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GLOBAL DYNAMICS FOR THE 3D MAXWELL-DIRAC SYSTEM

SEBASTIAN HERR, MIHAELA IFRIM, AND MARTIN SPITZ

ABSTRACT. The aim of these notes is to provide an overview of the ideas in the recent proof of global well-posedness for the massive Maxwell-Dirac system in the Lorenz gauge in $\mathbb{R}^{1+3}$, for small and decaying initial data of limiting regularity. The result also includes an in-depth study of the asymptotic dynamics of the global solutions, which can be described as modified scattering. While heuristically we exploit the close connection between the massive Maxwell-Dirac and the wave-Klein-Gordon equations, for the proof of the results we develop a novel approach which applies directly at the level of the Dirac equations. The modified scattering result follows from a precise description of the asymptotic behavior of the solutions inside the light cone, which is derived via the method of testing with wave packets of Ifrim-Tataru.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

We consider the Cauchy problem for the Maxwell–Dirac system on Minkowski space $\mathbb{R}^{1+3}$. In the Lorenz gauge, the system reads

$$\begin{cases} -i\,\gamma^\mu \partial_\mu \psi + \psi = \gamma^\mu A_\mu \psi, \\ \Box A_\mu = -\overline{\psi}\,\gamma_\mu \psi, \\ \partial^\mu A_\mu = 0. \end{cases} \tag{1.1}$$

This model, fundamental in relativistic field theory, describes the interaction of a spinor (electron) with its self-induced electromagnetic field. Our focus is on the long-time dynamics of solutions arising from initial data prescribed at time $t = 0$,

$$\psi(0,x) = \psi_0(x), \qquad A_\mu(0,x) = a_\mu(x), \qquad \partial_t A_\mu(0,x) = \dot{a}_\mu(x). \tag{1.2}$$

The unknowns in the system (1.1) are the spinor field $\psi = \psi(t,x)$, which takes values in $\mathbb{C}^4$, and the real-valued electromagnetic potentials $A_\mu(t,x)$, with $\mu \in \{0,1,2,3\}$. Without loss of generality, we normalize the mass in the Dirac equation to be equal to 1.

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The main results of the authors' paper [17] address two fundamental questions: (i) the global existence of solutions for small initial data subject to mild regularity and spatial decay assumptions, and (ii) a precise asymptotic description of those solutions. A closer analysis of the large-time behavior reveals a *modified scattering* phenomenon. Specifically, we show that inside the light cone:

(i) A decays at the rate t^{-1}, and
(ii) ψ decays at the dispersive rate $t^{-3/2}$, but with an additional logarithmic phase correction.

In comparison with earlier results, our work in [17] advance our understanding of this problem in several substantive ways:

- Although the system under consideration is semilinear in structure, the asymptotic analysis of its solutions reveals a modified scattering behavior that reflects a stronger and more intricate coupling between the Dirac and Maxwell components than one might anticipate from a superficial examination of the nonlinearity.
- To a large extent our estimates are Lorentz invariant, which reflects the full Lorentz symmetry of the Maxwell-Dirac system in the Lorenz gauge, and is a consequence of having derived the Lorentz vector fields that commute with the linear component of our system (1.1).
- We make no assumptions on the support of the initial data. Furthermore, we make very mild decay assumptions on the initial data at infinity. In particular, we use only three Lorentz vector fields in the analysis, which is close to optimal and significantly below anything that has been done before for this model.
- Rather than using arbitrarily high regularity, here we work with very limited regularity for the initial data, e.g. our three vector fields bound is simply in the energy space.
- In terms of methods, our work employs a combination of energy estimates localized to dyadic space-time regions, and pointwise interpolation type estimates within the same regions. This is akin to ideas previously used by Metcalfe-Tataru-Tohaneanu [30] in a linear setting, and then later refined to apply to a quasilinear setting in the work of Ifrim-Stingo [18].
- The asymptotic description of the spinor field ψ is obtained via the wave–packet testing method of Ifrim–Tataru [19–22], together with a new family of projections introduced here at the level of the Dirac equation. This uncovering of the intrinsic Dirac structure is, to our knowledge, the first of its kind and should not be confounded with the projectors employed in the work of D'Ancona *et al* [9]. Our analysis does not rely on a reduction to the Klein–Gordon framework; indeed, we deliberately avoid such a reduction because it is inefficient from the standpoint of regularity. The new projections are precisely what allow us to work at lower regularity while controlling the dynamics with a minimal number of Lorentz vector fields. In this sense, the present result is among the few in the literature that both lowers regularity thresholds for a wave-like model and carefully optimizes the use of vector fields.
- We identify an asymptotic system for ψ and A inside the light cone, which has a very clean expression in hyperbolic coordinates.

1.1. **Previous work.** A brief survey of previous results on the massive Maxwell–Dirac system and related equations is in order. We would like to include a more exhaustive list of works in order to create a context of ideas and results that have emerged in this line of research, including higher-dimensional settings, as well as works that address related models such as the massless Maxwell–Dirac system and the Maxwell–Klein–Gordon system.

We start with a brief survey of previous results on (1.1) and related equations, namely the early work on local well-posedness of (1.1) on $\mathbb{R}^{1+3}$ by Gross [16] and Bournaveas [5], followed by the more recent work of D'Ancona–Foschi–Selberg [9], who established local well-posedness of (1.1) on $\mathbb{R}^{1+3}$ in the Lorenz gauge $\partial^\mu A_\mu = 0$ for data $\psi(0) \in H^\varepsilon$, $A_\mu[0] \in H^{1/2+\varepsilon} \times H^{-1/2+\varepsilon}$, which is almost optimal. Critical for their approach is their discovery of a deep system null structure of (1.1) in the Lorenz gauge. We also mention the work on uniqueness by Masmoudi–Nakanishi [28]. In more recent work, Gavrus and Oh [14] obtained global well-posedness of the massless Maxwell–Dirac equation in the Coulomb gauge on $\mathbb{R}^{1+d}$ $(d \geq 4)$ for data with small scale-critical Sobolev norm, as well as modified scattering of solutions. In [26], Lee obtained linear scattering for solutions of (1.1) on $\mathbb{R}^{1+4}$.

In terms of global well-posedness, D'Ancona–Selberg [11] obtained a global result for (1.1) on $\mathbb{R}^{1+2}$ and proved global well-posedness in the charge class. Regarding work in $\mathbb{R}^{1+3}$ for (1.1), we also mention results by Georgiev [15], Flato–Simon–Taflin [13], and Psarelli [34] on global well-posedness for small, smooth, and localized data, as well as the works [1, 29] on the nonrelativistic limit and [27] on unconditional uniqueness at regularity $\psi \in C_t H^{1/2}$, $(A, \partial_t A) \in C_t(H^1 \times L^2)$ in the Coulomb gauge. Simplified versions of (1.1) were studied in [7, 8, 35]. Stationary solutions were constructed by Esteban–Georgiev–Séré [12].

The next two paragraphs discuss related models that have played a crucial role in the ideas that emerged in the study of the Maxwell–Dirac system. For example, a scalar counterpart of (1.1) is the *Maxwell–Klein–Gordon equations* (MKG). In Klainerman–Machedon [23], global well-posedness in the Coulomb and temporal gauges in $d = 3$ was proved. More recent work on these models includes local well-posedness results for (MKG) by Krieger–Sterbenz–Tataru [25], and, in the energy-critical case $d = 4$, global well-posedness for arbitrary finite-energy data was established by Oh and Tataru [31–33], and independently by Krieger–Lührmann [24].

Another model that contributed to the circle of ideas later circulating in this research direction is the *Dirac–Klein–Gordon system*. Recent work includes D'Ancona–Foschi [10], as well as the more recent result of Bejenaru and Herr [4], where, under a nonresonant condition on the masses, they proved global well-posedness and scattering for the massive Dirac–Klein–Gordon system with small initial data of subcritical regularity in $d = 3$.

Work on Dirac equations was also influential for results on the Maxwell–Dirac system. Notable recent results include optimal small-data global well-posedness for the *cubic Dirac equation* in $\mathbb{R}^{1+2}$ and $\mathbb{R}^{1+3}$ by Bejenaru–Herr [2, 3] (massive case) and Bournaveas–Candy [6] (massless case). The references in this paragraph make use of a feature that the Dirac equation possesses, namely a spinorial null structure.

Our work in [17] differs significantly from previous approaches in that it does not rely on a spinorial null structure, which has traditionally been developed to relate the Dirac equation to Klein–Gordon models and has been exploited in scattering results for Maxwell–Dirac equations. Instead, we work directly at the level of the Dirac equation to uncover the modified scattering behavior. In doing so, we reveal a new structural property of the Dirac equation that is better suited to global dynamical analysis, and in particular for deriving the asymptotic equation for the spinor field ψ.

Recent work of the second author with Tataru on modified scattering for a series of relevant models [19,20,22] played a crucial role in the novel approach we present here. A comprehensive expository account of recent developments on modified scattering is due to the second author and Tataru; see [22]. A second important reference that informs the energy and pointwise estimates we perform in [17] is the work of the second author with Stingo [18] on almost global existence for wave–Klein–Gordon systems.

1.2. **The Maxwell-Dirac system.** We consider the Maxwell-Dirac system on the Minkowski space-time $\mathbb{R}^{1+d}$ for space dimension $d = 3$. The space-time coordinates are denoted by x^α with $\alpha = \overline{0,3}$ and $t = x^0$, and the Minkowski metric and its inverse are

$$(g_{\alpha\beta}) := \mathrm{diag}(-1,1,1,1), \qquad (g^{\alpha\beta}) := \mathrm{diag}(-1,1,1,1),$$

with standard conventions for raising and lowering indices.

The Dirac equation is described using the "gamma matrices", which are 4×4 complex-valued matrices γ^μ with μ ranging from 0 to 3,

$$\gamma^0 := \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & -\mathbf{I}_2 \end{pmatrix}, \qquad \gamma^j := \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

with the Pauli matrices given by

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and satisfying the anti-commutation relations

$$(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = -2g^{\mu\nu}\,\mathbf{I}_4, \tag{1.3}$$

where $\mathbf{I}_4$ is the 4×4 identity matrix; if no confusion is created, a handy short hand notation we will be using is $\mathbf{I}_4 =: \mathbf{I}$.

Given a vector valued function (spinor field) ψ on $\mathbb{R}^{1+3}$ that takes values in $\mathbb{C}^4$, on which γ^μ acts as multiplication, we define the following *conjugation operation*

$$\overline{\psi} := \psi^\dagger\gamma^0, \tag{1.4}$$

where $\psi^\dagger$ is the Hermitian adjoint of ψ. The same conjugation relation defined for vectors in equation (1.4) extends to general 4×4 matrices γ

$$\overline{\gamma} := \gamma^0\gamma^\dagger\gamma^0. \tag{1.5}$$

In particular for the matrices γ^α above one easily verifies that

$$\overline{\gamma^\alpha} = \gamma^\alpha. \tag{1.6}$$

A *spinor field* ψ is a function on $\mathbb{R}^{1+3}$ or on any open subset of $\mathbb{R}^{1+3}$ that takes values in $\mathbb{C}^4$. Given a real-valued 1-form A_μ (connection 1-form), we introduce the *gauge covariant derivative* on spinors

$$\mathbf{D}_\mu\psi := \partial_\mu\psi + iA_\mu\psi,$$

and the associated *curvature 2-form*

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu = (\mathrm{d}A)_{\mu\nu}.$$

The Maxwell–Dirac equations describe the relativistic quantum electrodynamics of particles within self-consistent generated and external electromagnetic fields. The relativistic

Lagrangian field describing the interaction between a connection 1-form A_μ, representing an electromagnetic potential, and a spinor field ψ, modeling a charged fermionic field is a space-time integral that takes the form

$$\mathscr{S}[A_\mu,\psi] = \iint_{\mathbb{R}^{1+3}} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\langle\gamma^\mu \mathbf{D}_\mu\psi,\gamma^0\psi\rangle - \langle\psi,\psi\rangle\, dtdx.$$

Here $\langle\psi^1,\psi^2\rangle := (\psi^2)^\dagger\psi^1$ is the usual inner product on $\mathbb{C}^4$. The Euler–Lagrange equations for $\mathscr{S}[A_\mu,\psi]$ take the form

$$\begin{cases} \partial^\nu F_{\mu\nu} = -\langle\psi,\gamma^0\gamma^\mu\psi\rangle \\ i\gamma^0\gamma^\mu\mathbf{D}_\mu\psi = \gamma^0\psi. \end{cases} \tag{1.7}$$

We will refer to (1.7) as the *Maxwell–Dirac equations.*

A key feature of (1.7) is its invariance under gauge transformations meaning that given any solution (A,ψ) of (1.7) and a real-valued function χ, called gauge transformation, on $I\times\mathbb{R}^3$, the gauge transform $(\widetilde{A},\widetilde{\psi}) = (A-\mathrm{d}\chi, e^{i\chi}\psi)$ of (A,ψ) is also a solution to (1.7). This in fact says that relative to this gauge transform we should think of a solution as being an equivalence class of functions that are solutions to our problem.

In order to address the well-posedness theory we need to remove the ambiguity arising from this invariance, for our system (1.7), and fix the gauge. Traditionally there are several gauges that have been used to address this issue. This includes for instance the *Coulomb gauge* $\partial_j A_j = 0$, which leads to a mix of hyperbolic and elliptic equations. Another possible gauge choice is the *temporal gauge* $A_0 = 0$, which retains causality but loses some ellipticity.

In this work we impose the *Lorenz gauge* condition, which reads

$$\partial^\mu A_\mu = 0, \tag{1.8}$$

and has the advantage that it is Lorentz invariant, resulting in a more symmetric form of the equations (nonlinear wave equations) compared to the other choices discussed above.

When applied to (1.7), the Lorenz gauge leads us to the system

$$\begin{cases} -i\gamma^\mu\partial_\mu\psi + \psi = \gamma^\mu A_\mu\psi \\ \quad\Box A_\mu = -\overline{\psi}\gamma_\mu\psi \\ \quad\partial^\mu A_\mu = 0. \end{cases}$$

The main interest here is on the long time dynamics of the Cauchy problem with prescribed initial data at time $t=0$, given by (1.2).

If one considers only the (self-contained) system formed by the first two equations in (1.1), then the initial data above can be chosen arbitrarily. However, if in addition one also adds the third equation, then the initial data is required to satisfy the following constraint equations

$$\begin{cases} \dot{a}_0 = \ \partial_j a_j \\ \Delta a_0 = \ \partial_j\dot{a}_j - |\psi_0|^2, \end{cases} \tag{1.9}$$

which are then propagated to later times by the flow generated by the first two equations.

1.3. **Functional spaces.** In this section, we introduce the main function spaces we use to prove our main results. As a guideline we use the scaling of the massless Maxwell-Dirac system, which is known to be invariant under the scaling ($\lambda > 0$)

$$(\psi, A_\mu) \to (\lambda^{-3/2}\psi(\lambda^{-1}t, \lambda^{-1}x), \lambda^{-1}A_\mu(\lambda^{-1}t, \lambda^{-1}x)). \tag{1.10}$$

This leads to the critical Sobolev space $\mathscr{H}^0 := L^2 \times \dot{H}^{1/2} \times \dot{H}^{-1/2}$; the first space measures ψ and the remaining spaces measure position and velocity respectively. In terms of interesting quantities let us state the ones that are available for this model, but emphasize that none of them will play a role in our analysis:

(i) the **charge conservation**

$$\mathbf{J_0} := -\int |\psi|^2\, dx = -\|\psi_0\|^2_{L^2}, \tag{1.11}$$

(ii) the **energy**

$$\mathbf{E} := \int \overline{\gamma^j D_j \psi}\psi + \overline{\psi}\psi + \frac{1}{2}|\nabla A|^2\, dx. \tag{1.12}$$

In terms of terminology, our problem is called *charge critical*, and this is because the charge is measured in the critical space L^2. In $d = 4$, the critical Sobolev space would change and the energy will be expressed in terms of these critical Sobolev spaces, leading to the terminology *energy critical Maxwell-Dirac system.*

1.4. **Main results.** To study the small data long time well-posedness problem for the nonlinear evolution (1.1) one needs to add some decay assumptions for the initial data to the mix. Before doing so we need to introduce two small pieces of notations:

- we make the convention of using upper-case letters for multi-indices, e.g. $\partial_x^I = \partial_{x_0}^{i_0} \cdots \partial_{x_d}^{i_d}$ and $x^I = {x_0}^{i_0} \cdots {x_d}^{i_d}$, where $I = (i_0, \dots, i_d)$, and we write I_0 if $i_0 = 0$.
- we also recall the vector fields (denoted here by) $\Omega_{\alpha\beta}$,

$$\Omega_{\alpha\beta} := x_\alpha \partial_\beta - x_\beta x_\alpha, \quad \alpha, \beta = \overline{0,3},$$

  which represent the generators of the Lorentz group.

At this point we are ready to state the first main result in [17], which describes the type of initial data we are considering, and provides global pointwise bounds for the solutions:

Theorem 1.1. *Assume that the initial data $(\psi_0, a, \dot{a})$ for the system (1.1) satisfies the smallness and decay conditions*

$$\sum_{3|J_0|+|K_0|\le 9} \|x^{J_0}\partial_x^{K_0}\psi_0\|_{L^2} + \|x^{J_0}\partial_x^{J_0+K_0}\psi_0\|_{L^2} + \|x^{J_0}\partial_x^{J_0+K_0}a\|_{\dot{H}^{1/2}} + \|x^{J_0}\partial_x^{J_0+K_0}\dot{a}\|_{\dot{H}^{-1/2}} \le \varepsilon, \tag{1.13}$$

as well as the additional low frequency bound

$$\|a^\mu\|_{H^{1/2-\nu}} + \|\dot{a}^\mu\|_{H^{-1/2-\nu}} \le \varepsilon, \qquad \nu > 0. \tag{1.14}$$

If ε is small enough, then the solution (ψ, A) is global in time, and satisfies the vector field bounds

$$\sum_{3|J|+|K|\le 9} \|\Omega^J \partial^K \psi\|_{L^2} + \|\Omega^J \partial^K A\|_{\dot{H}^{1/2}} + \|\Omega^J \partial^K \partial_t A\|_{\dot{H}^{-1/2}} \le \varepsilon t^{C\varepsilon}, \tag{1.15}$$

as well as the pointwise bounds

$$|\psi(t,x)| \lesssim \frac{\varepsilon}{\langle t+|x|\rangle^{3/2-C\varepsilon}\langle t-|x|\rangle^{C\varepsilon}}, \qquad |A(t,x)| \lesssim \frac{\varepsilon}{\langle t+|x|\rangle}, \ (t>0) \tag{1.16}$$

and, in addition, when $|x|-t \geq t^{1/3}$,

$$|\psi(t,x)| \lesssim \frac{\varepsilon}{\langle t+|x|\rangle^{3/2} t^{\delta}}.$$

Remark 1.2. For smooth and localized initial data the existence of a unique global solution of (1.1) was shown in Theorem 1 in the work of Georgiev [15]. On the other hand the work in Psarelli in [34] provides a lower regularity global well-posedness result, though working with compactly supported initial data which is a very restrictive assumption to make. The same result also includes pointwise decay bounds for the solutions, however no asymptotic equations are derived. By contrast our result applies at low regularity without using any support assumptions, and additionally we derive clean asymptotic equations for the solutions; see Theorem 1.3 below.

We comment here on the decay rates for ψ and A in the above theorem. Beginning with ψ, we see that we have the standard dispersive decay rate of $t^{-3/2}$ inside the cone, but a better decay rate outside. The latter happens simply because of the initial data localization, as the group velocities for ψ waves lie inside the cone, and approach the cone only in the high frequency limit. However, because of the t^{-1} size of A there are strong nonlinear interactions that happen inside the cone which prevent standard scattering and instead remodulate the ψ waves, suggesting there should be a modified scattering asymptotic.

Turning our attention to A, if one were to naively think of the A equation as a linear homogeneous wave then the bulk of it would be localized near the cone, with better decay inside, and would have a minimal interaction with ψ. However, as it turns out, the bulk of A inside the cone comes from solving the wave equation with a ψ dependent quadratic source term. This is what produces the exact t^{-1} decay rate. However, we do get the expected decay estimates for ∇A both outside and inside the cone.

To capture the asymptotic behavior of ψ and A at infinity, and also understand the coupling between A and ψ in time-like directions, one needs to make the above heuristic discussion rigorous. We do this in the next theorem, which describes the asymptotics profiles for ψ and A as well as the modified scattering asymptotics.

Theorem 1.3. *There exist $\delta > 0$ so that, for all solutions (ψ, A) for the Maxwell-Dirac equations as in Theorem* 1.1, *there exist asymptotic profiles*

$$(\rho_\infty^\pm, a_\infty^\mu) \in C^{1/2}(B(0,1)), \tag{1.17}$$

vanishing at the boundary, so that inside the light cone we have the asymptotic expansions

$$A^\mu(t,x) = (t^2-x^2)^{-1/2} a_\infty^\mu(x/t) + O(\varepsilon\langle t\rangle^{-1}\langle t-r\rangle^{-\delta}), \tag{1.18}$$

respectively

$$\psi(t,x) = (t^2-x^2)^{-3/4} \sum_{\pm} e^{\pm i\sqrt{t^2-x^2}} e^{i\frac{x_\mu a_\infty^\mu(x/t)}{2\sqrt{t^2-x^2}}\log(t^2-x^2)} \rho_\infty^\pm(x/t) + O(\varepsilon\langle t\rangle^{-3/2}\langle t-r\rangle^{-\delta}), \tag{1.19}$$

where a_∞^μ is uniquely determined by $\rho_\infty^\pm$ via the elliptic equation

$$(-1-\Delta_H) a_\infty^\mu = -\overline{\rho_\infty^\pm}\gamma^\mu \rho_\infty^\pm, \tag{1.20}$$

for the hyperbolic Laplacian Δ_H in the Klein–Beltrami disk model.

The following comments should help clarify various aspects of this result.

(i) Modified scattering: the asymptotic expansion for ψ in (1.19) departs from the corresponding linear asymptotic due to the logarithmic phase correction. This is in turn generated by the exact t^{-1} decay rate for A inside the cone, which is also not consistent with the linear theory.

(ii) Gauge invariance: one may certainly remove the logarithmic phase correction in the ψ asymptotics with a change of gauge; however, this would merely switch the logarithmic correction to A.

(iii) Hyperbolic geometry: the asymptotic profiles should be best viewed as functions on the hyperbolic space H, with the Poincaré disk representation via the velocity coordinate $v = x/t \in B(0,1)$.

(iv) Profile regularity: the $C^{1/2}$ bound represents just the simplest common regularity property for $\rho^{\pm}$ and a^{μ}_{∞}, but in effect we prove an expanded set of bounds, which are best expressed in the hyperbolic setting, where the Lorentz vector fields Ω play the role of normalized derivatives:

$$|\Omega^{\leq 2} a^{\mu}_{\infty}(v)| \lesssim \varepsilon^2 (1-v^2)^{1/2}, \tag{1.21}$$

$$|\rho^{\pm}_{\infty}(v)| \lesssim \varepsilon (1-v^2)^{1-C\varepsilon}, \tag{1.22}$$

$$\|(1-v^2)^{-3/2+C\varepsilon} \Omega^{\leq 2} \rho^{\pm}_{\infty}\|_{L^2} \lesssim \varepsilon. \tag{1.23}$$

We refer the reader to the last section of these notes for more details.

(v) Higher regularity: If the initial data for (ψ, A) has additional regularity then the hyperbolic space regularity of $(\rho^{\pm}_{\infty}, a^{\pm}_{\infty})$ can be improved, as well as the decay rate for $\rho^{\pm}_{\infty}$ at the boundary of the unit ball. However, there is no improved decay rate for $a^{\pm}_{\infty}$; instead, $(1-v^2)^{-1/2} a^{\pm}_{\infty}$ will always have a nondegenerate limit at the boundary.

(vi) Low frequency assumption: the additional condition (1.14) on the initial data for A is necessary in order to obtain the expansion (1.18) even if $\psi = 0$. Otherwise, as $\nu \to 0$, we correspondingly must have $\delta \to 0$ in (1.18).

(vii) Connection to Klein-Gordon: the Dirac waves are closely related to Klein-Gordon waves, and this is reflected in the form of the asymptotic expansion for ψ. The two components $\rho^{\pm}_{\infty}$ correspond exactly to the two Klein-Gordon half-waves, as it can be readily seen by examining the phases of the associated terms in the ψ expansion. In a related vein, the ranges of $\rho^{\pm}_{\infty}(v)$ are restricted to v dependent but Lorentz invariant subspaces $V^{\pm}_{v}$, see (2.15), which are orthogonal with respect to the $\langle \cdot, \cdot \rangle_H$ inner product defined by

$$\langle \psi^1, \psi^2 \rangle_H = -\langle \gamma^0 \gamma^H \psi^1, \psi^2 \rangle. \tag{1.24}$$

With these notations, the source term in the coupling equation (1.20) takes the form

$$\overline{\rho^{\pm}_{\infty}} \gamma^{\mu} \rho^{\pm}_{\infty} = \frac{v^{\mu}}{\sqrt{1-v^2}} (\|\rho^{+}_{\infty}\|^2_H + \|\rho^{-}_{\infty}\|^2_H). \tag{1.25}$$

(viii) Charge conservation: this is reflected in the asymptotic profile via the identity

$$\|\rho^{+}_{\infty}\|^2_{L^2(H)} + \|\rho^{-}_{\infty}\|^2_{L^2(H)} = \|\psi_0\|^2_{L^2}. \tag{1.26}$$

(ix) The Landau notation in the above theorem means

$$\sup_{|x|<t} |A(t,x) - (t^2 - x^2)^{-1/2} A_\infty(x/t)| \lesssim t^{-1-\delta}$$

as $t \to \infty$ and analogously for ψ.

Finally we comment on the low frequency assumption (1.14):

Remark 1.4. The result in Theorem 1.1 also holds without the assumption (1.14) if one is willing to slightly relax the pointwise bound for A to

$$|A(t,x)| \lesssim \frac{\varepsilon}{\langle t+|x|\rangle} \log \frac{2\langle t+r\rangle}{\langle t-r\rangle}.$$

One venue to achieve this is to rely on the weaker BMO bound for A; this in turn would require replacing the L^∞ endpoint with a BMO endpoint in some of the vector field interpolation Lemmas. Alternatively, one can slightly rebalance the bootstrap bounds for A and ψ, from L^∞ and L^6 to $L^{\infty-}$ and L^{6+}, with appropriate changes in the powers of t. We chose not to pursue either alternative in [17] because on one hand this assumption turns out to be needed for Theorem 1.3, and on the other hand, it allows for a more streamlined argument.

2. Preliminaries and notations

2.1. **Notations.** The coordinates in $\mathbb{R}^{3+1}$ are denoted by $x := (x^0, x^1, x^2, x^3)$, and lower the indices using the Minkowski metric. For indices we have the following traditional convention: (i) Greek indices range over $0, 1, \dots, d$, (ii) Latin indices over $1, \dots, d$, (iii) Einstein summation convention of summing repeated upper and lower indices over these ranges, and (iv) raising and lowering indices is performed using the Minkowski metric. For the multi-index notation we use upper-case letters, e.g. $\partial_x^I = \partial_{x_0}^{i_0} \cdots \partial_{x_d}^{i_d}$ and $x^I = {x_0}^{i_0} \cdots {x_d}^{i_d}$, where $I = (i_0, \dots, i_d)$, and we write I_0 if $i_0 = 0$.

2.2. **Vector fields.** To describe the regularity of the solutions we use the vector fields associated to the symmetries of the Minkowski space-time. Precisely, the rotation vector fields and Lorentz boosts are denoted by $\Omega_{\alpha\beta}$,

$$\Omega_{\alpha\beta} := x_\beta \partial_\alpha - x_\alpha \partial_\beta, \qquad \alpha, \beta = \overline{0,3}. \tag{2.1}$$

Together with the translations, these Lorentz generators will be denoted by Γ,

$$\Gamma := \{\partial_0, \partial_1, \partial_2, \partial_3, \Omega_{\alpha\beta}\}. \tag{2.2}$$

As defined above, $\Omega_{\alpha\beta}$ do not commute with the linear component of the Dirac equation in (1.1) due to the vectorial structure of the spinors. Instead we need to consider a correction to the Lorentz vector fields, which represents the Lie derivative of the spinor field with respect to the Lorenz vector fields:

$$\widehat{\Omega}_{\alpha\beta} := \Omega_{\alpha\beta} + \frac{1}{2}\gamma_\alpha\gamma_\beta, \quad \text{for all } \ 0 \le \alpha < \beta \le 3. \tag{2.3}$$

This indeed satisfies

$$[\widehat{\Omega}_{\alpha\beta}, i\gamma^\mu \partial_\mu] = 0.$$

We will later apply $\widehat{\Omega}_{\alpha\beta}$ to the Dirac component of the Maxwell-Dirac system (1.1). However, this is not the end of the story as we want to apply these vector fields to the nonlinear system, which itself has Lorentz invariance. Explicitly, when applying $\widehat{\Omega}_{\alpha\beta}$ to the Maxwell-Dirac system (1.1) implies, for instance, that for the first equation we should formally be able expressed the RHS as follows

$$-i\gamma^\mu\partial_\mu\widehat{\Omega}_{\alpha\beta}\psi+\widehat{\Omega}_{\alpha\beta}\psi=\widehat{\Omega}_{\alpha\beta}\left(\gamma^\mu A_\mu\psi\right)=\widetilde{\Omega}_{\alpha\beta}A_\mu\gamma^\mu\psi+A_\mu\gamma^\mu\widehat{\Omega}_{\alpha\beta}\psi. \tag{2.4}$$

Here the only thing we did was to distribute $\widehat{\Omega}_{\alpha\beta}$, observing that one potential outcome would be to have the corresponding vector field applied to A_μ, which is naturally different from the vector field applied to ψ. At the same time, this new vector field, denoted here by $\widetilde{\Omega}_{\alpha\beta}$, should be commuting with the linear component of the second equation. More so, it should distribute itself according to the product rule in the nonlinearity of the wave equation, namely, we should have

$$\Box\widetilde{\Omega}_{\alpha\beta}A_\mu=-\overline{\widehat{\Omega}_{\alpha\beta}\psi}\gamma_\mu\psi-\overline{\psi}\gamma_\mu\widehat{\Omega}_{\alpha\beta}\psi. \tag{2.5}$$

Indeed, a direct computation leads to the following expressions for the generators of the Lorentz group of symmetries for the full Maxwell-Dirac system:

Lemma 2.1. *The family of vector fields* $\left\{\widehat{\Omega}_{\alpha\beta},\widetilde{\Omega}_{\alpha\beta}\right\}$, *with* $\alpha,\beta=\overline{0,3}$, *and so that*

$$\begin{cases}\widehat{\Omega}_{\alpha\beta}:=\Omega_{\alpha\beta}+\dfrac{1}{2}\gamma_\alpha\gamma_\beta\\ \widetilde{\Omega}_{\alpha\beta}A_\delta:=\Omega_{\alpha\beta}A_\delta+g_{\beta\delta}A_\alpha-g_{\alpha\delta}A_\beta,\end{cases} \tag{2.6}$$

commute with the linear Maxwell-Dirac equations and satisfy the product rule in (2.4), (2.5).

For both the Dirac and the wave components of (1.1) we have defined ten vector fields and in the following we denote these generalized vector fields by Γ_1 to Γ_{10} (omitting the hat and tilde) and employ multi-index notation in the following, i.e.,

$$\Gamma^J=\Gamma_1^{j_1}\cdots\Gamma_{10}^{j_{10}},\qquad J\in\mathbb{N}_0^{10}.$$

Separating derivatives and vector fields we write we will weight differently the two kinds of derivatives, and set

$$\Gamma^{\le k}=\{\Gamma^J\partial^I\}_{|I|+3|J|\le k}.$$

2.3. **Energies for the Dirac equation on hyperboloids and orthogonal decompositions in $\mathbb{C}^4$.** Suppose ψ is a solution for the homogeneous Dirac equation. We can write the L^2-conservation law for the inhomogeneous Dirac equation with a source term F in the density-flux form

$$\partial_t|\psi|^2+\partial_j(\psi^\dagger\gamma^0\gamma^j\psi)=-2\,\mathrm{Im}(\psi^\dagger\gamma^0F). \tag{2.7}$$

An immediate consequence of this is the conservation of the L^2 norm of the solution on time slices. However, in this article we will also need to use energy functionals on hyperboloids

$$H:=\{(t,x)\,|\,t^2-x^2=c^2>0\}.$$

In the homogeneous case $F=0$, integrating the density-flux relation within the region between H and the initial surface $t=0$ we obtain the energy relation

$$\|\psi(0)\|_{L^2}^2=E_H(\psi), \tag{2.8}$$

where the energy of ψ on the hyperboloid H is given by

$$E_H(\psi) = \int_{H_s} (\nu_0|\psi|^2 + \nu_j \psi^\dagger \gamma^0 \gamma^j \psi)\, d\sigma. \tag{2.9}$$

The density for this energy is

$$e_H(\psi) = \nu_0|\psi|^2 + \nu_j \psi^\dagger \gamma^0 \gamma^j \psi,$$

which is positive definite since the normal vector to the hyperboloid is time-like. We first diagonalize it with respect to the Euclidean metric, by writing

$$e_H(\psi) = \frac{t}{\sqrt{t^2+x^2}}|\psi|^2 + \frac{\langle \psi, x_j \gamma^0 \gamma^j \psi\rangle}{\sqrt{t^2+x^2}} = \frac{1}{\sqrt{t^2+x^2}}(t|\psi|^2 + |x|\langle \psi, \gamma^0 \gamma^\theta \psi\rangle),$$

where using polar coordinates we have denoted

$$\gamma^\theta := \theta_j \gamma^j, \qquad \theta = \frac{x}{|x|}. \tag{2.10}$$

To complete our diagonalization we need to consider the spectral properties of the matrix $\gamma^0\gamma^\theta$. The matrices γ^θ share with γ^j the following properties:

Lemma 2.2. *For each $\theta \in \mathbb{S}^2$, the matrix $\gamma^0\gamma^\theta$ is Hermitian and has double eigenvalues ± 1.*

Motivated by this lemma, in order to better describe the energy on hyperboloids it is useful to introduce projectors

$$P^\theta_\pm := \frac{1}{2}(I_4 \pm \gamma^0\gamma^\theta)$$

on the positive, respectively the negative eigenspaces of $\gamma^0\gamma^\theta$. Correspondingly, we split

$$\psi = \psi_+ + \psi_- := P^\theta_+ \psi + P^\theta_- \psi,$$

where we can think of the two components as "outgoing", respectively "incoming". Then we can rewrite the energy density on the hyperboloid H as

$$e_H(\psi) := \frac{t-r}{\sqrt{t^2+r^2}}|\psi_+|^2 + \frac{t+r}{\sqrt{t^2+r^2}}|\psi_-|^2. \tag{2.11}$$

The two components $\psi_\pm$ of ψ will play different roles in our decay bounds for the Dirac field.

Another interpretation of the energy density on the hyperboloids can be naturally obtained by using the hyperbolic metric and volume element. The invariant measure on the hyperbolic space is related to the above Euclidean measure by

$$d\sigma = \sqrt{t^2+x^2}\,(t^2-x^2)\, dV_H.$$

Then the above energy is rewritten in an invariant form as

$$E_H(\psi) = -\int_H (t^2-x^2)^{3/2} \langle \gamma^0 \gamma^H \psi, \psi\rangle\, dV_H, \qquad \gamma^H := \frac{x_\alpha \gamma^\alpha}{\sqrt{t^2-x^2}}.$$

Here it is natural to introduce the (positive definite) inner product on $\mathbb{C}^4$

$$\langle \psi^1, \psi^2\rangle_H = -\langle \gamma^0\gamma^H \psi^1, \psi^2\rangle. \tag{2.12}$$

Comparing this with (2.11) we can diagonalize this in terms of the $\psi_\pm$ decomposition as

$$\|\psi\|_H^2 = \frac{t-r}{\sqrt{t^2-r^2}}|\psi_+|^2 + \frac{t+r}{\sqrt{t^2-r^2}}|\psi_-|^2. \tag{2.13}$$

The matrix γ^H above will play an important role in the sequel. We begin with

Lemma 2.3. *The matrix γ^H satisfies $(\gamma^H)^2 = I_4$, and has double eigenvalues ± 1.*

Based on this property, we introduce the new set of projectors

$$2P_v^{\pm} := 1 \mp \gamma^H, \qquad v = x/t \in B(0,1). \tag{2.14}$$

These generate a decomposition of $\mathbb{C}^4$ as a direct sum of two subspaces $V^{\pm}$ defined as

$$V_v^{\pm} := \ker P_v^{\pm}. \tag{2.15}$$

Since γ^H is in general not symmetric, these projectors are no longer orthogonal in the Euclidean setting. However, the $\langle \cdot, \cdot \rangle_H$ inner product turns out instead to be the one with respect to which the projectors $P_v^{\pm}$ are indeed orthogonal:

Lemma 2.4. *The subspaces V^+ and V^- are orthogonal with respect to the $\langle \cdot, \cdot \rangle_H$ inner product, and $P^{\pm}$ are the corresponding orthogonal projectors.*

3. Outline of proofs of the main results

This section aims to provide an outline of the main ideas that go into the proofs of the results in [17]. We have structured the steps of the proof in a modular fashion, where each module can be understood separately and only the main result carries forward. We distinguish four main modules/steps:

- (i) energy estimates for the linearized equation,
- (ii) vector field energy estimates,
- (iii) pointwise bounds derived from energy estimates (sometimes called Klainerman-Sobolev inequalities),
- (iv) asymptotic and wave packet analysis.

While this may seem like a standard approach, there are a number of technical difficulties that prevent us from carrying a straightforward analysis, and also there are several improvements we bring to the analysis.

The proof of the global result is structured as a bootstrap argument. But unlike the classical approach where a large number of vector field bounds are needed, here our bootstrap assumption involves only pointwise bounds on the solutions, precisely it has the form

$$\|\psi(t)\|_{L^6} + \|A(t)\|_{L^\infty} \lesssim \frac{C\varepsilon}{\langle t \rangle}, \tag{3.1}$$

which is consistent with the linear dispersive decay bounds for the Dirac equation, respectively the wave equation. Then the final objective becomes to show that we can improve this bound. This is accomplished in several steps as noted above:

3.1. **Energy estimates for the linearized equation.**

These are relatively straightforward, as they are carried out in our base Sobolev space $L^2 \times \dot{H}^{1/2} \times \dot{H}^{-1/2}$. Nevertheless, their proof is still instructive in understanding how a minimal $t^{C\varepsilon}$ energy growth can be derived using only the above bootstrap assumptions.

The solutions for the linearized system around a solution (ψ, A) are denoted by (ϕ, B). Including also source terms, the linearized system takes the form

$$\begin{cases} -i\gamma^\mu \partial_\mu \phi + \phi = \gamma^\mu A_\mu \phi + \gamma^\mu B_\mu \psi + F \\ \Box B_\mu = -\overline{\phi}\gamma_\mu \psi - \overline{\psi}\gamma_\mu \phi + G_\mu \\ \partial^\mu B_\mu = 0. \end{cases} \tag{3.2}$$

For completeness we write down the energy estimates for the linearized system (3.2). These are obtained assuming the bootstrap hypothesis (3.1), which is consistent with having minimal assumptions on the control norms used in getting these energy estimates. To keep the ideas simple here, we assume this holds in a time interval $[0, T]$. However, our bootstrap argument for the full problem will instead be carried out in the regions $C_{<T}$ which we introduce in the next section, see (3.6).

Proposition 3.1. *Assuming the bootstrap bound* (3.1) *(on ψ), we have the estimate*

$$\begin{aligned} \|(\phi, B)(t)\|^2_{\mathscr{H}^0} \leq \|(\phi, B)(1)\|^2_{\mathscr{H}^0} + \int_1^t C_1 C_0 \varepsilon s^{-1} \|(\phi, B)(s)\|^2_{\mathscr{H}^0}\, ds \\ + \Big| \mathrm{Re} \int_1^t \int_{\mathbb{R}^3} (\phi \cdot \overline{i\gamma^0 F} + |D|^{-1}\partial_t B_\mu \overline{G_\mu})\, dxds \Big|. \end{aligned} \tag{3.3}$$

In particular, in the case $F = G = 0$, we get the energy estimate

$$\|(\phi, B)(t)\|_{\mathscr{H}^0} \lesssim \langle t \rangle^{c\varepsilon} \|(\phi, B)(0)\|_{\mathscr{H}^0}. \tag{3.4}$$

These energy estimates are central for our analysis. On one hand, they are partly responsible for the choice we make for the bootstrap assumption. On the other hand, they also provide the starting point for the vector fields bounds for the solution to (1.1), which are described in subsection 3.2.

3.2. **Energy estimates for the solutions.** This is again done under the above bootstrap condition (3.1), and it yields energy bounds with a $t^{C\varepsilon}$ growth. It also includes the vector field bounds, and for clarity they are separated into several steps. They are first proved for the solution and its higher derivatives, second for vector fields, and finally for both vector fields and derivatives applied to the solution. While using just interpolation inequalities and Gronwall type inequalities in time works in the first case, in order to obtain vector field energy bounds using only our bootstrap assumptions we work instead in dyadic time slabs denoted by C_T, which with the proper set-up enable us to optimize the interpolation of vector field bounds. In this we follow the lead of the earlier work of Ifrim-Stingo [18].

Returning to the main goal of this subsection, we recall that we want to establish energy bounds for (ψ, A) and their higher derivatives as well as energy bounds for the solution (ψ, A) to which we have applied a certain number of vector fields admissible to (1.1). These functions solve a system which is closely related to the linearized system, but the vector fields bring in additional difficulties which require a more complex argument.

Ideally, given such a bootstrap assumption in a time interval $[0, T]$, one would like to prove a vector field energy bound of form

$$\|\Gamma^{\leq k}(\psi, A)(t)\|_{\mathscr{H}^0} \lesssim \langle t \rangle^{c\varepsilon} \|\Gamma^{\leq k}(\psi, A)(0)\|_{\mathscr{H}^0}, \tag{3.5}$$

where $c \approx C_0$. Here we would like to use interpolation and Gronwall's inequality as in the previous section. But since our vector fields also involve time derivatives, the interpolation should happen in a space-time setting. Because of this we will no longer be able to apply directly Gronwall's inequality in time; instead it turns out that a dyadic time decomposition would address the issue; this is similar to the work of the second author in [18].

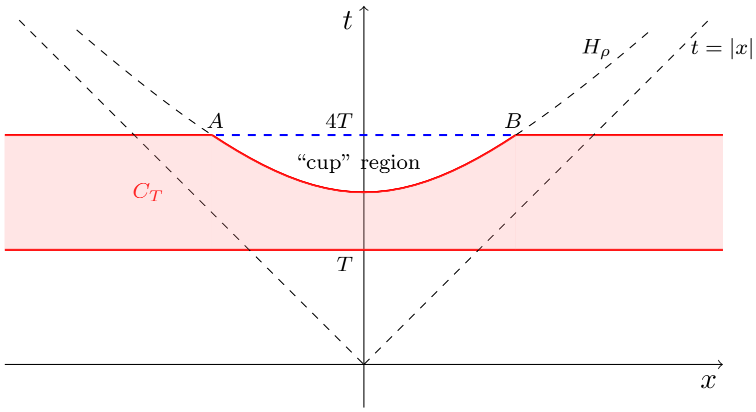

Figure 1. Region C_T in 3+1 space-time dimension; "cup" region definition

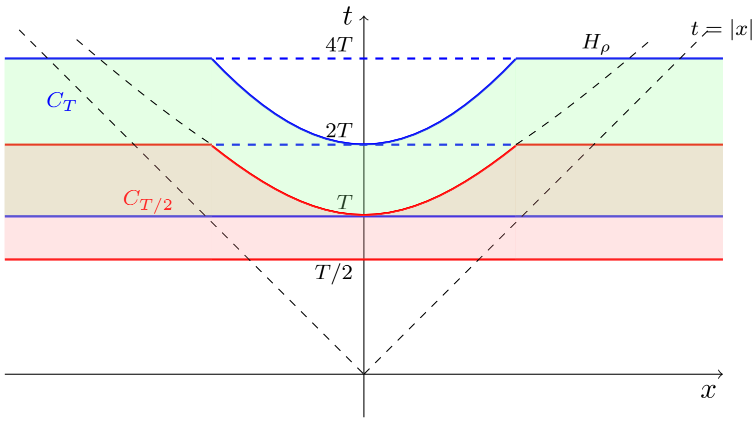

Figure 2. Overlapping C_T regions

Ideally one might want to work in the overlapping dyadic regions $[T, 4T] \times \mathbb{R}^3$, except that such regions cannot be well foliated by hyperboloids. So we define instead the regions

$$C_T := \{t \in [T, 4T]\,,\, t^2 - x^2 \leq 4T^2\},$$

and also

$$C_{<T} := \{t \in [0, 4T]\,,\, t^2 - x^2 \leq 4T^2\}. \tag{3.6}$$

For interpolation purposes, we will also use a slight enlargement C_T^+ of C_T where we add a lower cap, thus working with the region we define next

$$C_T^+ := C_T \cup \{(t, x) \in [T/2, T] \cap \mathbb{R}^3, t^2 - x^2 \geq T^2/4\}; \tag{3.7}$$

explicitly, this is a slab with a cap removed on top and with a similar cap added at the bottom.

Our main bootstrap argument in [17] will be carried out exactly in a region of the form $C_{<T}$, where we will assume that the bootstrap assumption (3.1) holds with a large constant C_0, and then show that we can improve the constant.

The aim here is to carry out the first step of this process, namely to prove that, under such a bootstrap assumption in $C_{<T}$, we can obtain vector field energy estimates in the same region $C_{<T}$. For this, the strategy will be to inductively prove the vector field bounds in the (overlapping) regions C_T for dyadic T, losing a $1 + C\varepsilon$ factor at every step.

To measure our solutions in C_T, it is convenient to introduce a stronger norm which does not contain only the energy, but also the size of the source terms in the corresponding linear equation. Thus we define the following norms in the regions C_T:

$$\|(\psi, A)\|_{X_T} := \|(\psi, A)(T)\|_{\mathscr{H}^0} + T^{1/2}\| - i\gamma^\mu \partial_\mu \psi + \psi\|_{L^2(C_T)} + T^{1/3}\|\Box A\|_{L^{3/2}(C_T)}. \tag{3.8}$$

After this discussion, we are ready to state our main energy estimates:

Proposition 3.2. *Assume that the bootstrap bounds* (3.1) *hold in $C_{<T}$. Then in any dyadic region $C_{T_1} \subset C_{<T}$ we have the energy estimates*

$$\|\Gamma^{\leq k}(\psi, A)\|_{X_{T_1}} \lesssim \langle T_1\rangle^{c\varepsilon}\|\Gamma^{\leq k}(\psi, A)(0)\|_{\mathscr{H}^0}, \qquad T_1 \leq T, \tag{3.9}$$

holding for a total of $k = 9$ vector fields and derivatives, with $c \approx C_0$.

This in particular implies the fixed time energy bounds in (3.5), but also provides additional information which we will use later on for the Klainerman-Sobolev inequality.

Remark 3.3. The proof of this proposition splits the analysis in three parts. First we derive the bounds if only translation vector fields are applied, then if only Ω (this is a shorthand notation for $\Omega_{\alpha\beta}$ which we will frequently use throughout) vector fields are applied, and finally if a mix of translation and Ω vector fields is applied.

3.3. **Pointwise (Klainerman-Sobolev) bounds.** These are derived from the previous energy bounds, and are akin to classical Sobolev embeddings but on appropriate scales. For this purpose we separate the dyadic time slabs C_T above into smaller sets, namely the dyadic regions $C^{\pm}_{TS}$, where T stands for dyadic time, S for the dyadic distance to the cone, and $\pm$ for the interior/exterior cone, plus an additional interior region C^{int}_T and an exterior region C^{ext}_T. Then it becomes important, as an intermediate step, to derive space-time L^2 *local energy bounds*, localized to these sets. Once this is done, our pointwise bounds are akin to Sobolev embeddings or interpolation inequalities in these regions, with the extra step of also using the linear equation in several interesting cases. We note that these bounds inherit the $t^{C\varepsilon}$ extra growth from the energy estimates, so they do not suffice in order to close the bootstrap.

A first step in recovering the bootstrap bounds on the global time scale is to prove appropriate Klainerman-Sobolev inequalities, where the aim is to obtain pointwise bounds from the integral X_T type bounds in Proposition 3.2. By itself this does not suffices globally in time because the time growth $t^{C\varepsilon}$ from the energy estimates will carry over. Instead it only suffices almost globally in time. Nevertheless, the bounds we establish here will suffice in order to estimate the errors in the asymptotic equations in later sections.

Our main result here is linear and applies at a fixed dyadic scale T. Because of this, we omit the $T^{C\varepsilon}$ factor in Proposition 3.2. An intermediate step is to show that

Theorem 3.4. *Assume that in a time dyadic region $C_T \cup C_{T/2}$ we have*

$$\|\Gamma^{\leq k}(\psi, A)(t)\|_{X_T} \leq 1, \qquad k = 9. \tag{3.10}$$

Then in C_T we have

$$|\psi| \lesssim t^{-3/2}\langle t-r\rangle_-^{-\delta}, \tag{3.11}$$

$$|\partial A| \lesssim t^{-1}\langle t-r\rangle^{-1/2}, \tag{3.12}$$

$$|\mathscr{T} A| \lesssim t^{-3/2}, \tag{3.13}$$

where $\delta > 0$. In addition, inside the cone we have an improved bound for $\mathscr{T} A$, namely

$$|\mathscr{T} A| \lesssim \langle t\rangle^{-3/2}\langle t-r\rangle^{-1/2}. \tag{3.14}$$

Remark 3.5. What is missing here is the uniform t^{-1} bound for A, which would be too much to ask for at this point, using only the information given in the hypothesis of the theorem above. Instead, we will prove pointwise bounds for A later on, by using the wave equation for A and the pointwise bounds for ψ.

Remark 3.6. The improved bound (3.14) is due to the fact that out baseline spaces for the wave equation are $\dot H^{1/2} \times \dot H^{-1/2}$, as opposed to $\dot H^1 \times L^2$. With additional work one should be able to obtain a similar improved bound for ∇A

$$|\nabla A| \lesssim \langle t\rangle^{-1}\langle t-r\rangle^{-1}, \tag{3.15}$$

but this would require some adjustments to the X_T spaces.

The space-time decomposition. It suffices to prove the desired pointwise bounds in the region C_T^+, separately the Dirac and the wave component. Our strategy is to reduce the proof of the theorem to standard Sobolev embeddings in regions which, in suitable coordinates, have unit size. To place ourselves in this situation, we decompose the region C_T^+ into smaller regions which have fixed geometry, as follows:

$$C_T^+ := C_T^{int} \bigcup C_T^{ext} \bigcup_{\pm} \bigcup_{1\leq S\leq T} C_{TS}^{\pm}. \tag{3.16}$$

We now describe the sets in this decomposition:

- The interior region C_T^{int} is defined as
$$C_t^{int} := ([T/2, 4T] \times \mathbb{R}^3) \cap \{T^2/4 \leq t^2 - x^2 \leq 4T^2\}.$$
This region can be foliated with large sections of hyperboloids.
- The exterior region C_T^{ext} is far outside the cone, and is described as
$$C_T^{ext} := \{(t,x); t \in [T, 4T]; r \geq 2T\}.$$
- The region around the cone, we dyadically decompose with respect to the size of $t-r$, which measures how far or close we are to the cone
$$\begin{aligned} &C_{TS}^+ := \{(t,x) \,:\, S \leq t-r \leq 2S,\, T \leq t \leq 2T\}\,, \text{ where } 1 \leq S \lesssim T,\\ &C_{TS}^- := \{(t,x) \,:\, S \leq r-t \leq 2S,\, T \leq t \leq 2T\}\,, \text{ where } 1 \leq S \lesssim T; \end{aligned} \tag{3.17}$$
see Figure 3.

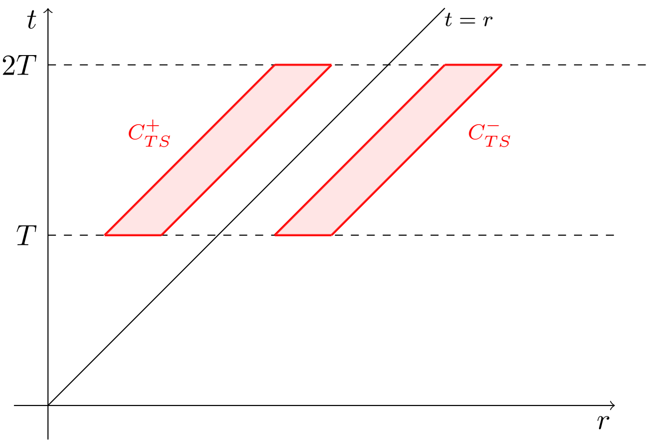

FIGURE 3. 1D vertical section of space-time regions $C^{\pm}_{TS}$

Here C^+_{TS} represents a spherically symmetric dyadic region inside the cone with width S, distance S from the cone, and time length T. C^-_{TS} is the similar region outside the cone where, far from the cone, we would have $T \lesssim S$. To simplify the exposition we will use the notation C_{TS} as a shorthand for either C^+_{TS} or C^-_{TS}. These regions are also well foliated with sections of hyperboloids. Such a decomposition has been introduced before by Metcalfe-Tataru-Tohaneanu [30] in a linear setting; we largely follow their notations.

In the above definition of the C_{TS} sets we limit S to $S \geq 1$ because our assumptions are invariant with respect to unit size translations. In particular, this leaves out a conical shell region along the side of the cone $t = r$, which intersects both the interior and the exterior of the cone. To also include this region in our analysis we redefine

$$C_{T1} := \{(t,x) \,:\, |t-r| \leq 2,\, T \leq t \leq 2T\}\,, \text{ where } S \sim 1. \tag{3.18}$$

Localized energy bounds. These represent a key intermediate step in the proof of the pointwise bounds, and differ depending on whether we are inside or outside the cone.

a) *Inside the cone:* This is the more favorable case, where the hyperboloids are space-like and thus we have energy estimates on the hyperboloids. For the Dirac component ψ, these estimates represent an extension of (2.8), and, by (2.11), have the form

$$\|\psi_+\|_{L^2(H\cap C_{TS})} \lesssim T^{1/2}S^{-1/2}\|\psi\|_{X^T}, \qquad \|\psi_+\|_{L^2(H\cap C_{TS})} \lesssim \|\psi\|_{X^T}. \tag{3.19}$$

For the wave equation it is convenient to work at the H^1 level rather than at the $H^{1/2}$ level, in which case the energy estimates on hyperboloids have the form

$$\|\nabla A\|_{L^2(H\cap C_{TS})} \lesssim T^{1/2}S^{-1/2}\|A\|_{\tilde{X}^T}, \qquad \|\mathscr{T} A\|_{L^2(H\cap C_{TS})} \lesssim \|A\|_{\tilde{X}^T}. \tag{3.20}$$

where

$$\|A\|_{\tilde{X}^T} = \|\nabla A(T)\|_{L^2} + \|\Box A\|_{L^1L^2}.$$

a) *Outside the cone:* Here the hyperboloids are time-like so it is no longer possible to obtain energy bounds on hyperboloids. Instead, we can prove L^2 bounds in the regions C^-_{TS}:

Lemma 3.7. *We have the following estimates:*

$$\|\psi_+\|_{L^2_{t,x}(C_{TS})} \lesssim T^{1/2}\|\psi\|_{X_T}, \qquad \|\psi_-\|_{L^2_{t,x}(C_{TS})} \lesssim S^{1/2}\|\psi\|_{X_T}, \tag{3.21}$$

respectively

$$\|\nabla A\|_{L^2_{t,x}(C_{TS})} \lesssim T^{1/2}\|A\|_{\tilde{X}_T}, \qquad \|\mathscr{T} A\|_{L^2_{t,x}(C_{TS})} \lesssim S^{1/2}\|A\|_{\tilde{X}_T}. \tag{3.22}$$

As stated in the lemma, similar bounds also hold on C^+_{TS}, but there they can be seen as direct consequences of the energy estimates on hyperboloids.

The pointwise bounds. The localized energy bounds describe above can be applied to the functions $\Gamma^{\leq 9}(\psi, A)$ in each of the sets $C^{int}_T, C^{\pm}_{TS}$. Once this is done, the proof of the pointwise bounds for (A, ψ) is restricted to each of these sets, whose analysis is at this point completely decoupled, without any remaining global considerations. In these dyadic sets, the pointwise bounds are broadly obtained by a careful application of Sobolev embeddings and interpolation. The implementation of this idea, however, brings forth a number of technical difficulties. To describe the strategy of the proof, we begin with some general considerations:

- (i) Each of the regions $C^{int}_T, C^{\pm}_{TS}$ is foliated by hyperboloids, whose intersection with the corresponding region has a unit size with respect to the hyperbolic metric.
- (ii) All Lorentz boosts $\Omega_{\alpha\beta}$ involve derivatives along hyperboloids which, have a unit size with respect to the hyperbolic metric. To obtain a basis in the tangent space we complement these Lorentz vector fields with the radial derivative ∂_r.
- (iii) The vector field bounds for A are all at the $\dot{H}^{1/2}$ level, while the localized energy bounds are at the H^1 level. To shift between the two, we simply use a spatial Littlewood-Paley decomposition.

Taking the above ideas into account, we can now discuss separately the strategy to prove pointwise bounds in each of our regions:

- (a) The region C^{int}_T is the easiest to consider, as there it suffices to use the Sobolev embeddings on each of the hyperboloid sections $H \cap C^{int}_T$.
- (b) In the regions C^+_{TS} we still have energy bounds on hyperboloids, so the Sobolev embeddings on the hyperboloid sections $H \cap C^{int}_T$ are still applicable. This suffices for the bounds for ψ_+ and $\mathscr{T} A$. However, in the case of ψ_- and ∇A we also need to access their r derivatives, which are gained from the Dirac, respectively the wave equation by expressing them in the frame involving only vector field derivatives and ∂_r derivatives.
- (c) In the region C^-_{TS} we no longer have direct access to traces on hyperboloids, so we need to use the localized L^2 bounds in the entire region. We view the tangent space of these regions as spanned by ∂_r and by vector field derivatives. As in the C^+_{TS} case, the ∂_r derivatives are accessed via Dirac, respectively the wave equation by expressing them in this frame. In the Dirac case we obtain an elliptic system in the r direction, which yields better decay bounds than inside the cone. In the wave case we obtain control over $\partial_r^2 A$, which can be used in an interpolation argument combined with the vector field bounds.
- (d) In the exterior region C^{ext}_{TS} we can use the finite speed of propagation and straightforward energy estimates followed by Sobolev embeddings, without any need for vector field bounds.

3.4. **Asymptotic analysis for ψ and A.** Combining the vector field energy bounds with the Klainerman-Sobolev inequalities yields pointwise decay bounds for ψ and A but with an additional $t^{c\varepsilon}$ factor, which does not allow one to close the bootstrap, and even less to describe their asymptotics. This is rectified via a careful analysis of the asymptotic behavior of both A and ψ, which is done in several stages.

The asymptotic profiles and the asymptotic equation for ψ. Heuristically one expects a Klein-Gordon type asymptotic expansion for the spinor field,

$$\psi(t,x) = t^{-3/2} \sum_{\pm} e^{\pm i\sqrt{t^2-x^2}} \rho^{\pm}(t,x) + O(t^{-3/2-\delta}), \qquad \delta > 0, \tag{3.23}$$

with well chosen slower varying *asymptotic profiles* $\rho^{\pm}$. In the case of the linear Dirac flow one may choose $\rho^{\pm} = \rho^{\pm}(v)$ to depend only on the velocity $v = x/t$, with the added restriction that $\rho^{\pm} \in V^{\pm}$, where these two H-orthogonal subspaces are defined in (2.15). However, for our nonlinear flow this is no longer possible, and instead we need to allow the profiles to also have a slow dependence on t,

$$\rho^{\pm} = \rho^{\pm}(t,v), \qquad \rho^{\pm} \in V^{\pm}.$$

Then the objectives are

(i) to identify good *asymptotic profiles*, and
(ii) to study their time dependence on rays (*asymptotic equation*).

The asymptotic profiles are defined using the method of wave packet testing of Ifrim-Tataru [19], [21], [20], [22]. However, the wave packet analysis is carefully adapted to the Dirac system, which is novel and quite interesting. The profiles constructed in this manner are shown to provide a good approximation to the spinor field ψ in the sense of (3.23), and to satisfy an appropriate asymptotic equation, which turns out to be an ode of the form

$$i\partial_t \rho^{\pm}(t,v) = v_\alpha A^\alpha \rho^{\pm}(t,v) + O(t^{-1-\delta}), \qquad \delta > 0. \tag{3.24}$$

In both cases the errors are estimated both in L^2 and in L^∞ norms, based on the vector field energy estimates and the matching pointwise bounds; these exhibit $t^{c\varepsilon}$ growth, but that is harmless in the proof of the error bounds.

Since the connection coefficients A^α are real, the asymptotic equation (3.24) allows us to propagate uniform pointwise bounds for $\rho^{\pm}$, which are then transferred to ψ using (3.23). Thus, by the end of this section we are able to prove $t^{-3/2}$ decay for ψ on rays $x = vt$, and thus to close the ψ part of the bootstrap loop.

Uniform bounds for A. The t^{-1} decay bounds for A are obtained directly from the wave equation for A, using the standard bounds for the fundamental solution for the d'Alembertian. Here one needs to separately estimate the contributions of the initial data and of the source term, where for the latter we use the $t^{-3/2}$ decay bounds for ψ from the previous step. This closes the A part of the bootstrap loop, and thus completes the proof of the global well-posedness result in Theorem 1.1.

Radiation profiles for ψ and A inside the cone. These are constructed in the last section of the [17], whose final objective is to prove the modified scattering result in Theorem 1.3. This is achieved in several steps, where we successively construct

(a) an initial radiation profile $\rho_\infty^\pm$ for ψ, which corresponds to an exact solution to (3.24), without source term. This is accurate only up to a phase rotation, but suffices for the next step, which requires only the profile size $\|\rho_\infty^\pm\|_H$.
(b) a radiation profile $a_\infty^\mu(v)$ for A, which can be thought of as the limit of $(t^2 - x^2)A^\mu$ along rays $x = vt$. This is obtained by solving the inhomogeneous wave equation with a -3-homogeneous source term which corresponds to replacing $\rho^\pm$ with its radiation profile in (3.23). Expressed in hyperbolic coordinates, this yields exactly the equation (1.20).
(c) Using the result in part (b) we refine the choice of the radiation profile $\rho_\infty^\pm$ for ψ, removing the phase rotation ambiguity in (a). This is achieved by replacing A with its radiation profile in the asymptotic equation (3.24).

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