

One may consider the Leray projection to solenoidal vector fields as normalization to ensure uniqueness. Above the square root growth we choose a different normalization: $u(t, 0) = 0$ and we use a frame in which the particle at the origin does not move. The gauge $\gamma(t)$ then solves the ODE $\dot{\gamma} + u(t, -\gamma) = 0$. This part is work in progress.

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2. THE SUB-SQUARE-ROOT CASE

In this case we continuously extend the Euler flow from L^2 to Y^α , $0 \leq \alpha < 1/2$ where the space is defined by

$$Y^\alpha = \left\{ u : \nabla \times u \in L^\infty, \|u_0\|_{L_\alpha^2} := \|(1 + |x|^2)^{(1+\alpha)/2} u_0\|_{L^2(\mathbb{R}^2)} < \infty \right\}.$$

Here $\nabla \times u = \partial_1 u^2 - \partial_2 u^1$ is again the curl. Observe that all regular data satisfying $u_0(x) = O(|x|^\alpha)$ as $|x| \rightarrow \infty$ lie in this space. As discussed above the crucial question is the definition (or normalization) of the pressure. Consider

$$-\Delta \pi = \partial_{ij}^2 (u^i u^j).$$

The condition $u \in L^\infty$ is not sufficient to define the pressure π as a function. This is because $\pi = (-\Delta)^{-1} \partial_{ij}^2 (u^i u^j)$ is the image of L^∞ functions by Riesz transforms, and therefore are BMO functions, which are only defined up to the addition of a constant. However, only the gradient $\nabla \pi$ enters the equation and we can fully define $\nabla \pi$ as long as $u \in L_\alpha^2$ for $0 \leq \alpha < 1/2$. This is because the kernel of $\nabla \Delta^{-1} D^2$ decays like $|x|^{-3}$ away from $x = 0$ and we need sublinear growth of $u \otimes u$ for the convolution integral to be defined. In this case we can write

$$(6) \quad P(u_t + u \cdot \nabla u) = 0, \quad \nabla \cdot u = 0, \quad u(0) = u_0$$

where P is again the (extension of the) Leray projection to the solenoidal vector fields in L^2 .

Theorem 1 (Sub-square-root growth). *Let $0 < \alpha < 1/2$. Given $u_0 \in Y^\alpha$ there is a unique weak solution $u \in C((0, T); L_\alpha^2)$ to the projected equation (6) for all $T > 0$. Moreover*

$$(7) \quad \|\nabla \times u(t)\|_{L^\infty} = \|\nabla \times u_0\|_{L^\infty},$$

$$(8) \quad \|u(t)\|_{L_\alpha^2} \leq \|u_0\|_{L_\alpha^2} + c_\alpha t^{(1+\alpha)/(1-\alpha)} \|u_0\|_{L_\alpha^2}^{2/(1-\alpha)} + c_\alpha t^{2(1+\alpha)/(1-\alpha)} \|u_0\|_{L_\alpha^2}^{2/(1-\alpha)}.$$

Let $\alpha < \gamma < 1 - \alpha$, u, v two solutions, with

$$K(T) = \|\nabla \times u\|_{L^\infty} + \|\nabla \times v\|_{L^\infty} + \sup_{0 \leq t \leq T} \left(\|u(t)\|_{L_\alpha^2} + \|v(t)\|_{L_\alpha^2} \right).$$

Then there exists a constant c_α so that

$$\|u(t) - v(t)\|_{L_\gamma^2} \leq c_\alpha \|u(0) - v(0)\|_{L_\gamma^2}^{e^{-c_\alpha K(t)t}} K^{1-c_\alpha K(t)t}.$$

In particular $u \in C([0, \infty); L_\alpha^2)$.

There are two different parts: The local energy estimate and a nontrivial adaption of Yudovich's uniqueness proof – more precisely we need a continuity estimate for the dependence of the solution on the initial data. Then we can define an extension of the flow map to Y^α from the Yudovich setting $\nabla \times u_0 \in L^\infty \cap L^r$, $u_0 \in L^2$.

We consider weak solutions u which satisfy for all $T > 0$

$$(9) \quad \|\nabla \times u\|_{L^\infty((0, \infty) \times \mathbb{R})} < \infty, \quad \sup_{0 < t < T} \|u(t)\|_{L_\alpha^2} < \infty$$

3. THE WEIGHTED ENERGY ESTIMATE

It will be convenient to have a variant of the function space L_α^2 for $R \geq 1$

$$\|u_0\|_{L_\alpha^2(R)} = \|(R^2 + |x|^2)^{-1-\alpha} u_0\|_{L^2}.$$

The norm is decreasing in R .

Proposition 2. *Suppose that u is a weak solution which satisfies (9) and $R > 1$. Then for almost all t*

$$(10) \quad \left| \frac{d}{dt} \|u(t)\|_{L_\alpha^2(R)}^2 \right| \leq c(\alpha) \left(R^{\alpha-1} \|u(t)\|_{L_\alpha^2(R)} + (R^{\alpha-1} \|u(t)\|_{L_\alpha^2(R)} \|\nabla \times u(t)\|_{L^\infty})^{1/2} \right) \|u(t)\|_{L_\alpha^2(R)}^2$$

and $\|\nabla \times u(t)\|_{L^\infty}$ is independent of t .

Proof. We compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int (R^2 + |x|^2)^{-1-\alpha} |u|^2 dx &= \int (R^2 + |x|^2)^{-1-\alpha} u \cdot u_t dx \\ &= - \int (R^2 + |x|^2)^{-1-\alpha} (u \cdot (\nabla \cdot u) u + u \cdot \nabla \pi) dx \\ &= \int \left(2u^i |u|^2 \partial_i (R^2 + |x|^2)^{-1-\alpha} - (R^2 + |x|^2)^{-\alpha} u^i \partial_i \pi \right) dx. \end{aligned}$$

The derivative of the weight gains decay

$$\left| \nabla (R^2 + |x|^2)^{-1-\alpha} \right| \leq 2(1+\alpha) |x| (R^2 + |x|^2)^{-2-\alpha}.$$

We bound the first term by

$$(11) \quad 4 \|(R^2 + |x|^2)^{-\frac{1}{2}} u\|_{L^\infty} \|u\|_{L_\alpha^2}^2.$$

The second term is more involved. We use two lemmas.

Lemma 3. *The pressure can be written as*

$$(12) \quad \nabla \pi = \nabla(\pi_<) + (\nabla \pi)_0 + (\nabla \pi)_>$$

so that

$$\|(\nabla \pi)_0\|_{L_\alpha^2} + \|\pi_<\|_{L_{1+\alpha}^2(R)} \lesssim R^{(\alpha-1)/2} \|(R^2 + |x|^2)^{(1+\alpha)/4} u\|_{L^\infty} \|u\|_{L_\alpha^2(R)}$$

and

$$\|(\nabla\pi)_{>}\|_{L^2_\alpha(R)} \lesssim R^{1-\alpha}\|u\|_{L^2_\alpha(R)}^2.$$

Using this lemma we bound the terms with the pressure:

$$\begin{aligned} \left| \int (R^2 + |x|^2)^{-(1+\alpha)} \nabla \cdot (u\pi_{<}) dx \right| &= \left| \int u\pi_{<} \nabla (R^2 + |x|^2)^{-(1+\alpha)} dx \right| \\ &\leq 4\|u\|_{L^2_\alpha(R)} \|\pi_{<}\|_{L^2_{1+\alpha}(R)} \\ &\leq cR^{(\alpha-1)/2} \|u\|_{L^2_\alpha(R)}^2 \|(R^2 + |x|^2)^{(1+\alpha)/4} u\|_{L^\infty}, \end{aligned}$$

and

$$\begin{aligned} \left| \int (R^2 + |x|^2)^{-1-\alpha} u(\nabla\pi)_0 dx \right| &\leq \|u\|_{L^2_\alpha(R)} \|(\nabla\pi)_0\|_{L^2_\alpha(R)} \\ &\leq cR^{(\alpha-1)/2} \|u\|_{L^2_\alpha(R)}^2 \|(R^2 + |x|^2)^{(1+\alpha)/4} u\|_{L^\infty}, \end{aligned}$$

similarly

$$\left| \int (R^2 + |x|^2)^{-1-\alpha} u(\nabla\pi)_{>} dx \right| \leq \|u\|_{L^2_\alpha(R)} \|(\nabla\pi)_{>}\|_{L^2_\alpha(R)} \leq cR^{\alpha-1} \|u\|_{L^2_\alpha(R)}^3.$$

We collect the estimates

$$\left| \frac{d}{dt} \|u\|_{L^2_\alpha(R)}^2 \right| \leq c \left(R^{\alpha-1} \|u\|_{L^2_\alpha(R)} + R^{(\alpha-1)/2} \|(R^2 + |x|^2)^{(1+\alpha)/4} u\|_{L^\infty} \right) \|u\|_{L^2_\alpha(R)}^2.$$

Lemma 4 (Interpolation estimate). *The following estimate holds*

$$\|(R^2 + |x|^2)^{\frac{1+\alpha}{4}} u\|_{L^\infty} \leq c \left(\|u\|_{L^2_\alpha(R)}^{1/2} \|\nabla \times u\|_{L^\infty}^{1/2} + R^{-\frac{1-\alpha}{2}} \|u\|_{L^2_\alpha(R)} \right).$$

Together

$$(13) \quad \left| \frac{d}{dt} \|u\|_{L^2_\alpha(R)}^2 \right| \leq c \left(R^{\alpha-1} \|u\|_{L^2_\alpha(R)} + R^{(\alpha-1)/2} \|u\|_{L^2_\alpha(R)}^{1/2} \|\nabla \times u_0\|_{L^\infty}^{1/2} \right) \|u\|_{L^2_\alpha(R)}^2.$$

Proof of Lemma 3. We fix $\eta_0 \in C_c^\infty(\mathbb{R}^2)$, supported in a ball of radius 1, identically 1 on ball of radius $1/2$ and define $\eta_j(x) = \eta_0(2^{-j}x) - \eta_0(2^{1-j}x)$. Then

$$1 = \sum_{j=0}^{\infty} \eta_j(x).$$

We decompose (we are only interested in the values on $B^{2^j}(0)$)

$$\pi = \pi_{j,<} + \pi_{j,>}$$

by

$$\Delta \nabla \pi_{j,>} = \nabla \partial_{ij}^2 \left[1 - \eta_0(2^{-j-4}x) u^i u^j \right]$$

which we normalize by $\pi_{j,>}(0) = 0$ (recall that the solution of the above $\nabla \pi_{j,>}$ is only defined up to a constant) and

$$\Delta \pi_{j,<} = \partial_{ij}^2 \left[\eta_0(2^{-j-4}x) u^i u^j \right].$$

We define

$$\begin{aligned} \nabla \pi_{>}(x) &= \sum_{j=0}^{\infty} \nabla(\eta_j(x) \pi_{j,>}(x)) - \sum_{j=0}^{\infty} (\nabla \eta_j(x)) \pi_{j,>}(x), \\ (\nabla \pi)_0 &= - \sum_{j=0}^{\infty} (\nabla \eta_j(x)) \pi_{j,<}(x) \end{aligned}$$

and

$$\pi_{<}(x) = \sum_{j=0}^{\infty} \eta_j(x) \pi_{j,<}(x).$$

The desired estimates are immediate. $\square$

Proof of Lemma 4. Fix $x \in \mathbb{R}^2$, either in $B_R(0)$, or in an annulus $B_{2^j R} \setminus B_{2^{j-1} R}$. Let $v(x) = \eta_0(x/(2^{j+1}R))u$. Then

$$\nabla \times v = \eta_0\left(\frac{x}{2^{j+4}R}\right) \nabla \times u + (u^2 \partial_1 - u^1 \partial_2) \eta_0\left(\frac{x}{2^{j+4}R}\right)$$

and

$$\nabla \cdot v = u \cdot \nabla \eta_0\left(\frac{x}{2^{j+4}R}\right).$$

We represent v by singular integrals similar to the arguments above. $\square$

Since the velocity is log Lipschitz and grows slower than linearly at infinity the L^∞ norm of the vorticity is conserved. There are several immediate consequences of (13). Suppose that

$$u \in L^\infty((0, T); L_\alpha^2(R)) \quad \text{with} \quad \nabla \times u \in L^\infty((0, T) \times \mathbb{R}^2)$$

is a weak solution to the Euler equations in the form (2). Then $u : (0, T) \rightarrow L_\alpha^2$ is weakly continuous and by the bound on the derivative of $\|u(t)\|_{L_\alpha^2(R)}^2$ the map $(0, T) \rightarrow \|u(t)\|_{L_\alpha^2(R)}$ is continuous. Thus $(0, T) \ni t \rightarrow u(t) \in L_\alpha^2(R)$ is uniformly continuous and we can assign an initial data $u_0 \in L_\alpha^2(R)$. The norms $L_\alpha^2(R)$ are all equivalent for different $R \geq 1$ and we obtain continuity in L_α^2 and preservation of $\|\nabla \times u(t)\|_{L^\infty}$, hence (7). Let T be large and

$$R \geq c \max \left\{ (T^2 \|u_0\|_{L_\alpha^2} \|\nabla \times u_0\|_{L^\infty})^{1/(1-\alpha)}, (T \|u_0\|_{L_\alpha^2})^{1/(1-\alpha)} \right\}$$

then, by Proposition 2

$$R^{-1-\alpha} \sup_{0 \leq t \leq T} \|u(t)\|_{L_\alpha^2} \leq \sup_{0 \leq t \leq T} \|u(t)\|_{L_\alpha^2(R)} \leq 2 \|u_0\|_{L_\alpha^2(R)} \leq 2 \|u_0\|_{L_\alpha^2}.$$

This implies the estimate (8). $\square$

Suppose $u_0 \in L^2$ and $\nabla \times u_0 \in L^2 \cap L^\infty$. Then there exists a unique global solution to which we can apply the estimates of the previous section. Now let $0 < \alpha < 1/2$ and $u_0 \in L_\alpha^2$, $\nabla \times u_0 \in L^\infty$. We choose a sequence $u_0^n \in L^2$ which converges to u_0 in L_α^2 so that $\sup_n \|\nabla \times u_0^n\|_{L^\infty} < \infty$. We obtain a sequence u^n so that $\|\nabla \times u^n\|_{L^\infty} = \|\nabla \times u_0^n\|_{L^\infty}$ is uniformly bounded (with respect to n) and there exists $K(t)$ so that

$$\|u^n\|_{L_\alpha^2} < K(t).$$

Thus a subsequence converges weakly to a solution with initial data u_0 . Moreover u is a solution with

$$\|\nabla \times u\|_{L^\infty} = \|\nabla \times u_0\|_{L^\infty}, \quad \|u(t)\|_{L_\alpha^2} \leq K(t).$$

This yields existence of a solution, given initial data $u_0 \in Y_\alpha$.

The following lemma will be useful for showing continuity of the initial data to solution map.

Lemma 5. *Let u be a weak solution which satisfies (9). For all $T > 1$*

$$\lim_{R \rightarrow \infty} \sup_{0 \leq t \leq T} \|u(t)\|_{L_\alpha^2(R)} = 0$$

Proof. The statements holds for the initial data. If $T \geq 1$ and R is sufficiently large then

$$\sup_{0 \leq t \leq T} \|u(t)\|_{L_\alpha^2(R)} \leq 2\|u(0)\|_{L_\alpha^2(R)}.$$

This implies the claim of the lemma. $\square$

4. THE CONTINUITY ESTIMATE

We turn to the continuity estimate which is proven by a variant of Yudovich's argument.

Proposition 6. *Let u_1 and u_2 be weak solutions which satisfy (9). Let*

$$\begin{aligned} K_0 &= \|\nabla \times u_1(0)\|_{L^\infty} + \|\nabla \times u_2(0)\|_{L^\infty} \\ K(T) &= K_0 + \sup_{0 \leq t \leq T} (\|u_1(t)\|_{L_\alpha^2} + \|u_2(t)\|_{L_\alpha^2}). \end{aligned}$$

Then

$$\|u_2(t) - u_1(t)\|_{L_\alpha^2} \leq e^{tK(t)} \|u_2(0) - u_1(0)\|_{L_\alpha^2}^{e^{-ct}} K^{1-e^{-ct}}$$

Proof. The function $u = u^2 + u^1$ satisfies

$$\|\nabla \times u(t)\|_{L^\infty} \leq K_0, \quad \sup_{0 \leq t \leq T} \|u(t)\|_{L_\alpha^2} \leq K(T).$$

The difference $v = u^2 - u^1$ satisfies

$$\begin{aligned} v_t + u \nabla v + v \nabla u + \nabla \pi &= 0, \quad \nabla \cdot v = 0 \\ -\Delta \pi &= \partial_{ij}^2 (u^i v^j) \end{aligned}$$

with initial data $v(0) = u_2(0) - u_1(0)$. We first assume $0 < \alpha < 1/3$, choose $(1 + \alpha)/2 \leq \gamma < 1 - \alpha$ and compute

$$\frac{d}{dt} \|v\|_{L_\gamma^2}^2 = - \int (1^2 + |x|^2)^{1+\gamma} v^i v^j \partial_j u^i dx + J$$

where computing as above gives

$$|J| \leq c \left(\|u(t)\|_{L_\alpha^2} + \|u(t)\|_{L_\alpha^2}^{1/2} \|\nabla \times u_0\|_{L^\infty}^{1/2} \right) \|v\|_{L_\gamma^2}^2 \leq cK(t) \|v(t)\|_{L_\gamma^2}^2.$$

We decompose ∇u into two different parts.

Lemma 7. *The gradient of u on $B_r(0)$ for $r \geq 1$ can be decomposed as*

$$\nabla u = (\nabla u)_< + (\nabla u)_>$$

where for $2 \leq p < \infty$

$$\|(\nabla u)_<\|_{L^p} \leq cpr^{2/p} \|\nabla \times u\|_{L^\infty}$$

and

$$\|(\nabla u)_>\|_{L^\infty(B_r)} \leq cr^{-2} \|u\|_{L^2(B_{2r})}.$$

The proof of Lemma 7 is similar to the pressure estimate in Lemma 3. Using Lemma 7 we estimate

$$\int (1 + |x|^2)^{-\gamma} |v|^2 |(\nabla u)_>| dx \leq c \|u\|_{L_\alpha^2} \|v\|_{L_\gamma^2}^2.$$

and by Hölder's inequality, the interpolation estimate and Lemma 7

$$\begin{aligned}
 \int_{B_{2^k} \setminus B_{2^{k-1}}} |v|^2 |(\nabla u)_<| dx &\leq \|v\|_{L^q(B_{2^k} \setminus B_{2^{k-1}})}^2 \|\nabla u\|_{L^p(B_{2^k} \setminus B_{2^{k-1}})} \\
 &\leq p 2^{\frac{2k}{p}} \|v\|_{L^2(B_{2^k} \setminus B_{2^{k-1}})}^{2-2/p} \|u\|_{L^\infty(B_{2^k})}^{2/p} \|\nabla \times u_0\|_{L^\infty} \\
 &\leq cpK_0 \left(\frac{K(t) 2^{k((1+\alpha)/2-\gamma)}}{\|v(t)\|_{L_\gamma^2}} \right)^{2/p} \|v\|_{L^2(B_{2^k} \setminus B_{2^{k-1}})}^2.
 \end{aligned}$$

The exponent $(1+\alpha)/2 - \gamma$ is negative. We choose

$$p = \left(\frac{2K(t)}{\|v(t)\|_{L_\gamma^2}} \right)$$

and obtain

$$(14) \quad \left| \frac{d}{dt} \|v(t)\|_{L_\gamma^2}^2 \right| \leq cK(t) \left(1 + \log \left(\frac{2K}{\|v\|_{L_\gamma^2(R)}} \right) \right) \|v(t)\|_{L_\gamma^2}^2.$$

Next we consider $1/3 \leq \alpha < 1/2$. The argument above does not work as such, because the power $2^{k((1+\alpha)/2-\gamma)}$ no longer decays as $k \rightarrow \infty$. Instead, we will let the exponent γ depend on time to obtain additional decay. We choose $\gamma(0)$ as above and set $\gamma(t) = \gamma_0 + \gamma_1 t$ and fix T so that $\gamma(T) < 1 - \alpha$. The calculation has to be slightly modified:

$$\begin{aligned}
 &\frac{d}{dt} \|v(t)\|_{L_{\gamma(t)}^2}^2 + \gamma_1 \int (1 + |x|^2)^{-1-\gamma(t)} \log(1 + |x|^2) |v(t)|^2 dx \\
 &= \int (1 + |x|^2)^{1+\gamma(t)} v^i(t) v^j(t) \partial_j u^i(t) dx + J
 \end{aligned}$$

The J is estimated as above. We estimate the last term on the right hand side as above and optimize p on each annulus. We choose (recall $1 + \alpha - 2\gamma > 0$)

$$p = \log \left(\frac{2^{k((1+\alpha)/2-\gamma(t))} 2K(t)}{\|v(t)\|_{L_{\gamma(t)}^2}} \right)$$

and obtain

$$\begin{aligned}
 &\frac{d}{dt} \|v\|_{L_{\gamma(t)}^2}^2 + \gamma_1 \int (1 + |x|^2)^{-1-\gamma(t)} \log(1 + |x|^2) v^2 dx \\
 &\leq cK(t) \left(1 + \log \left(\frac{2K(t)}{\|v(t)\|_{L_{\gamma(t)}^2}} \right) \right) \|v\|_{L_{\gamma(t)}^2}^2 \\
 &\quad + c \int \log(1 + |x|^2) (1 + |x|^2)^{\gamma(t)} |v|^2 dx.
 \end{aligned}$$

We choose γ_1 sufficiently large, subtract the last term on the RHS from both sides and obtain

$$\begin{aligned}
 \left| \frac{d}{dt} \|v(t)\|_{L_{\gamma(t)}^2}^2 \right| &\leq cK(t) \left(1 + \log \left(\frac{2K(t)}{\|v(t)\|_{L_{\gamma(t)}^2}} \right) \right) \|v(t)\|_{L_{\gamma(t)}^2}^2 \\
 &\leq CK(t) \log \left(\frac{4K(t)}{\|v(t)\|_{L_{\gamma(t)}^2}} \right) \|v(t)\|_{L_{\gamma(t)}^2}^2.
 \end{aligned}$$

This differential inequality (and (14) if $\alpha < 1/3$ implies the bound

$$\|v\|_{L^2_{\gamma(t)}} \leq \|v(0)\|_{L^2_{\gamma(0)}} e^{-cK(t)t} (4K)^{1-e^{-cK(t)t}}$$

local in time bound, for interval of a length T which is determined by α , γ_0 and $K(t)$. At time T we interpolate

$$\|v(T)\|_{L^2_{\gamma_0}} \leq \|v(T)\|_{L^2_{\gamma_0+T\gamma_1}}^{(\gamma_0-\alpha)/(\gamma_0+T\gamma_1-\alpha)} K^{T\gamma_1/(\gamma_0+T\gamma_1-\alpha)}$$

to obtain the desired bound for $\|v(t)\|_{L^2_{\gamma_0}}$. We iterate this procedure. $\square$

Let $L^2_{\alpha,L} = \{u \in L^2_{\alpha} : \|\nabla \times u\|_{L^{\infty}} \leq L\}$. We claim that

$$L^2_{\alpha,L} \ni u_0 \rightarrow u(t) \in L^2_{\alpha}$$

is continuous in the following sense: Let $u_0 \in Y_{\alpha}$, $\varepsilon > 0$. Then there exists $\delta > 0$ so that

$$\|\nabla v_0\|_{L^{\infty}} \leq L, \quad \|v_0 - u_0\|_{L^2_{\alpha}} < \delta \implies \|v(t) - u(t)\|_{L^2_{\alpha}} < \varepsilon.$$

Let $\varepsilon > 0$. There exists R so that

$$\|u_0\|_{L^2_{\alpha}(R)} < \varepsilon/8.$$

If $\|u_0 - v_0\|_{L^2_{\alpha}(R)} < \varepsilon/8$ then $\|v_0\|_{L^2_{\alpha}(R)} < \varepsilon/4$. If R is sufficiently large then by Lemma 5

$$\sup_{0 \leq t \leq T} \|v(t)\|_{L^2_{\alpha}(R)} < \varepsilon/2.$$

If in addition

$$(15) \quad \|u(t) - v(t)\|_{L^2_{\gamma}} \leq \frac{\varepsilon}{2} R^{\gamma-\alpha}$$

then

$$\|u(t) - v(t)\|_{L^2_{\alpha}} < \varepsilon.$$

By the continuity argument above there exists δ so that (15) holds if

$$\|\nabla \times v_0\|_{L^{\infty}} \leq L, \quad \|v_0 - u_0\|_{L^2_{\alpha}} < \varepsilon/4, \quad \|v_0 - u_0\|_{L^2_{\gamma}} < \delta.$$

This implies continuity.

5. THE SUBLINEAR CASE

Let $0 < \alpha < 1$. We search solutions in the homogeneous space (always assuming bounded vorticity) defined by $\||x|^{-1-\alpha}u\|_{L^2}$. Since u is solenoidal (divergence free) and has bounded vorticity it is Hölder continuous for all Hölder exponents between 0 and 1. As a consequence $u(0, x) = 0$. This is an implicit normalization, an alternative to the Leray projection, which breaks the Galilean symmetry.

Theorem 8 (Sublinear growth). *Let $0 < \alpha < 1$ and suppose that $|x|^{-1-\alpha}u_0 \in L^2(\mathbb{R}^2)$ and $\nabla \times u_0 \in L^{\infty}$. Then there exists a unique global in time solution with*

$$\nabla \times u(t) \in L^{\infty}([0, \infty) \times \mathbb{R}^2), \quad |x|^{-1-\alpha}u \in C([0, \infty); L^2).$$

Moreover

$$\|\nabla \times u(t)\|_{L^{\infty}} = \|\nabla \times u_0\|_{L^{\infty}}$$

and

$$\||x|^{-1-\alpha}u(t)\|_{L^2} \leq c(1 + |t|)^{\mu} \||x|^{-1-\alpha}u_0\|_{L^2(\mathbb{R}^2)}.$$

If u_1 and u_2 are two solutions we define

$$K(t) = \|\nabla \times u_1(0)\|_{L^\infty} + \|\nabla \times u_1(0)\|_{L^\infty} + \sup_{0 \leq s \leq t} \| |x|^{-1-\alpha} (|u_1(s)| + |u_2(s)|) \|_{L^2}.$$

Let $\alpha < \gamma < 1$. Then, with constants c depending on $K(t)$ and α

$$\| |x|^{-1-\gamma} (u_2(t) - u_1(t)) \|_{L^2} \leq \| |x|^{-1-\gamma} (u_1(0) - u_2(0)) \|_{L^2(\mathbb{R}^2)}^{e^{-ct}} (4K)^{1-ce^{ct}}.$$

The starting point is a weighted elliptic existence statement with a Poisson equation estimate, which allows to overcome the limitation of the Leray projection. Note that the lemma below is not a Riesz transform estimate *per se*, as the weight $|x|^{1-\alpha}$ is not Muckenhoupt. We are able to prove (16) by replacing the Riesz transform by a corrected operator, which enforces cancellation of the solution u and some of its derivatives at $x = 0$.

Lemma 9. Consider $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and

$$-\Delta u = f$$

$$\| |x|^{1-\alpha} f \|_{L^2(\mathbb{R}^2)} < \infty.$$

Then there exists a unique solution u with $u(0) = 0$, $\nabla u(0) = 0$ and

$$(16) \quad \| |x|^{-1-\alpha} u \|_{L^2} + \| |x|^{-\alpha} \nabla u \|_{L^2} + \| |x|^{1-\alpha} D^2 u \|_{L^2} \leq \frac{c}{d(\alpha, \mathbb{Z})} \| |x|^{1-\alpha} f \|_{L^2}.$$

Proof. By localization it suffices to prove

$$\| |x|^{-1-\alpha} u \|_{L^2} \leq \frac{c}{d(\alpha, \mathbb{Z})} \| |x|^{1-\alpha} f \|_{L^2}.$$

We use a dyadic decomposition of $\mathbb{R}^n$ and decompose $f = \sum_{\lambda \in 2^{\mathbb{Z}}} f_\lambda$ and $u = \sum_{\lambda} u_\lambda$ where

$$\Delta u_\lambda = f_\lambda.$$

By scaling it suffices to consider $\lambda = 1$ and to construct a unique solution which satisfies

$$\sup_R \left\| (|x|^{-[\alpha]-1} + |x|^{-[\alpha]}) u_1 \right\|_{L^2(B_{2R} \setminus B_R)} \leq \frac{c}{d(\alpha, \mathbb{Z})} \| |x|^{1-\alpha} f_1 \|_{L^2}.$$

We define u_1 by the convolution with a fundamental solution and subtract the Taylor expansion of order $[\alpha]$ (resp. the logarithmic term) at 0 if $\alpha > -2$, resp. the Taylor expansion (resp. the logarithmic term) at ∞ of order $[\alpha] + 1$ if $\alpha < -1$, up to an order depending on $[\alpha]$. The result follows by summation. $\square$

Now the proof of Theorem 8 follows the same line as the proof of Theorem 1. Given $R \geq 1$ we decompose the pressure into two parts with right hand side supported in B_{2R} and outside B_R . For the second we use Lemma 5. We bound similar to above $\|u(t)\|_{L^2_\alpha(R)}$. However, we loose the condition $u(t, 0) = 0$ and reinstate it by a Galilei transform. Checking the L^∞ norm on a suitable ball we obtain estimates for $\| |x|^{-1-\alpha} u(t) \|_{L^2}$.

For the continuity estimate we work as above, again with the decomposition of the pressure. Then we adjust by a Galilei transforms, and control the errors we get.

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