Séminaire Laurent Schwartz

EDP et applications

Année 2025-2026

Exposé n° XIII (5 mai 2026)

Rémi Carles, Quentin Chauleur, and Guillaume Ferriere

DEPENDENCE OF THE NONLINEAR SCHRÖDINGER FLOW UPON THE NONLINEARITY

https://doi.org/10.5802/slsedp.183

Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
http://www.ihes.fr/

Centre de mathématiques Laurent Schwartz
CMLS, CNRS, École polytechnique,
Institut Polytechnique de Paris
F-91128 PALAISEAU CEDEX
https://cmls.ip-paris.fr/

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2266-0607

Séminaire Laurent-Schwartz — EDP et applications
Centre de mathématiques Laurent Schwartz, 2025-2026
Exposé n$^{\text{o}}$ XIII, 1-13

DEPENDENCE OF THE NONLINEAR SCHRÖDINGER FLOW UPON THE NONLINEARITY

RÉMI CARLES, QUENTIN CHAULEUR, AND GUILLAUME FERRIERE

Abstract. We consider the defocusing nonlinear Schrödinger equation in the energy-subcritical case, and investigate the dependence of the solution upon the power of the nonlinearity. Special attention is paid to the global in time description. The main three aspects addressed, in the decreasing order of difficulty, are the limit when the total power tends to one, along with the connection with the logarithmic Schrödinger equation, the description when long range effects may be present, and the continuity of the scattering operator in the short range case. This text resumes the presentation given by the first author at École polytechnique for the Laurent Schwartz seminar, in May 2026.

1. Introduction

We consider the Cauchy problem associated to the defocusing nonlinear Schrödinger equation with power-like nonlinearity,

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^{2\sigma}u \quad ; \quad u_{|t=0} = \phi, \tag{1.1}$$

for $x \in \mathbb{R}^d$, $d \geqslant 1$, in the energy-subcritical case, $0 < \sigma < 2/(d-2)_+$. For clarity, we denote by ϕ_σ the initial data, by u_σ the above solution, as the main goal is to understand the dependence of u_σ with respect to σ, locally in time and, especially, globally in time.

Several papers have considered the continuity of the flow map with respect to the initial data, see e.g. [12, 10, 23, 24] for positive results, and [25, 15, 7, 9] for lack of continuity (according to the function space considered). In the present paper, we address the dependence of the flow map $\phi \mapsto u$ with respect to the nonlinearity, that is with respect to σ. Since the nonlinearity is defocusing, the following quantities are formally conserved by the flow, providing useful a priori estimates,

$$\begin{aligned}
&\text{Mass:} \quad \frac{d}{dt}\|u_\sigma(t)\|^2_{L^2(\mathbb{R}^d)} = 0,\\
&\text{Energy:} \quad \frac{d}{dt}\left(\frac{1}{2}\|\nabla u_\sigma(t)\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{\sigma+1}\|u_\sigma(t)\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^d)}\right) = 0.
\end{aligned}$$

The results from [20] make this statement rigorous: for $\phi_\sigma \in H^1(\mathbb{R}^d)$, we have a unique, global, solution $u_\sigma \in C(\mathbb{R}, H^1)$, and this solution satisfies the above conservation laws. If in addition

$$\phi_\sigma \in \Sigma = \{f \in H^1(\mathbb{R}^d),\ x \mapsto xf(x) \in L^2(\mathbb{R}^d)\},$$

then $u_\sigma \in C(\mathbb{R}, \Sigma)$. The questions we address are threefold, by decreasing order of motivation (and, we believe, of originality):

(1) What can we say about the limit $\sigma \to 0$? More precisely, up to recasting (1.1), do we have a convergence to the solution of the logarithmic Schrödinger equation,

$$i\partial_t u + \frac{1}{2}\Delta u = u \ln |u|^2?$$

(2) When $\sigma \leqslant 1/d$, we know that the nonlinear dynamics cannot be compared to the linear dynamics in the large time régime (long range scattering): can we have a uniform in time continuity of the flow map, which would yield some hints on the nature of long range effects?
(3) When $\sigma > \sigma_0(d)$, the Strauss exponent, do we have continuity of the nonlinear scattering operator with respect to σ?

We provide rather complete answers to these three questions. The corresponding statements are given below, as well as a flavor of the associated proofs. Details can be found in [6], on which the talk is based.

2. Preliminary

Resuming the estimates used in order to prove global existence in $H^1(\mathbb{R}^d)$, based on a fixed point argument relying on Strichartz estimates, Hölder inequality, and possibly Sobolev embedding (when $\sigma > 2/d$), it is not difficult to prove:

Proposition 2.1 (Local in time continuity)**.** *Let $0 < \sigma < 2/(d-2)_+$ and $(\phi_\nu)_{|\nu-\sigma|\leqslant\varepsilon}$ bounded in $H^1(\mathbb{R}^d)$ for some $\varepsilon > 0$ such that $0 < \sigma - \varepsilon$ and $\sigma + \varepsilon < 2/(d-2)_+$. Let $T > 0$:*

- *There exists C such that, as $\nu \to \sigma$,*

$$\sup_{t\in[-T,T]} \|u_\nu(t) - u_\sigma(t)\|_{L^2(\mathbb{R}^d)} \leqslant C\|\phi_\nu - \phi_\sigma\|_{L^2(\mathbb{R}^d)} + C|\nu - \sigma|,$$

$$\sup_{t\in[-T,T]} \|u_\nu(t) - u_\sigma(t)\|_{H^1(\mathbb{R}^d)} \leqslant C\|\phi_\nu - \phi_\sigma\|_{H^1(\mathbb{R}^d)} + C|\nu - \sigma|^\theta,$$

  for some $\theta \in (0, 1]$, which can be taken equal to one if $2\sigma > 1$.
- *If in addition $(\phi_\nu)_{|\nu-\sigma|\leqslant\varepsilon}$ is bounded in Σ, then*

$$\sup_{t\in[-T,T]} \|u_\nu(t) - u_\sigma(t)\|_\Sigma \leqslant C\|\phi_\nu - \phi_\sigma\|_\Sigma + C|\nu - \sigma|^\theta,$$

  for the same θ as above.

Scheme of the proof. The function to estimate satisfies

$$i\partial_t (u_\nu - u_\sigma) + \frac{1}{2}\Delta (u_\nu - u_\sigma) = \underbrace{|u_\nu|^{2\nu}u_\nu - |u_\sigma|^{2\nu}u_\sigma}_{\text{"as usual"}} + \underbrace{|u_\sigma|^{2\nu}u_\sigma - |u_\sigma|^{2\sigma}u_\sigma}_{\text{source term}}.$$

The first group on the right hand side is estimated "as usual", with the technical ingredients recalled above, implying that the difference $u_\nu - u_\sigma$ is controlled by the norm of the source term. On the other hand, Taylor formula yields

$$y - y^{1+h} = -hy \ln y \int_0^1 y^{sh} ds,$$

hence

$$\left|y - y^{1+h}\right| \lesssim |h| \left(y^{1-\eta} + y^{1+\eta}\right),$$

for $\eta > 0$ arbitrarily small. One may think of y as $2\sigma + 1$ homogeneous in u_σ, and $h = 2(\nu - \sigma)$ (possibly negative, but always small in absolute value), and the

L^2-estimate follows easily. The possible presence of θ for the H^1 estimate is due typically to the fact that when differentiating the above equation in space, one faces terms like

$$|u_\sigma|^{2\nu}\nabla u_\sigma - |u_\sigma|^{2\sigma}\nabla u_\sigma,$$

and the map $y \mapsto |y|^{2\sigma}$ is only Hölder continuous when $2\sigma < 1$. □

3. Continuity of the scattering operator

In the case where the power σ is sufficiently large, the previous continuity can be made uniform in time. The Strauss exponent is given by

$$\sigma_0(d) = \frac{2-d+\sqrt{d^2+12d+4}}{4d},$$

and we only emphasize the bounds $1/d < \sigma_0(d) < 2/d$.

Theorem 3.1 (Global continuity in the (very) short range case)**.** *Let $\sigma_0(d) < \sigma < 2/(d-2)_+$ and $(\phi_\nu)_{|\nu-\sigma|\leqslant\varepsilon}$ bounded in Σ with $\varepsilon > 0$ such that $\sigma - \varepsilon > \sigma_0(d)$ and $\sigma + \varepsilon < 2/(d-2)_+$. Then we have the global in time estimate:*

$$\sup_{t\in\mathbb{R}} \left\| e^{-i(t/2)\Delta}\left(u_\nu(t) - u_\sigma(t)\right)\right\|_\Sigma \leqslant C\|\phi_\nu - \phi_\sigma\|_\Sigma + C|\nu-\sigma|^\theta,$$

for the same θ as in Proposition 2.1.

We briefly recall the notion of nonlinear scattering operator. For a given asymptotic state u_-, consider the equation (1.1) where the initial value is replaced by a final value (at infinite time),

$$e^{-i(t/2)\Delta}u(t)\Big|_{t=-\infty} = u_-.$$

If this new problem has a solution defined up to $t=0$, then the map $W_- : u_- \mapsto u_{|t=0}$ is called wave operator. Conversely, if the solution to the Cauchy problem (1.1) behaves asymptotically linearly,

$$u(t) \underset{t\to+\infty}{\sim} e^{i(t/2)\Delta}u_+,$$

for some asymptotic state u_+, then $u_+ = W_+^{-1}u_{|t=0}$ and the scattering operator $S = W_+^{-1}\circ W_-$ is the map $u_- \mapsto u_+$. This map is well defined on Σ for $\sigma > \sigma_0(d)$, as established initially in [19] (see also [13, 26]). In view of the present context, we emphasize its dependence upon σ by using the notation S_σ.

Corollary 3.2 (Continuity of the scattering operator)**.** *Let $\sigma_0(d) < \sigma < 2/(d-2)_+$ and $(u_{-,\nu})_{|\nu-\sigma|\leqslant\varepsilon}$ be a family in Σ for some $\varepsilon > 0$. The scattering operator is continuous at σ in the sense that if*

$$\|u_{-,\sigma} - u_{-,\nu}\|_\Sigma \underset{\nu\to\sigma}{\longrightarrow} 0,$$

then

$$\|S_\sigma u_{-,\sigma} - S_\nu u_{-,\nu}\|_\Sigma = \|u_{+,\sigma} - u_{+,\nu}\|_\Sigma \underset{\nu\to\sigma}{\longrightarrow} 0.$$

The extra argument compared to Proposition 2.1 relies on global explicit decay in time of quantities of the form $\|u_\sigma(t)\|_{L^p(\mathbb{R}^d)}$, provided by the a priori estimates stemming from the pseudo-conformal conservation law, discovered in [19].

4. Uniform in time convergence, including long range cases

It is well known that as soon as $\sigma \leqslant 1/d$, the nonlinear dynamics generated by (1.1) and the linear dynamics $e^{i(t/2)\Delta}$ can be compared only in the case of the trivial solution ([2]):

$$\|u_\sigma(t) - e^{i(t/2)\Delta}u_+\|_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0 \iff \phi_\sigma = u_\sigma = u_+ = 0.$$

In the critical case $\sigma = 1/d$ and for small data, as established initially in [28, 18, 22], the large time behavior of u is described at leading order by a nonlinear phase modification of the free dynamics.

This implies for instance that the flow map cannot be continuous uniformly in time at $\sigma = 1/d$ (as long range effects are absent for $\sigma > 1/d$, see [4]). The general consensus is that even for $\sigma < 1/d$, modified scattering should be characterized at leading order by a (nonlinear) phase modification, even though no general proof of this fact seems to be available so far. Our approach supports this motto: instead of considering u, we first introduce a time dependent rescaling which counterbalances the linear dispersive behavior, and we consider the squared modulus of this new function, that is we set

$$\rho_\sigma(t,y) = \langle t\rangle^d \, |u_\sigma(t, y\langle t\rangle)|^2 \|\phi_\sigma\|_{L^2}^{-2}, \quad \text{where} \quad \langle t\rangle = \sqrt{1+t^2}, \tag{4.1}$$

and the last factor ensures, thanks to the conservation of mass, that $\rho_\sigma(t,\cdot)$ is a probability density. Our global in time convergence result involves the Wasserstein (or Kantorovich–Rubinstein) distance W_1, which can be characterized, for μ_1 and μ_2 probability measures on $\mathbb{R}^d$, by

$$W_1(\mu_1,\mu_2) := \sup\left\{ \int_{\mathbb{R}^d} \psi d(\mu_1-\mu_2) \;\middle|\; \psi \in C(\mathbb{R}^d,\mathbb{R}) \text{ and } \|\psi\|_{\mathrm{Lip}} \leqslant 1 \right\},$$

where the Lipschitz semi-norm is defined by

$$\|\psi\|_{\mathrm{Lip}} = \sup_{x\neq y} \frac{|\psi(x)-\psi(y)|}{|x-y|}.$$

See for instance [30].

Theorem 4.1 (Global continuity for the rescaled modulus)**.** *Let $0 < \sigma < 2/(d-2)_+$ and $(\phi_\nu)_{|\nu-\sigma|\leqslant\varepsilon}$ bounded in Σ for some $\varepsilon > 0$ such that $0 < \sigma - \varepsilon$ and $\sigma + \varepsilon < 2/(d-2)_+$. If*

$$\phi_\nu \underset{\nu\to\sigma}{\longrightarrow} \phi_\sigma \quad \text{in } L^2(\mathbb{R}^d),$$

then

$$\sup_{t\in\mathbb{R}} W_1\left(\rho_\nu(t), \rho_\sigma(t)\right) \underset{\nu\to\sigma}{\longrightarrow} 0.$$

Remark 4.2. As proven in [21, Lemma 2.1], for any $s > (d+1)/2$, there exists C such that for any probability densities f and g,

$$\|f-g\|_{H^{-s}(\mathbb{R}^d)} \leqslant C W_1(f,g)^{1/2}.$$

This implies other convergence results than in Theorem 4.1, involving Sobolev or Lebesgue spaces, which we do not detail here, see [6].

Informal proof. The heuristics for the proof of the above result goes as follows: for T arbitrarily large but fixed, we may invoke the local result Proposition 2.1 on $[-T,T]$. Pretending that instead of working with the Wasserstein distance W_1, we considered a normed space X, write, in view of the Fundamental Theorem of Calculus and Minkowski inequality, for $t > T$,

$$\|\rho_\nu(t) - \rho_\sigma(t)\|_X \leqslant \|\rho_\nu(T) - \rho_\sigma(T)\|_X + \int_T^t \|\partial_t \rho_\nu(s)\|_X ds + \int_T^t \|\partial_t \rho_\sigma(s)\|_X ds.$$

The normalized density ρ_σ solves a non-autonomous continuity equation,

$$\partial_t \rho_\sigma + \frac{1}{\langle t\rangle^2} \operatorname{div} j_\sigma = 0,$$

where j_σ is given by

$$j_\sigma = \frac{1}{\|\phi_\sigma\|_{L^2}^2} \operatorname{Im}\left(\overline{v}_\sigma \nabla v_\sigma\right),$$

and v_σ is related to u_σ via the formula

$$u_\sigma(t,x) = \frac{1}{\langle t\rangle^{d/2}} v_\sigma\left(t, x/\langle t\rangle\right) e^{i(t/(1+t^2))|x|^2/2},$$

and solves

$$i\partial_t v_\sigma + \frac{1}{2\langle t\rangle^2}\Delta v_\sigma = \frac{|y|^2}{2\langle t\rangle^2} v_\sigma + \frac{1}{\langle t\rangle^{d\sigma}} |v_\sigma|^{2\sigma} v_\sigma \quad ; \quad v_{\sigma|t=0} = \phi_\sigma.$$

One can prove the estimates

$$\|v_\sigma(t)\|_{L^2} = \|u_\sigma(t)\|_{L^2} = \|\phi_\sigma\|_{L^2},$$
$$\|\nabla v_\sigma(t)\|_{L^2} \lesssim \langle t\rangle^{\max(0,1-d\sigma/2)},$$

and thus, by Cauchy-Schwarz inequality,

$$\frac{1}{\langle t\rangle^2}\|j_\sigma(t)\|_{L^1(\mathbb{R}^d)} \lesssim \frac{1}{\langle t\rangle^2} \langle t\rangle^{\max(0,1-d\sigma/2)},$$

which is integrable in time provided that $\sigma > 0$. We refer to [6] for the complete proof. □

We note in passing that setting formally $\sigma = 0$ in the above argument, we lose integrability in time. The limit $\sigma \to 0$ turns out to require a totally different approach.

5. The limit $\sigma \to 0$

In the limit $\sigma \to 0$, at points where $u \neq 0$,

$$|u|^{2\sigma} = \exp\left(\sigma \ln |u|^2\right) = 1 + \sigma \ln |u|^2 + \mathcal{O}(\sigma^2).$$

Up to a gauge transform ($u \to ue^{it}$), we may replace the nonlinearity in (1.1) with $\left(|u|^{2\sigma} - 1\right) u$. We may then proceed like in [17] where the stationary, focusing case is considered, and let $\sigma \to 0$ in

$$i\partial_t \mathbf{u}_\sigma + \frac{1}{2}\Delta \mathbf{u}_\sigma = \frac{1}{\sigma}\left(|\mathbf{u}_\sigma|^{2\sigma} - 1\right)\mathbf{u}_\sigma \quad ; \quad \mathbf{u}_{\sigma|t=0} = \phi_\sigma. \tag{5.1}$$

Formally, if $\phi_\sigma \to \phi_0$ as $\sigma \to 0$, $\mathbf{u}_\sigma$ is expected to converge to the solution of the logarithmic Schrödinger equation

$$i\partial_t \mathbf{u}_0 + \frac{1}{2}\Delta \mathbf{u}_0 = \mathbf{u}_0 \ln(|\mathbf{u}_0|^2) \quad ; \quad \mathbf{u}_{0|t=0} = \phi_0. \tag{5.2}$$

This equation was introduced in [3], and the mathematical study of the Cauchy problem (5.2) started in [11].

Remark 5.1. One may adopt an alternative point of view. If u solves (1.1), then

$$\tilde{u}_\sigma(t,x) = \sigma^{1/(2\sigma)} u(t,x) e^{it/\sigma}$$

solves the PDE in (5.1), but with initial value $\sigma^{1/(2\sigma)}\phi$. The fact that the factor $\sigma^{1/(2\sigma)}$ is unbounded as $\sigma \to 0$ is an alternative evidence that nonlinear effects must be enhanced in the nonlinear Schrödinger equation in order to get some convergence toward the logarithmic Schrödinger equation.

Several aspects indicate that the limit $\sigma \to 0$ in (5.1) is more involved than the previous limit $\nu \to \sigma > 0$. Indeed, it was proven in [8] that solutions to (5.2) enjoy properties which are in sharp contrast with the dynamics associated to (1.1). We emphasize three of them:

- The dispersion is enhanced by the nonlinearity: the decay is of the order $\tau_0(t)^{-d/2} \approx t^{-d/2} (\ln t)^{-d/4}$ instead of $t^{-d/2}$ in the linear case.
- Like for the linear heat equation (and as opposed to the scattering theory for (1.1)), there is a universal Gaussian profile, if $\phi_0 \in \Sigma \setminus \{0\}$,

  $$\tau_0(t)^d \left|\mathbf{u}_0\left(t, y\tau_0(t)\right)\right|^2 \|\phi_0\|^{-2}_{L^2(\mathbb{R}^d)} \underset{t\to+\infty}{\longrightarrow} \Gamma,$$

  in Wasserstein distance W_2, where Γ is given by

  $$\Gamma(y) = \frac{e^{-|y|^2}}{\pi^{d/2}}.$$

- The Sobolev norms of any nontrivial solution are unbounded in the large time limit, and we have precisely

  $$\|u_0(t)\|_{\dot{H}^s(\mathbb{R}^d)} \underset{t\to+\infty}{\sim} c(s) \left(\ln t\right)^{s/2}, \quad \forall s \in [0,1].$$

The local in time convergence $\mathbf{u}_\sigma \to \mathbf{u}_0$ turns out to be rather easy, thanks to a property discovered by Cazenave and Haraux [11].

Theorem 5.2. *Let $\varepsilon > 0$ and $(\phi_\sigma)_{0\leqslant\sigma\leqslant\varepsilon}$ bounded in Σ. Let $\mathbf{u}_\sigma$ denote the solution to* (5.1)*, and $\mathbf{u}_0$ the solution to* (5.2)*. There exists $\varepsilon_0 > 0$ such that for all $\sigma \in (0, \varepsilon_0)$, the following holds. There exist $C_0, C_1 > 0$ such that for any $T > 0$,*

$$\sup_{t\in[-T,T]} \|\mathbf{u}_\sigma(t) - \mathbf{u}_0(t)\|_{L^2(\mathbb{R}^d)} \leqslant \left(C_1\sigma + \|\phi_\sigma - \phi_0\|_{L^2(\mathbb{R}^d)}\right) e^{C_0 T}.$$

The convergence implied by Theorem 5.2 can be understood as a convergence up to some Ehrenfest time, by analogy with the definition from the semiclassical propagation of coherent states in (linear) Schrödinger equations, see e.g. [29], inasmuch as if $\|\phi_\sigma - \phi_0\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\sigma)$, the right hand side goes to zero as $\sigma \to 0$, up to $T = c\ln(1/\sigma)$ for any $c \in (0, 1/C_0)$.

Sketch of proof. Denote by $w = \mathbf{u}_\sigma - \mathbf{u}_0$ the error. It solves

$$i\partial_t w + \frac{1}{2}\Delta w = \frac{1}{\sigma}\left(|\mathbf{u}_\sigma|^{2\sigma} - 1\right)\mathbf{u}_\sigma - \mathbf{u}_\sigma \ln(|\mathbf{u}_\sigma|^2) + \mathbf{u}_\sigma \ln(|\mathbf{u}_\sigma|^2) - \mathbf{u}_0 \ln(|\mathbf{u}_0|^2), \tag{5.3}$$

with initial value $w_{|t=0} = \phi_\sigma - \phi_0$. The source term is

$$S_\sigma = \frac{1}{\sigma}\left(|\mathbf{u}_\sigma|^{2\sigma} - 1\right)\mathbf{u}_\sigma - \mathbf{u}_\sigma \ln(|\mathbf{u}_\sigma|^2). \tag{5.4}$$

Recall an identity discovered in [11]:

Lemma 5.3 (From Lemma 1.1.1 in [11]). *There holds*

$$\left|\mathrm{Im}\left(\left(z_2\log|z_2|^2 - z_1\log|z_1|^2\right)(\overline{z_2}-\overline{z_1})\right)\right| \leqslant 2\,|z_2-z_1|^2\,,\quad \forall z_1,z_2\in\mathbb{C}.$$

Proceeding as usual to derive L^2 estimates in Schrödinger equations, we multiply (5.3) by $\overline{w}$, integrate in space, and take the imaginary part. This yields, in view of Cauchy-Schwarz inequality and Lemma 5.3,

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 \leqslant \|S_\sigma\|_{L^2}\|w\|_{L^2} + 2\|w\|_{L^2}^2.$$

The theorem follows from Grönwall lemma, provided that we can show that $\|S_\sigma(t)\|_{L^2} = \mathcal{O}(\sigma)$ on $[-T,T]$. Taylor formula applied to $\sigma\mapsto (|z|^{2\sigma}-1)z = (e^{\sigma\ln|z|^2}-1)z$ yields

$$S_\sigma = \sigma\mathbf{u}_\sigma\left(\ln(|\mathbf{u}_\sigma|^2)\right)^2\int_0^1(1-\theta)|\mathbf{u}_\sigma|^{2\theta\sigma}d\theta, \tag{5.5}$$

hence the pointwise bound

$$|S_\sigma| \lesssim \sigma\left(\ln(|\mathbf{u}_\sigma|^2)\right)^2\left(|\mathbf{u}_\sigma| + |\mathbf{u}_\sigma|^{2\sigma+1}\right).$$

For $\eta>0$ sufficiently small, we have the uniform pointwise estimate (for $0<\sigma\leqslant\sigma_0\ll 1$),

$$|S_\sigma| \lesssim \sigma\left(|\mathbf{u}_\sigma|^{1-\eta} + |\mathbf{u}_\sigma|^{1+\eta}\right),$$

where the implicit constant depends on $\eta>0$. We then use the classical embeddings

$$H^1(\mathbb{R}^d)\hookrightarrow L^{2+2\eta}(\mathbb{R}^d),\quad \mathcal{F}H^1(\mathbb{R}^d)\hookrightarrow L^{2-2\eta}(\mathbb{R}^d),$$

provided that $\eta>0$ is sufficiently small, in terms of d. The theorem then follows from Σ bounds for $\mathbf{u}_\sigma$, stemming from Lemma 5.4 below. □

Consider the family of ordinary differential equations indexed by $\sigma\geqslant 0$,

$$\ddot{\tau}_\sigma = \frac{1}{2\tau_\sigma^{d\sigma+1}},\quad \tau_\sigma(0)=1,\quad \dot{\tau}_\sigma(0)=0. \tag{5.6}$$

The case $\sigma=0$ was considered in [8] to describe the large time behavior of $\mathbf{u}_0$. For $\sigma>0$, such equations were introduced in [5] (up to some scaling described below) in order to analyze the large time dynamics of solutions to equations from compressible fluid mechanics or nonlinear Schrödinger equations. In a first step, we wish to emphasize the following properties:

- For $\sigma>0$, $\tau_\sigma(t)\underset{t\to+\infty}{\sim} t/\sqrt{\sigma}$.
- For $\sigma=0$, $\tau_0(t)\underset{t\to+\infty}{\sim} t\sqrt{\ln t}$.
- As proven in [6], there exists $C>0$ such that

  $$|\tau_\sigma(t)-\tau_0(t)| \leqslant C\sigma t\left(\ln(t+2)\right)^{3/2},\quad \forall t\geqslant 0.$$

  In particular, for $\sigma\approx\frac{1}{\ln t}$, τ_σ,τ_0 and the above right hand side have the same order of magnitude, suggesting that a transition occurs in this régime.

Roughly speaking, the transition between the (long range) scattering behavior associated to (5.1) and the specific dynamics associated to (5.2), recalled above, is given mostly by the transition at the level of the ordinary differential equations, as we will see in Theorem 5.5. Before this final statement and a short description of the arguments of the proof, we complement the understanding of τ_σ, and provide the announced uniform estimates for $\mathbf{u}_\sigma$.

Consider, for $\alpha>0$, the ordinary differential equation introduced in [5],

$$\ddot r_\alpha = \frac{\alpha}{2r_\alpha^{\alpha+1}},\quad r_\alpha(0)=1,\quad \dot r_\alpha(0)=0.$$

Multiplying by $\dot r_\alpha$ and integrating, we find

$$(\dot r_\alpha)^2 = 1-\frac{1}{r_\alpha^\alpha}.$$

This readily shows that $r_\alpha(t)\geqslant 1$ for all $t\in\mathbb{R}$. Since $\ddot r_\alpha\geqslant 0$, $\dot r_\alpha\geqslant 0$ on $[0,+\infty)$, and r_α is nondecreasing. If it was bounded, then $\ddot r_\alpha$ would be bounded from below away from zero, hence a contradiction after integration. Since $r_\alpha(t)\to+\infty$ as $t\to+\infty$, $\dot r_\alpha(t)\to 1$, hence

$$r_\alpha(t)\underset{t\to+\infty}{\sim} t.$$

One may be surprised at this stage, as this asymptotic behavior is independent of $\alpha>0$. We note the identity $r_2(t)=\langle t\rangle$, the dispersive rate considered in (4.1) and Theorem 4.1

Seeking formally an asymptotic expansion for r_α of the form $r_\alpha(t)=t+w_\alpha(t)+$ h.o.t., we come up with

$$2\dot w_\alpha = -\frac{1}{t^\alpha},\ \text{hence}\ w_\alpha(t)=\frac{1}{2(\alpha-1)}t^{1-\alpha}.$$

We observe that $w_\alpha(t)$ becomes of the same order as the leading term t for $\alpha\approx 1/\ln t$, meeting the above remark. Finally, we relate r_α and τ_σ via the scaling

$$\tau_\sigma(t)=r_{d\sigma}\left(t/\sqrt{\sigma}\right),$$

and we note that the limit $\sigma\to 0$ in (5.6) is regular, while the limit $\alpha\to 0$ for r_α is singular.

The error in Theorem 5.2 stops being small for $T=(1/C_0)\ln(1/\sigma)$, that is way before the transition at the ODE level, which occurs for $\sigma\approx 1/\ln T$. This suggests that the analysis requires a different approach to go up to this kind of time, not to mention infinite time, which is addressed in our main result, Theorem 5.5 below.

Generalizing the change of unknown function introduced in [8], let v_σ given by

$$\mathbf{u}_\sigma(t,x)=\frac{1}{\tau_\sigma(t)^{d/2}}v_\sigma\left(t,\frac{x}{\tau_\sigma(t)}\right)\exp\left(i\frac{\dot\tau_\sigma(t)}{\tau_\sigma(t)}\frac{|x|^2}{2}\right). \tag{5.7}$$

It solves

$$i\partial_t v_\sigma+\frac{1}{2\tau_\sigma^2}\Delta v_\sigma=\frac{|y|^2}{4\tau_\sigma^{d\sigma}}v_\sigma+\frac{1}{\sigma\tau_\sigma^{d\sigma}}|v_\sigma|^{2\sigma}v_\sigma-\frac{1}{\sigma}v_\sigma.$$

Up to another gauge transform, we get

$$i\partial_t v_\sigma+\frac{1}{2\tau_\sigma^2}\Delta v_\sigma=\frac{|y|^2}{4\tau_\sigma^{d\sigma}}v_\sigma+\frac{1}{\sigma\tau_\sigma^{d\sigma}}\left(|v_\sigma|^{2\sigma}-1\right)v_\sigma. \tag{5.8}$$

This means that we have replaced the initial v_σ with

$$\tilde v_\sigma(t,y)=v_\sigma(t,y)\exp\left(-i\frac{t}{\sigma}+i\int_0^t\frac{ds}{\sigma\tau_\sigma(s)^{d\sigma}}\right). \tag{5.9}$$

We have dropped the tildas to lighten notations.

Lemma 5.4. *Let $\varepsilon \in (0, 1/d)$, and $(\phi_\sigma)_{0\leqslant\sigma\leqslant\varepsilon}$ bounded in Σ. There exists C independent of $t \geqslant 0$ and $\sigma \in (0, \varepsilon]$ such that*

$$\frac{1}{\tau_\sigma(t)^{2-d\sigma}}\|\nabla v_\sigma(t)\|_{L^2}^2 + \|yv_\sigma(t)\|_{L^2}^2 + \int_{\mathbb{R}^d}\left|\frac{|v_\sigma(t,y)|^{2\sigma}-1}{\sigma}\right||v_\sigma(t,y)|^2dy \leqslant C,$$

and

$$\int_0^\infty \frac{\dot\tau_\sigma(t)}{\tau_\sigma(t)^{3-d\sigma}}\|\nabla v_\sigma(t)\|_{L^2}^2dt \leqslant C.$$

Sketch of proof. Introduce the pseudo-energy

$$\begin{aligned}\mathcal{E}_\sigma(t) = &\frac{1}{2\tau_\sigma(t)^2}\|\nabla v_\sigma(t)\|_{L^2}^2 + \frac{1}{4\tau_\sigma(t)^{d\sigma}}\|yv_\sigma(t)\|_{L^2}^2\\ &+ \frac{1}{(\sigma+1)\tau_\sigma(t)^{d\sigma}}\int_{\mathbb{R}^d}\left(\frac{|v_\sigma(t,y)|^{2\sigma}-1}{\sigma}\right)|v_\sigma(t,y)|^2dy.\end{aligned}\tag{5.10}$$

Since the purely time dependent phase function in (5.9) does not affect $\mathcal{E}_\sigma$, we may consider that v_σ solves (5.8). We compute

$$\begin{aligned}\dot{\mathcal{E}}_\sigma(t) = -\frac{\dot\tau_\sigma(t)}{\tau_\sigma(t)}\Bigg(&\frac{1}{\tau_\sigma(t)^2}\|\nabla v_\sigma(t)\|_{L^2}^2 + \frac{d\sigma}{4\tau_\sigma(t)^{d\sigma}}\|yv_\sigma(t)\|_{L^2}^2\\ &+ \frac{d\sigma}{(\sigma+1)\tau_\sigma(t)^{d\sigma}}\int_{\mathbb{R}^d}\left(\frac{|v_\sigma(t,y)|^{2\sigma}-1}{\sigma}\right)|v_\sigma(t,y)|^2dy\Bigg).\end{aligned}$$

We note that $\mathcal{E}_\sigma$ is not sign-definite, and decompose it as $\mathcal{E}_\sigma = \mathcal{E}_\sigma^+ - \mathcal{E}_\sigma^-$, where

$$\begin{aligned}\mathcal{E}_\sigma^+(t) &= \frac{1}{2\tau_\sigma(t)^2}\|\nabla v_\sigma(t)\|_{L^2}^2 + \frac{1}{4\tau_\sigma(t)^{d\sigma}}\|yv_\sigma(t)\|_{L^2}^2\\ &\quad+ \frac{1}{(\sigma+1)\tau_\sigma(t)^{d\sigma}}\int_{|v_\sigma|\geqslant 1}\left(\frac{|v_\sigma(t,y)|^{2\sigma}-1}{\sigma}\right)|v_\sigma(t,y)|^2dy,\\ \mathcal{E}_\sigma^-(t) &= \frac{1}{(\sigma+1)\tau_\sigma(t)^{d\sigma}}\int_{|v_\sigma|<1}\left(\frac{1-|v_\sigma(t,y)|^{2\sigma}}{\sigma}\right)|v_\sigma(t,y)|^2dy.\end{aligned}$$

Now $\mathcal{E}_\sigma^+$ is the sum of three nonnegative terms, and $\mathcal{E}_\sigma^-$ is nonnegative. Taylor formula for the function $f(\sigma) = y^\sigma$ yields

$$\frac{1-y^\sigma}{\sigma} = \ln\frac{1}{y}\int_0^1 y^{\theta\sigma}d\theta.$$

We infer

$$\begin{aligned}\mathcal{E}_\sigma^-(t) &\leqslant \frac{1}{(\sigma+1)\tau_\sigma(t)^{d\sigma}}\int_{|v_\sigma|<1}|v_\sigma(t,y)|^2\ln\frac{1}{|v_\sigma(t,y)|^2}dy\\ &\lesssim \frac{C(\eta)}{(\sigma+1)\tau_\sigma(t)^{d\sigma}}\int_{\mathbb{R}^d}|v_\sigma(t,y)|^{2-2\eta}dy,\end{aligned}$$

where $\eta > 0$ is arbitrarily small. In view of the conservation of the mass and the embedding $\mathcal{F}H^1 \hookrightarrow L^{2-2\eta}$, this implies

$$\mathcal{E}_\sigma^-(t) \lesssim \frac{1}{\tau_\sigma(t)^{d\sigma}}\|yv_\sigma(t)\|_{L^2}^{d\eta/(2-2\eta)} \lesssim \frac{1}{\tau_\sigma(t)^{d\sigma}}\left(\tau_\sigma^{d\sigma}\mathcal{E}_\sigma^+\right)^{d\eta/(1-1\eta)}.\tag{5.11}$$

We infer in particular that $\mathcal{E}_\sigma(0)$ is bounded uniformly in $\sigma \in (0, 1/d)$. In view of the derivative of $\mathcal{E}_\sigma$, we also have

$$\frac{d}{dt}\left(\tau_\sigma^{d\sigma}\mathcal{E}_\sigma\right) = -\dot\tau_\sigma\tau^{d\sigma-1}\left(1-\frac{d\sigma}{2}\right)\frac{1}{\tau_\sigma^2}\|\nabla v_\sigma\|_{L^2}^2 \leqslant -\dot\tau_\sigma\tau_\sigma^{d\sigma-1}\frac{1}{2\tau_\sigma^2}\|\nabla v_\sigma\|_{L^2}^2,$$

where we have used the facts that $\tau_\sigma, \dot\tau_\sigma \geqslant 0$ and $\sigma < 1/d$. We obtain the uniform bound

$$\mathcal{E}_\sigma(t) \leqslant \frac{\mathcal{E}_\sigma(0)}{\tau_\sigma(t)^{d\sigma}} \leqslant \frac{C}{\tau_\sigma(t)^{d\sigma}}.$$

Invoking (5.11), we have

$$\tau_\sigma^{d\sigma}\mathcal{E}_\sigma^+ \leqslant C + \tau_\sigma^{d\sigma}\mathcal{E}_\sigma^- \lesssim 1 + \left(\tau_\sigma^{d\sigma}\mathcal{E}_\sigma^+\right)^{d\eta/(1-\eta)}.$$

Taking $\sigma > 0$ such that $d\eta/(1-\eta) < 1$ shows that $\tau_\sigma^{d\sigma}\mathcal{E}_\sigma^+$ is bounded uniformly in $t \geqslant 0$ and $\sigma \in (0, 1/d)$. Again from (5.11), this implies that so is $\tau_\sigma^{d\sigma}\mathcal{E}_\sigma^-$, hence $\tau_\sigma^{d\sigma}\left(\mathcal{E}_\sigma^+ + \mathcal{E}_\sigma^-\right) \leqslant C$, which is the first claim of the lemma. We infer that $\tau_\sigma^{d\sigma}\mathcal{E}_\sigma$ is uniformly bounded from below, so its derivative is integrable,

$$\int_0^\infty \frac{\dot\tau_\sigma(t)}{\tau_\sigma(t)^{3-d\sigma}}\|\nabla v_\sigma(t)\|_{L^2}^2 dt \leqslant C,$$

which completes the proof. □

Our final result is the following uniform in time convergence:

Theorem 5.5. *Let $\varepsilon > 0$, $(\phi_\sigma)_{0\leqslant\sigma\leqslant\varepsilon}$ bounded in Σ, with $\phi_\sigma \to \phi_0$ in $L^2(\mathbb{R}^d)$ as $\sigma \to 0$. Define, for $0 \leqslant \sigma \leqslant \varepsilon$*

$$\varrho_\sigma(t,y) = \tau_\sigma(t)^d \left|\mathbf{u}_\sigma\left(t, y\tau_\sigma(t)\right)\right|^2 \|\phi_\sigma\|_{L^2}^{-2}.$$

We have the uniform in time convergence in Wasserstein distance:

$$\sup_{t\geqslant 0} W_1\left(\varrho_\sigma(t), \varrho_0(t)\right) \underset{\sigma\to 0}{\longrightarrow} 0.$$

Moreover, if $\|\phi_\sigma - \phi_0\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\sigma)$, we have the convergence rate

$$\sup_{t\geqslant 0} W_1\left(\varrho_\sigma(t), \varrho_0(t)\right) \lesssim \frac{1}{\sqrt{\ln\ln(1/\sigma)}}.$$

We note that the convergence of ϱ_σ toward ϱ_0 holds uniformly in time, and so supersedes the range on which τ_σ converges to τ_0.

We only describe some steps of the proof and evoke the main technical ingredients of the rather long proof given in [6]. First, we consider the Madelung transform associated to v_σ:

$$\rho_\sigma = |v_\sigma|^2, \quad J_\sigma = \operatorname{Im}\left(\overline{v}_\sigma\nabla v_\sigma\right).$$

The hydrodynamical unknowns solve

$$\begin{cases} \partial_t\rho_\sigma + \dfrac{1}{\tau_\sigma^2}\operatorname{div} J_\sigma = 0, \\ \partial_t J_\sigma + \dfrac{1}{(\sigma+1)\tau_\sigma^{d\sigma}}\nabla\rho_\sigma^{\sigma+1} + \dfrac{2}{\tau_\sigma^{d\sigma}}y\rho_\sigma = \dfrac{1}{4\tau_\sigma^2}\Delta\nabla\rho_\sigma - \dfrac{1}{\tau_\sigma^2}\operatorname{div}\nu_\sigma, \end{cases}$$

where $\nu_\sigma = \operatorname{Re}\left(\nabla v_\sigma \otimes \nabla\overline{v}_\sigma\right)$. We eliminate the momentum J_σ by considering $\partial_t\left(\tau_\sigma^2\partial_t\rho_\sigma\right)$:

$$\partial_t\left(\tau_\sigma^2\partial_t\rho_\sigma\right) = \frac{1}{(\sigma+1)\tau_\sigma^{d\sigma}}\Delta\rho_\sigma^{\sigma+1} + \frac{2}{\tau_\sigma^{d\sigma}}\operatorname{div}(y\rho_\sigma) - \frac{1}{4\tau_\sigma^2}\Delta^2\rho_\sigma + \frac{1}{\tau_\sigma^2}\operatorname{div}\operatorname{div}\nu_\sigma.$$

We develop
$$\partial_t\left(\tau_\sigma^2\partial_t\rho_\sigma\right) = 2\tau_\sigma\dot\tau_\sigma\partial_t\rho_\sigma + \tau_\sigma^2\partial_t^2\rho_\sigma,$$
and recall that in the case $\sigma = 0$, the term $\tau_\sigma^2\partial_t^2\rho_\sigma$ is negligible in the large time limit (see [8] for an argument based on compactness, and [16] for a quantitative proof). Taking into account the time dependent factor in front of the first two terms on the right hand side of the equation for $\partial_t\left(\tau_\sigma^2\partial_t\rho_\sigma\right)$, we change the time variable t to s, in order to have
$$\frac{\partial}{\partial s} = (1-d\sigma)\dot\tau_\sigma(t)\tau_\sigma(t)^{d\sigma+1}\frac{\partial}{\partial t}.$$
The factor $1-d\sigma$ is somehow cosmetic, to shorten formulas in [6]. Recalling (5.6), $\dot\tau_\sigma(t)\tau_\sigma(t)^{d\sigma+1} = \dot\tau_\sigma(t)/(2\ddot\tau_\sigma(t))$, so we compute "explicitly"
$$s = s_\sigma(t) = \frac{1}{2(1-\partial\sigma)}\ln\dot\tau_\sigma(t) = \frac{1}{4(1-d\sigma)}\ln\left(\frac{1}{d\sigma}\left(1-\frac{1}{\tau_\sigma(t)^{d\sigma}}\right)\right).$$
For fixed $\sigma > 0$, the time variable has been compactified, as
$$s_\sigma(t)\underset{t\to+\infty}{\longrightarrow}\frac{1}{4(1-d\sigma)}\ln\left(1/d\sigma\right),$$
but this compact becomes unbounded in the limit $\sigma \to 0$. The new unknown function $\tilde\rho_\sigma$ given by
$$\tilde\rho_\sigma(s_\sigma(t), y) = \rho_\sigma(t, y)$$
solves an equation of the form
$$\partial_s\tilde\rho_\sigma = \frac{1}{\sigma+1}\Delta\tilde\rho_\sigma^{\sigma+1} + 2\operatorname{div}\left(y\tilde\rho_\sigma\right) + \mathcal{R},$$
where $\mathcal{R}$ is expected to be negligible in the limit $\sigma \to 0$, for sufficiently large time. Making this loose statement rigorous turns out to be the core of the proof of Theorem 5.5. The connection to the large time behavior of $\mathbf{u}_0$ can then be guessed as follows: forgetting $\mathcal{R}$, we get the porous medium equation,
$$\partial_s f = \frac{1}{\sigma+1}\Delta f^{\sigma+1} + 2\operatorname{div}(yf),$$
for which it was proven initially in [27] that the solution converges, in the large time limit, to a Barenblatt profile, in Wasserstein distance. On the other hand, as $\sigma \to 0$, this Barenblatt profile converges to the Gaussian Γ (up to suitable renormalization), see e.g. [14]. The proof in [6] consists indeed in measuring the distance between $\tilde\rho_\sigma$ and Γ, by using the following tools and properties:

- Formally setting $\sigma = 0$, the right hand side of the porous medium equation involves the harmonic Fokker-Planck operator L,
$$Lf = \Delta f + 2\operatorname{div}(yf).$$
- Under suitable assumptions on ϕ, the following convergence is classical (see e.g. [1]),
$$W_1\left(e^{sL}\phi, \Gamma\right) \leqslant W_2\left(e^{sL}\phi, \Gamma\right) \leqslant e^{-2s}W_2\left(\phi, \Gamma\right).$$
- In view of [16],
$$W_1\left(\varrho_0(t), \Gamma\right) \lesssim \frac{1}{\sqrt{\ln t}}.$$

This explains the rate announced in Theorem 5.5, where, among others, we match the error estimate from Theorem 5.2 with this estimate, typically at $T = (1/2C_0)\ln(1/\sigma)$.
- The duality and regularizing techniques introduced in [16] for the case $\sigma = 0$ are resumed to deal with the case $0 < \sigma \ll 1$.
- We use fine estimates for τ_σ, $\tau_\sigma - \tau_0$ and v_σ.

References

[1] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[2] J. E. Barab. Nonexistence of asymptotically free solutions for nonlinear Schrödinger equation. *J. Math. Phys.*, 25:3270–3273, 1984.

[3] I. Białynicki-Birula and J. Mycielski. Nonlinear wave mechanics. *Ann. Physics*, 100(1-2):62–93, 1976.

[4] N. Burq, V. Georgiev, N. Tzvetkov, and N. Visciglia. H^1 scattering for mass-subcritical NLS with short-range nonlinearity and initial data in Σ. *Ann. Henri Poincaré*, 24(4):1355–1376, 2023.

[5] R. Carles, K. Carrapatoso, and M. Hillairet. Large-time behavior of compressible polytropic fluids and nonlinear Schrödinger equation. *Quart. Appl. Math.*, 80(3):549–574, 2022.

[6] R. Carles, Q. Chauleur, and G. Ferriere. On the dependence of the nonlinear Schrödinger flow upon the power of the nonlinearity. Preprint, archived at `https://hal.science/hal-05288185`, 2025.

[7] R. Carles, E. Dumas, and C. Sparber. Geometric optics and instability for NLS and Davey-Stewartson models. *J. Eur. Math. Soc. (JEMS)*, 14(6):1885–1921, 2012.

[8] R. Carles and I. Gallagher. Universal dynamics for the defocusing logarithmic Schrödinger equation. *Duke Math. J.*, 167(9):1761–1801, 2018.

[9] R. Carles and L. Gassot. Pathological set with loss of regularity for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 42(3):715–753, 2025.

[10] T. Cazenave, D. Fang, and Z. Han. Continuous dependence for NLS in fractional order spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(1):135–147, 2011.

[11] T. Cazenave and A. Haraux. Équations d'évolution avec non linéarité logarithmique. *Ann. Fac. Sci. Toulouse Math. (5)*, 2(1):21–51, 1980.

[12] T. Cazenave and F. Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Anal. TMA*, 14(10):807–836, 1990.

[13] T. Cazenave and F. Weissler. Rapidly decaying solutions of the nonlinear Schrödinger equation. *Comm. Math. Phys.*, 147:75–100, 1992.

[14] Q. Chauleur. The isothermal limit for the compressible Euler equations with damping. *Discrete Contin. Dyn. Syst. Ser. B*, 27(12):7671–7687, 2022.

[15] M. Christ, J. Colliander, and T. Tao. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. *Amer. J. Math.*, 125(6):1235–1293, 2003.

[16] G. Ferriere. Convergence rate in Wasserstein distance and semiclassical limit for the defocusing logarithmic Schrödinger equation. *Anal. PDE*, 14(2):617–666, 2021.

[17] M. Gallo, S. Mosconi, and M. Squassina. Power law convergence and concavity for the Logarithmic Schrödinger equation. *Math. Ann.*, 395(21), 2026.

[18] J. Ginibre and T. Ozawa. Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$. *Comm. Math. Phys.*, 151(3):619–645, 1993.

[19] J. Ginibre and G. Velo. On a class of nonlinear Schrödinger equations. II Scattering theory, general case. *J. Funct. Anal.*, 32:33–71, 1979.

[20] J. Ginibre and G. Velo. On a class of nonlinear Schrödinger equations. I The Cauchy problem, general case. *J. Funct. Anal.*, 32:1–32, 1979.

[21] M. Hauray and S. Mischler. On Kac's chaos and related problems. *J. Funct. Anal.*, 266(10):6055–6157, 2014.

[22] N. Hayashi and P. Naumkin. Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations. *Amer. J. Math.*, 120(2):369–389, 1998.

[23] T. Kato. On nonlinear Schrödinger equations. II. H^s-solutions and unconditional well-posedness. *J. Anal. Math.*, 67:281–306, 1995.

[24] T. Kato. Correction to: "On nonlinear Schrödinger equations. II. H^s-solutions and unconditional well-posedness". *J. Anal. Math.*, 68:305, 1996.

[25] C. Kenig, G. Ponce, and L. Vega. On the ill-posedness of some canonical dispersive equations. *Duke Math. J.*, 106(3):617–633, 2001.

[26] K. Nakanishi and T. Ozawa. Remarks on scattering for nonlinear Schrödinger equations. *NoDEA Nonlinear Differential Equations Appl.*, 9(1):45–68, 2002.

[27] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.

[28] T. Ozawa. Long range scattering for nonlinear Schrödinger equations in one space dimension. *Comm. Math. Phys.*, 139:479–493, 1991.

[29] D. Robert and M. Combescure. *Coherent states and applications in mathematical physics*. Theoretical and Mathematical Physics. Springer, Cham, 2021. Second edition.

[30] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
Email address: `Remi.Carles@math.cnrs.fr`

Univ. Lille, CNRS, Inria, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France
Email address: `quentin.chauleur@inria.fr`

Univ. Lille, CNRS, Inria, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France
Email address: `guillaume.ferriere@inria.fr`