Séminaire Laurent Schwartz

EDP et applications

Année 2024-2025

Exposé n° XVII (18 mars 2025)

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https://doi.org/10.5802/slsedp.181

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Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2266-0607

Séminaire Laurent-Schwartz — EDP et applications
Centre de mathématiques Laurent Schwartz, 2024-2025
Exposé n^{o} XVII, 1-13

ASYMPTOTIC STABILITY FOR SELF-SIMILAR BLOWUP OF MASS-SUPERCRITICAL NLS

ZEXING LI

Abstract. We consider self-similar blowup for (NLS) $i\partial_t u + \Delta u + u|u|^{p-1} = 0$ in $d \geq 1$, focusing on the slightly mass-supercritical range $0 < s_c := \frac{d}{2} - \frac{2}{p-1} \ll 1$. The existence and stability of such dynamics [39] and construction of suitable profiles [1] lead to the question of asymptotic stability. In this note, we review the background and recent results [25, 26, 27] on the asymptotic stability, with particular emphasis on mode stability and linear stability.

1. Introduction

The focusing nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + u|u|^{p-1} = 0, \quad u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}, \tag{NLS}$$

describes nonlinear propagation phenomena in physics including electromagnetic beams and Langmuir waves in plasma. The singularity here is termed filamentation or collapse [31, 47]. Mathematically, (NLS) is a fundamental nonlinear dispersive equation that admits linear dispersion and nonlinear soliton behavior. Compared with semilinear heat or wave equations, singularity formation for (NLS) is more challenging due to its system nature and more subtle dispersive decay.

In this note, we will discuss the self-similar singularity in the mass-supercritical and energy-subcritical setting, based on the works [25, 26, 27].

1.1. Preliminaries on NLS and its blowup.

Symmetry, conservation laws and criticality. The nonlinear Schrödinger equation has a $(2d+2)$-dimensional symmetry group related to space translation, phase rotation, scaling and Galilean invariance

$$\tilde{u}(t,x) := \lambda^{2/p-1} u(\lambda^2 t, \lambda x - \lambda^2 tv - x_0) e^{i\left(\frac{\lambda x\cdot v}{2} - \frac{\lambda^2|v|^2}{4}t + \gamma_0\right)}; \tag{1.1}$$

and the mass and energy functionals

$$M(u(t)) = \|u(t)\|^2_{L^2}, \quad E(u(t)) = \frac{1}{2}\|\nabla u(t)\|^2_{L^2} - \frac{1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}},$$

are formally conserved by the flow. In particular, the scaling symmetry $u \mapsto u_\lambda := \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$ for $\lambda > 0$ implies the critical L^2-based space

$$\|u_\lambda(t)\|_{\dot{H}^{s_c}} = \|u(\lambda^2 t)\|_{\dot{H}^{s_c}} \quad \text{for } s_c := \frac{d}{2} - \frac{2}{p-1}. \tag{1.2}$$

The case $s_c = 0$ or 1 is referred to as mass-critical or energy-critical respectively.

Ground state. Let us mention one important nonlinear feature of (NLS): the existence of ground state solutions. For (NLS) with $s_c < 1$, there exists a unique positive, radial H^1 solution of the nonlinear elliptic equation

$$\Delta Q - Q + Q|Q|^{p-1} = 0,$$

and then $u(t,x) = Q(x)e^{it}$ becomes a time-periodic special solution of (NLS). We call Q the *ground state* of (NLS).

Long-time dynamics: Global well-posedness and blowup. In mass-subcritical case $s_c < 0$, the mass and energy conservation laws plus a simple Gagliardo-Nirenberg interpolation imply a uniform a priori H^1 bound, resulting in global well-posedness of (NLS). For the mass-(super)critical case $s_c \geq 0$, singularity could appear from regular initial data. It is proven by a convexity argument based on the Virial identity, a special algebraic structure of (NLS). Nevertheless, this argument does not provide any detailed characterization of blowup (profile, blowup rate, asymptotics).

Type I and Type II blowup. A general a priori lower bound on the blowup rate can be derived from the local well-posedness theory and scaling invariance. For $0 \leq s_c < 1$, all blowup solutions satisfy [6]

$$\|u(t)\|_{\dot{H}^\sigma} \gtrsim (T-t)^{-(\sigma - s_c)/2} \tag{1.3}$$

for every $s_c < \sigma \leq 1$.[1] We call a blowup solution *self-similar* or *type I* if it saturates the self-similar law

$$\|u(t)\|_{\dot{H}^\sigma} \sim (T-t)^{-(\sigma - s_c)/2}, \quad s_c < \sigma \leq 1, \tag{1.4}$$

and otherwise *type II* if the norm blows up strictly faster than (1.4).

1.2. **Type I blowup for NLS.** From now on, we will focus on the self-similar blowup in mass-supercritical and energy-subcritical range

$$0 < s_c < 1.$$

We also mention that Type II solutions have been constructed both in the critical case $s_c = 0$, [41, 34, 36, 32, 33, 35, 42], and the supercritical case through the derivation of “ring" solutions which concentrate on a circle, [17, 43, 40, 20]. We refer to [44] as a comprehensive survey.

1.2.1. *Motivation: Existence and stability of Type I blowup.* With an increasing interest in the wave collapse, particularly for the $3D$ cubic (NLS) as a limit of Zakharov system for Langmuir waves in plasmas, there have been many numerical investigations on the Type I blowup [31, 24, 23, 47, 50], which strongly suggests the existence and stability of Type I profile in the range $0 < s_c < 1$. Moreover, this seems to be the only possible regime of singularity formation via bubbling at one point, because of an a priori log lower bound in the critical norm $L^{p_c} \supset \dot{H}^{s_c}$

$$\|u(t)\|_{L^{p_c}} \geq |\log(T-t)|^\gamma, \quad \text{as } t \to T, \tag{1.5}$$

for any radial blowup solution to (NLS) proven by Merle-Raphaël [37]. Here the constant $\gamma = \gamma(d,p)$ is universal, while the sharp constant is unclear even formally. Notice that a Type II bubbling is expected to have uniformly bounded critical norm due to the localization of the blowup profile.

The first rigorous construction result regarding Type I blowup for (NLS) was obtained by Merle-Raphaël-Szeftel [39]. They proved the existence and stability in slightly supercritical range $0 < s_c \ll 1$ by bifurcating the log-log analysis of the mass-critical case [32, 36].

[1] That is a formal manifestation of the concentration rate $\lambda(t) \sim (T-t)^{1/2}$ due to the scaling $\partial_t \sim \Delta_x$, supposing $u(t)$ has the asymptotic form $u(t) \approx \lambda(t)^{-2/(p-1)} W(\cdot/\lambda(t))$ for some profile $W \in \dot{H}^\sigma$.

Theorem 1.1 (Existence and stability of self-similar blowup for $s_c \ll 1$, [39])**.** *For $1 \le d \le 10$, $0 < s_c \ll 1$, there exists an open set of initial data in H^1 s.t.*

$$u(t,x) = \lambda(t)^{-2/(p-1)}(Q + \varepsilon(t))\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)}, \tag{1.6}$$

with

$$\lambda(t) \sim (T-t)^{-1/2}, \quad \|\nabla\varepsilon(t)\|_{L^2} \le \delta$$

for some $\delta \ll 1$. Moreover, there exists $u^* \in H^\sigma$ for $\sigma \in [0, s_c)$, $u^* \notin H^{s_c}$ such that

$$u \xrightarrow{t \to T} u^* \quad \text{in } H^\sigma, \quad \forall \sigma \text{ with } 0 \le \sigma < s_c.$$

To identify this as a type I blowup, we can derive from the decomposition (1.6), behavior of $\lambda(t)$ and smallness of $\varepsilon(t)$ in $\dot{H}^1$ that the self-similar law (1.4) holds for $u(t)$ with $\sigma = 1$. The open set of initial data leading to this dynamics is referred to as stability, and the last property confirms the existence of a limiting profile in subcritical topology (where the singularity is invisible). Moreover, the ground state Q serves as a rough approximate self-similar profile.

However, since Q is merely an *approximate* profile, the perturbation $\varepsilon(t)$ does not vanish as $t \to T$. That opens the question of *sharp description* of this blowup scenario, which involves two questions:

(1) Existence of an exact self-similar profile Q_b: Q_b is a stationary solution of (1.8).
(2) Asymptotic stability of such self-similar profile: $u = (Q_b + \varepsilon(t))_{\lambda(t)} e^{i\gamma(t)}$ with $\varepsilon(t) \to 0$ in some topology.

We remark that these questions can be asked for the slightly supercritical range $0 < s_c \ll 1$, or more generally, the full intercritical range $0 < s_c < 1$.

1.2.2. *Question 1: existence of self-similar profile.* To extract the self-similar profile, we rewrite the equation in self-similar coordinates. More precisely, for (NLS), the self-similar renormalization

$$u(t,x) = \frac{e^{i\gamma(t)}}{(\lambda(t))^{2/(p-1)}} v(\tau, y)$$

where

$$\lambda(t) = \sqrt{2b(T-t)}, \quad \gamma(t) = \tau(t), \quad \tau(t) = -\frac{1}{2b}\ln(T-t), \quad y = \frac{x}{\lambda(t)} \tag{1.7}$$

with $b > 0$ as a constant. This maps (NLS) onto the renormalized flow

$$i\partial_\tau v + \Delta v - v + \frac{ib}{2}\left(\frac{2}{p-1}v + y \cdot \nabla v\right) + v|v|^{p-1} = 0. \tag{1.8}$$

Then the self-similar profile will be a finite-energy stationary solution of (1.8).

Conjecture 1.2 (Existence of suitable self-similar profiles)**.** *Let $d \ge 1$ and $0 < s_c < 1$. Then there exists $b > 0$ and a smooth radially symmetric profile Q_b with the following properties.*

(i) Equation: *Q_b is a stationary solution to* (1.8)*:*

$$\Delta Q_b - Q_b + ib\left(\frac{2}{p-1}Q_b + y \cdot \nabla Q_b\right) + Q_b|Q_b|^{p-1} = 0. \tag{1.9}$$

(ii) Non-vanishing*:*

$$Q_b(x) \ne 0 \quad \forall x \in \mathbb{R}^d. \tag{1.10}$$

(iii) Self-similar decay*:*

$$\lim_{r\to\infty} r^{2/p-1}|Q_b(r)| = c_{d,s_c} > 0, \quad \limsup_{r\to\infty} r^{(p+1)/(p-1)}|Q_b'(r)| < \infty. \tag{1.11}$$

Remark 1.3. An ODE analysis of the linearization of (1.9) implies the behavior (1.11) so that $E(Q_b)$ is well-defined. The tail behavior also yields $Q_b \in \dot{H}^\sigma \Leftrightarrow \sigma > s_c$ and $E(Q_b) = 0$. We further remark that the non-vanishing property is inherited in (i) and (iii) by [48, Lemma 2.2].

Remark 1.4. We stress that the construction of self-similar profiles is central in the study of singularity formation mechanisms and has been mostly addressed in the energy supercritical case $s_c > 1$, see [11] for the heat equation, [12] for the wave equation, [19, 38] for compressible fluids and [15, 16, 8] for incompressible fluids.

The first rigorous existence result was obtained by Bahri-Martel-Raphaël [1] using a ODE bifurcation argument from the mass-critical ground state, inspired from the pioneering work [21] on generalized KdV equation.

Theorem 1.5 (Existence of suitable self-similar profile for $0 < s_c \ll 1$, [1])**.** *For $d \geq 1$, $0 < s_c \ll 1$, Conjecture 1.2 holds true. In particular, Q_b and b satisfy*

$$b \to 0, \quad \text{and} \quad Q_b \to Q \text{ in } \dot{H}^1, \qquad \text{as } s_c \to 0. \tag{1.12}$$

For bigger s_c, extending this bifurcation branch is a delicate nonlinear ODE problem. On the other hand, the latest progress came from Donninger-Shörkhuber [14], who rigorously constructed the suitable profile for the physical scenario $d = p = 3$ with computer assistance.

1.2.3. *Question 2: Asymptotic stability.* Using the profiles from Theorem 1.5, our main result from [25, 26, 27] answers the asymptotic stability question in the same slightly supercritical setting $0 < s_c \ll 1$ as Theorem 1.1.

Theorem 1.6 (Asymptotical stability for $s_c \ll 1$, [25, 26, 27].)**.** *Let $1 \leq d \leq 10$, $0 < s_c \ll 1$ small enough and Q_b with $b = b(s_c)$ be the self-similar profile from Theorem 1.5. Then for $0 < \sigma - s_c \ll 1$, there exists an open set of initial data $\mathcal{O} \subset H^1$ such that its solution to* (NLS) *blows up at $T = \lambda_0^2/2b$ and satisfies*

$$u(t,x) = \frac{1}{(\lambda_0^2 - 2bt)^{1/(p-1)}}(Q_b + \varepsilon)\left(t, \frac{x - x_0}{\sqrt{\lambda_0^2 - 2bt}}\right) e^{-i\left[\frac{\ln(\lambda_0^2/2b - t)}{2b} + \theta_0\right]}, \tag{1.13}$$

with $(\lambda_0, x_0, \theta_0) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$. Moreover, we have the decay of perturbation as

$$\|\varepsilon(t)\|_{\dot{H}^\sigma \cap \dot{H}^1} \lesssim (T - t)^{(\sigma - s_c)/2}, \tag{1.14}$$

and there exists $u_* \in \dot{H}^\sigma \cap \dot{H}^1$ and $u^* \in H^{\tilde{\sigma}}$ for every $0 \leq \tilde{\sigma} < s_c$ such that

$$u(t) - \frac{1}{\lambda(t)^{2/(p-1)}} Q_b\left(x/\lambda(t)\right) e^{i\tau(t)} \to u_* \qquad \text{in } \dot{H}^\sigma \cap \dot{H}^1 \quad \text{as } t \to T, \tag{1.15}$$

$$u(t) \to u^* \qquad \text{in } H^{\tilde{\sigma}} \quad \text{as } t \to T. \tag{1.16}$$

Remark 1.7. Comparing with Theorem 1.1, the scaling parameter $\lambda(t) = \left(\frac{T-t}{2b}\right)^{1/2}$ indicates the Type I nature, and with the exact profile Q_b, we obtain the decay of $\varepsilon(t)$ in supercritical norm $\dot{H}^\sigma \cap \dot{H}^1$. Moreover, apart from the limiting profile u^* in subcritical topology, we also show the existence of a limiting profile u_* in supercritical topology after removing the singularity. Thus, Theorem 1.6 verifies the asymptotic stability of Q_b in [1] and provides the sharp description of the blowup regime in Theorem 1.1.

Remark 1.8 (Regularity). The asymptotic stability also holds in $\dot{H}^\sigma$ topology for any $0 < \sigma - s_c \ll 1$, which is almost sharp in the homogeneous Sobolev space in view of the regularity of Q_b. However, the stability in critical topology $\dot{H}^{s_c}$ is still open.

Remark 1.9 (Dimension restriction). The restriction $1 \le d \le 10$ follows from the numerical verification of a spectral property, which was also used in Theorem 1.1. See discussion in Section 2.2.2.

Remark 1.10 (Unconditional finite-codimensional asymptotic stability). For all $d \ge 1$, $0 < s_c < 1$ and every admissible profile Q_b satisfying the Conjecture 1.2, a similar finite-codimensional asymptotic stability result holds true.

2. Ingredients of the proof of Theorem 1.6

In this section, we discuss the main ingredients of the proof of our main result, Theorem 1.6.

Notice that the asymptotic stability of self-similar blowup for (NLS) is equivalent to asymptotic stability of Q_b as a stationary solution for the self-similar flow (1.8). Our proof follows a straightforward linearization strategy:

(1) *Set up*: Rewriting the perturbation equation as
$$i\partial_\tau Z + \mathcal{H}_b Z = N(Z). \tag{2.1}$$
(2) *Mode stability* [26, 27]: the only unstable directions of $\mathcal{H}_b$ are generated from symmetry.
(3) *Linear stability* [25]: Strichartz estimate for $e^{it\mathcal{H}_b}$.
(4) *Nonlinear asymptotic stability* [25].

For the semilinear equation (NLS), the nonlinearity can be controlled easily by the Strichartz estimate (plus some technicalities about fractional derivatives and localization). Thus we will mainly focus on the linear analysis, namely Step (2) and (3) below.

2.1. Set up and basic linearization analysis.

Let $v = Q_b + \varepsilon$. Evolution of ε in renormalized coordinates (τ, y):
$$i\partial_\tau\varepsilon + \Delta_b\varepsilon - (1 + ibs_c)\varepsilon + W_{1,b}\varepsilon + W_{2,b}\overline{\varepsilon} + N(\varepsilon, \overline{\varepsilon}) = 0$$
where N is the nonlinearity, the potential
$$W_{1,b} = \frac{p+1}{2}|Q_b|^{p-1}, \quad W_{2,b} = \frac{p-1}{2}|Q_b|^{p-3}Q_b^2,$$
decay as r^{-2} near infinity, and the deformed Laplacian is
$$\Delta_b = \Delta + ib\left(\frac{d}{2} + y\cdot\nabla\right) = e^{-ib|x|^2/4} \circ \left(\Delta + \frac{b^2|x|^2}{4}\right) \circ e^{ib|x|^2/4}. \tag{2.2}$$
Let $Z = \binom{\varepsilon}{\overline{\varepsilon}}$, then Z satisfies (2.1) with the linearized operator
$$\mathcal{H}_b = \begin{pmatrix} \Delta_b - 1 & \\ & -\Delta_{-b} + 1 \end{pmatrix} - ibs_c + \begin{pmatrix} W_{1,b} & W_{2,b} \\ -\overline{W}_{2,b} & -W_{1,b} \end{pmatrix}. \tag{2.3}$$

Self-similar propagation. To understand the linearized operator $\mathcal{H}_b$, we begin by analyzing the deformed Laplacian Δ_b with $b \neq 0$:

(1) *Self-adjointness*: $\Delta_b = \Delta + ib\big(\frac{d}{2} + y\cdot\nabla\big)$ is self-adjoint in $L^2(\mathbb{R}^d)$, $\sigma(\Delta_b) = \mathbb{R}$. In particular, $e^{it\Delta_b}$ forms a unitary semigroup on L^2.

(2) $\dot{H}^\sigma$*-semigroup decay*: $\|e^{it\Delta_b}\|_{\dot{H}^\sigma\to\dot{H}^\sigma} = e^{-b\sigma t}$, $\sigma(\Delta_b\big|_{\dot{H}^\sigma}) = \mathbb{R} + ib\sigma$.
(3) *Self-similar dispersion*: For $2 < p \le \infty$,

$$\|e^{it\Delta_b}\|_{L^{p'}\to L^p} \lesssim_{d,p} \begin{cases} |t|^{-(d/2-d/p)} & |t| \le |b|^{-1} \\ (|b|^{-1}e^{|bt|})^{-(d/2-d/p)} & |t| > |b|^{-1} \end{cases}$$

These properties are formally evident from the definition (2.2) and the self-similar renormalization process. Indeed, one can obtain an explicit representation formula $\widehat{e^{it\Delta_b}u_0}(\xi) = e^{bdt/2}\hat{u}_0(e^{bt}\xi)e^{-i((e^{2bt}-1)/2b)|\xi|^2}$, which was first computed in [5]. We emphasize that the exponential improvement compared to $e^{it\Delta}$ appears not only in the semigroup decay when taking derivatives, but also in the dispersive estimate.

Lastly, we present two more important properties as standard corollaries of (3), which will be crucially used in the proof of linear stability:

(4) *Strichartz for* $e^{it\Delta_b}$:

$$\left\| e^{b\sigma t} e \int_0^t e^{i(t-s)\Delta_b} F(s)ds \right\|_{L_t^{q_2'}\dot{W}_x^{\sigma,p_2'}} \lesssim \left\| e^{b\sigma t} F \right\|_{L_t^{q_1}\dot{W}_x^{\sigma,p_1}} \tag{2.4}$$

where $\sigma \ge 0$ and (q_i, p_i) for $i = 1, 2$ satisfies

$$(q,p,d) \neq (2,\infty,d), \quad (q,p) \in \left\{ q \ge 2, 2 < p \le \infty, \frac{2}{q} + \frac{d}{p} \ge \frac{d}{2} \right\} \cup \{(\infty, 2)\}. \tag{2.5}$$

(5) *Extended resolvent families*: The resolvent of Δ_b can be defined across the real line as $L^{p'} \to L^p$ $(p > 2)$ operators.

$$R_b^\pm(z) = \pm i \int_0^\infty e^{\pm it\Delta_b} e^{\pm itz} dt, \quad \text{for } \pm \Im z > -b \min\{d/2, 1\}. \tag{2.6}$$

We remark that these properties hold for arbitrary $b \neq 0$.

2.2. **Mode stability.** From the property (1)-(2) and that relative compactness of potential in $\mathcal{H}_b$ with respect to Δ_b, we can obtain $\sigma_{\text{ess}}(\mathcal{H}_b\big|_{(\dot{H}^\sigma)^2}) = \mathbb{R} + ib(\sigma - s_c)$ for $\sigma > s_c$ (see [25, Proposition 4.5]). Thereafter, the mode stability can be stated as

$$\sigma_{\text{disc}}\left(\mathcal{H}\big|_{(\dot{H}^\sigma(\mathbb{R}^d))^2}\right) \cap \{z \in \mathbb{C} : \Im z < b(\sigma - s_c)\} = \{0, -bi, -2bi\}, \tag{2.7}$$

with the corresponding Riesz projections has $(d+2)$-dimensional range, generated by phase rotation, spatial translation, and scaling symmetry, respectively.

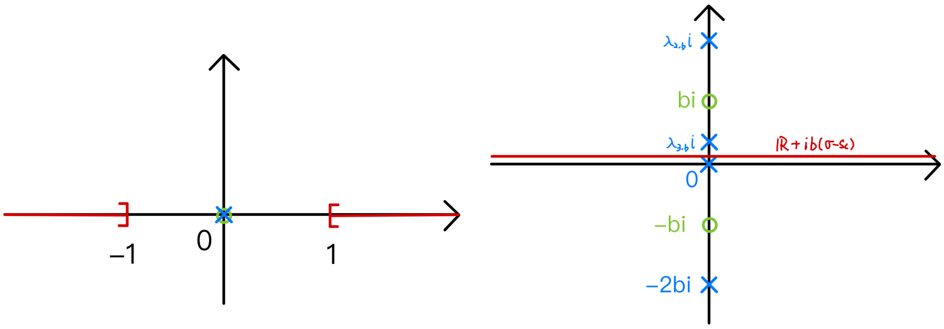
In [26, 27], we verify (2.7) for profiles in Theorem 1.5 with $1 \le d \le 10$, $0 < s_c \ll 1$ and $0 < \sigma - s_c \ll 1$. Due to the asymptotics behavior (1.12) as $s_c \to 0$, the operator $\mathcal{H}_b$ can be viewed as a bifurcation from $\mathcal{H}_0$, the linearized operator around the ground state Q in mass-critical NLS

$$\mathcal{H}_0 = \begin{pmatrix} \Delta - 1 & \\ & -\Delta + 1 \end{pmatrix} + \begin{pmatrix} W_1 & W_2 \\ -W_2 & -W_1 \end{pmatrix} \tag{2.8}$$

$$W_1 = \frac{p_0+1}{2} Q^{p_0-1}, \quad W_2 = \frac{p_0-1}{2} Q^{p_0-1}. \tag{2.9}$$

Therefore, the main task is to show there are no other eigenmodes below the essential spectrum $\sigma_{\text{ess}}(\mathcal{H}_b) = \mathbb{R} + ib(\sigma - s_c)$, which also requires a complete understanding of the bifurcation of $(2d+4)$-dimensional generalized kernel of $\mathcal{H}_0$[2] Although it is strongly expected from the stability result [39], the spectrum is difficult to obtain for the following reasons:

[2]In addition to the $(2d+2)$-dimensional symmetry group (1.1), there is one special pseudo-conformal symmetry in the mass-critical case $s_c = 0$, which brings two more generalized eigenmodes.



(A) $\sigma(\mathcal{H}_0|_{(L^2)^2})$ (B) $\sigma(\mathcal{H}_b|_{(\dot{H}^\sigma)^2})$

FIGURE 1. Indication of the spectrum of $\mathcal{H}_0$ and $\mathcal{H}_b$ near the origin: red line for σ_{ess}, blue cross for eigenpairs in radial class, and green circle for eigenpairs in first spherical class.

- The operator $\mathcal{H}_b$ is a non-self-adjoint matrix Hamiltonian. Even for $s_c = 0$ case, $\mathcal{H}_0$ as the linearization around ground state, the characterization of spectrum is open for $d \geq 2$. The $d = 1$ case is proven in [22] using the explicit formula for the ground state.
- $\mathcal{H}_b$ is a degenerate and non-relatively-bounded perturbation of $\mathcal{H}_0$, as manifested by the drastic change of essential spectrum. That forbids the usual Riesz projection argument plus Rouché's theorem [22, 46].
- The scalar operator relates to $\mathcal{H}_b$ is no longer elliptic, and the system cannot be anti-diagonalized like $L_\pm$ because Q_b is $\mathbb{C}$-valued. This causes great trouble when constructing bifurcation modes as in [7].

To prove the mode stability (2.7), we divide the unstable spectral half-plane $\{\lambda \in \mathbb{C} : \Im\lambda < b(\sigma - s_c)\}$ into two parts depending on the absolute value of spectral point: the low-energy part with $|\lambda| \leq \delta_0$, and the high-energy part with $|\lambda| \geq \delta_0$, where $\delta_0 \ll 1$ independent of s_c. They are treated in [26] and [27] respectively with different strategies, which we discuss below.

2.2.1. *Low-energy spectrum.* In this range, the $(2d+4)$-dim generalized kernel of $\mathcal{H}_0$ bifurcates with eigenvalues distributed on both sides of the essential spectrum. Therefore, we consider a larger region $\{\lambda \in \mathbb{C} : |\lambda| \leq \delta_0 \ll 1, \Im\lambda < 10b\}$ to include all the bifurcated spectral points, and use ODE arguments to construct and prove uniqueness of bifurcated eigenmodes.

Specifically, we decompose into spherical harmonic classes, on which $(\mathcal{H}_b - \lambda)Z = 0$ becomes an ODE system with a matrix Schrödinger operator after suitable conjugation:

$$(\mathbb{H}_{b,\nu} - \lambda)\Phi = 0 \tag{2.10}$$

where $\nu = l + (d-1)/2$ and

$$\mathbb{H}_{b,\nu} = \left(\partial_r^2 - 1 + \frac{b^2r^2}{4} - \frac{\nu^2 - 1/4}{r^2}\right)\sigma_3 + \begin{pmatrix} -ibs_c + W_{1,b} & e^{ibr^2/2}W_{2,b} \\ -e^{-ibr^2/2}\overline{W_{2,b}} & -ibs_c - W_{1,b} \end{pmatrix}.$$

For each class l, we construct 4 admissible solutions to (2.10): $f_{1,b,l}(\lambda), f_{2,b,l}(\lambda)$ regular at 0 and $g_{1,b,l}(\lambda), g_{2,b,l}(\lambda)$ admissible at $+\infty$ (via WKB approximation). Then the detection of eigenmodes becomes the matching problem between these branches. Based on that, we brief main ideas of the three parts of proof:

(1) Existence of bifurcated eigenmodes: matching asymptotics. Motivated by similar construction in the spectral problem related to Type II blowup [10, 9], we will set up an asymptotic expansion ansatz with the coefficient depending nonlinearly on λ, and solve for the residual term and λ by constructing families of interior and exterior solutions, and matching asymptotics at $x_* \sim |\log b|$. This is done in [26, Lemma 6.1, Proposition 6.2]. In particular, we verify that both bifurcated eigenvalues are stable.

(2) Uniqueness of bifurcated eigenmodes in low spherical classes: Jost function argument. To detect the matching of interior and exterior branches for equation (2.10), we define the *Jost function* as

$$F_b(\lambda) := \det \begin{pmatrix} \mathcal{W}[f_{1,b,l}(\lambda), g_{1,b,l}(\lambda)] & \mathcal{W}[f_{1,b,l}(\lambda), g_{2,b,l}(\lambda)] \\ \mathcal{W}[f_{2,b,l}(\lambda), g_{1,b,l}(\lambda)] & \mathcal{W}[f_{2,b,l}(\lambda), g_{2,b,l}(\lambda)] \end{pmatrix},$$

where $\mathcal{W}[f, g] = f \cdot g' - f' \cdot g$ is the Wronskian of two fundamental solutions. Then $F_b(\lambda)$ is analytic w.r.t. λ and continuous w.r.t. b for

$$(b, \lambda) \in [0, b_0] \times \{|\lambda| \leq \delta_l, \Im\lambda \leq 10b\}$$

for some $\delta_l \ll 1$ only depends on l. The Jost function characterizes the spectrum in the following sense:

- λ is eigenvalue $\Leftrightarrow F_b(\lambda) = 0$.
- If $F_b(\lambda) = 0$, the vanishing order of $F_b(\lambda)$ is the algebraic multiplicity of λ, namely

$$\min_n \left\{ \partial_\lambda F_b(\lambda)\big|_{\lambda=\lambda_0} \neq 0 \right\} = \dim \cup_{\mathrm{k}\geq 1} \left(\ker(\mathcal{H}_\mathrm{b} - \lambda)^\mathrm{k} \right). \tag{2.11}$$

Thereafter, the uniqueness of eigenvalue turns into continuity w.r.t. b of the number of zeros (counting multiplicity) of analytic functions $\{F_b\}_{0\leq b\ll 1}$ in $\{|\lambda| \leq \delta_l, \Im\lambda \leq 10b\}$. This is proven in [26, Lemma 7.4] using elementary complex analysis, which particularly avoids contour integration argument due to the degeneracy of regions.

We remark that the application of Jost function is classical for self-adjoint operators (see [45, Chap. XI, 8.E]), while for non-self-adjoint operators, its property as an indicator of generalized eigenspace was claimed and exploited in the pioneering works by Buslaev and Perelman [4, 41]. In our case, we provide a direct and elementary proof of (2.11) as [26, Lemma 7.2].

(3) Non-existence of bifurcated eigenmode in high spherical classes: almost free asymptotics. For high spherical classes $l \gg 1$, we show the potential $-(\nu^2 - 1)/r^2$ from spherical Laplacian will dominate the potential $W_{1,b}$, $W_{2,b}$ in the operator, leading to interior and exterior admissible solutions asymptotically free and mismatching. We employ the Turán type estimates of modified Bessel functions [2] in the interior region ([26, Lemma 3.3 (4)]) and construct new WKB approximate solutions ([26, Section 4.2]) in the exterior region.

2.2.2. *High-energy spectrum.* Our goal is to prove non-existence of eigenvalue in the high-energy spectrum $\{\lambda \in \mathbb{C} : |\lambda| \geq \delta_0, \Im\lambda < b(\sigma - s_c)\}$. The ODE method for low-energy spectrum seems less effective due to the large range of λ and the lack of knowledge of $\sigma(\mathcal{H}_0)$ to bifurcate from in $d \geq 2$.

The overall strategy we applied is called *linear Liouville argument*, originated from Martel-Merle [29, 28] on gKdV soliton stability. We consider eigenfunction as stationary solution of the linear evolution, so as to apply modulation argument and nonlinear dynamical control to prove its rigidity. The controlling laws are energy and Virial identities for the linearized flow, which were also used in the proof of Theorem 1.1. We stress that the coercivity of Virial commutator was proven with numerical help in [34, 18, 49] for $1 \le d \le 10$.

To give some flavor of the linear Liouville argument, we sketch the proof of mode stability of $\mathcal{H}_0$ (2.8) with $1 \le d \le 10$ [27, Theorem 1.1], and comment on the adaption to $\mathcal{H}_b$.

Suppose $Z_0 \in (L^2)^2$ solves $\mathcal{H}_0 Z_0 = \lambda Z_0$ with $\lambda \neq 0$. Then we have

$$Z_0 \perp \cup_{k\ge 1} \ker(\mathcal{H}_0^*)^k, \tag{2.12}$$

Consider its linear evolution

$$i\partial_t Z + \mathcal{H}_0 Z = 0, \quad Z\Big|_{t=0} = Z_0, \quad \Rightarrow \quad Z(t) = e^{i\lambda t} Z_0. \tag{2.13}$$

We will show (2.12)-(2.13) imply $Z_0 \equiv \vec{0}$. By a standard anti-diagonalization of $\mathcal{H}_0$, we can reformulate (2.12)-(2.13) as

$$u_0 \perp Q, |x|^2 Q, xQ, \quad w_0 \perp \Lambda_0 Q, L_+^{-1}(|x|^2 Q), \nabla Q. \tag{2.14}$$

$$\left| \begin{array}{l} \partial_t u = L_- w, \\ \partial_t w = -L_+ u, \end{array} \right. \qquad \left| \begin{array}{l} u = e^{i\lambda t} u_0, \\ w = e^{i\lambda t} w_0, \end{array} \right. \tag{2.15}$$

where $L_\pm = -\Delta + 1 - W_1 \mp W_2$, and our aim becomes $u_0 = w_0 = 0$.

For the linear evolution (2.15), compute the energy identity

$$\begin{aligned} E(t) &:= (L_+ u, u)_{L^2} + (L_- w, w)_{L^2} = e^{-2\Im\lambda t}\left[(L_+ u_0, u_0)_{L^2} + (L_- w_0, w_0)_{L^2}\right] \\ \partial_t E(t) &= = -2\Re(L_+ u, L_- w) + 2\Re(L_- w, L_+ u) = 0, \end{aligned}$$

and the Virial identity

$$\begin{aligned} I(t) &:= \Re \int_{\mathbb{R}^d} x \cdot \left(-\nabla u(t) \cdot \overline{w(t)} + \nabla w(t) \cdot \overline{u(t)} \right) dx \\ &= -2\Re(\Lambda_0 u, w)_{L^2} = -2e^{-2\Im\lambda t}\Re(\Lambda_0 u_0, w_0)_{L^2} \\ \partial_t I(t) &= -2\Re(\Lambda_0 L_- w, w)_{L^2} - 2\Re(\Lambda_0 u, -L_+ u)_{L^2} \\ &= ([L_-, \Lambda_0]w, w)_{L^2} + ([L_+, \Lambda_0]u, u)_{L^2} \\ &= e^{-2\Im\lambda t}\left[([L_-, \Lambda_0]w_0, w_0)_{L^2} + ([L_+, \Lambda_0]u_0, u_0)_{L^2}\right]. \end{aligned}$$

Thereafter, it is easy to observe that the coercivity of quadratic forms in $E(t)$ and $\partial_t I(t)$ under the corresponding orthogonal conditions (2.14) would imply $u_0 = w_0 = 0$ for $\Im\lambda \neq 0$ and $\Im\lambda = 0$ cases respectively. However, as proven in [34, 18, 49], the coercivity of Virial commutator holds true under a different orthogonal condition[3]

$$u_0 \perp Q, \Lambda_0 Q, xQ, \quad w_0 \perp \Lambda_0 Q, \Lambda_0^2 Q, \nabla Q. \tag{2.16}$$

To resolve this, we naturally apply modulation argument to go back to the original conditions (2.14) as in [34].

As for the adaptation to $\mathcal{H}_b$ case, the $b\Lambda_0$ term brings the Virial commutator into the time-derivative of energy, so that we only consider (weighted and truncated) energy to apply the same coercivities.

Lastly, it is possible to formulate the whole argument in a time-independent way, but it would greatly cost the clarity of the Virial identity, the modulation argument, and the analysis for $\mathcal{H}_b$.

[3] It is not true under (2.14) at least for $d = 1$ ([30, Section 4.2.3–4.2.4]).

2.3. **Linear stability.** The main result of linear stability can be stated as the following Strichartz estimate

$$\left\| e^{b(\sigma-s_c)t} e^{\int_0^t e^{i(t-s)\mathcal{H}_b}} P_{\text{ess}} F(s) ds \right\|_{L_t^{q_2'} \dot{W}_x^{\sigma,p_2'}} \lesssim \left\| e^{b(\sigma-s_c)t} P_{\text{ess}} F \right\|_{L_t^{q_1} \dot{W}_x^{\sigma,p_1}} \tag{2.17}$$

where $0 < \sigma - s_c \ll 1$, p_i, q_i satisfy the same requirement (2.5) as in (2.4), and $P_{\text{ess}} = 1 - P_{\text{disc}}$ is the Riesz projection onto essential spectrum of $\mathcal{H}_b\big|_{\dot{H}^\sigma}$. Notice that if we decompose $\mathcal{H}_b$ as[4]

$$\mathcal{H}_b = \mathring{\mathcal{H}}_b + V, \quad \mathring{\mathcal{H}}_b = \begin{pmatrix} \Delta_b - 1 & \\ & -\Delta_{-b} + 1 \end{pmatrix} - ibs_c,$$

then $e^{it\mathring{\mathcal{H}}_b}$ satisfies (2.17) without P_{ess} from the free Strichartz estimate (2.4). Hence our goal is to generalize the free case by adding the non-self-adjoint potential V.

The core strategy is Beceanu's approach [3] which originally serves for deriving Strichartz estimate for matrix Schrödinger operator with non-self-adjoint potential. Surprisingly, this abstract framework is robust enough to be adapted for our self-similar flow in non-radial setting. We mention that Strichartz estimates were derived by Donninger in the pioneering work [13] to control the flow around self-similar blowup of nonlinear wave equations in the renormalized light cone under radial symmetry.

We now sketch Beceanu's framework adapted to our problem.

Step 1. Reduction to invertibility of a space-time operator. Consider the linear evolution $i\partial_t Z + \mathcal{H}_b Z = F$ with $\mathcal{H}_b = \mathring{\mathcal{H}}_b + V$. Treating VZ as source term and applying Duhamel's formula yield

$$Z(t) = e^{it\mathring{\mathcal{H}}_b} Z_0 - i\int_0^t e^{i(t-s)\mathring{\mathcal{H}}_b}(F - VZ)ds = U\big(\delta_{t=0} \otimes Z_0 - iF + iVZ\big) \tag{2.18}$$

where

$$U = \int_0^t e^{i(t-s)\mathring{\mathcal{H}}_b} ds.$$

Decompose $V = V_1 V_2$, then we have

$$V_2 Z = V_2 U\big(\delta_{t=0} \otimes Z_0 - iF\big) + (iV_2 U V_1) \circ (V_2 Z). \tag{2.19}$$

Suppose

$$I - iV_2 U V_1 \text{ is invertible in } L_t^2\big(\mathbb{R}, e^{b(\sigma-s_c)t}\mathcal{L}(\dot{H}^\sigma)\big), \tag{Cond-1}$$

then we can invert this operator in (2.19), and plug in (2.18) to obtain

$$Z = \left[U + iUV_1(I - iV_2UV_1)^{-1}V_2 U\right](\delta_{t=0} \otimes Z - iF)$$

and the Strichartz estimate (2.17) follows the free one (2.4) for U and boundedness of V_1, V_2[5], and $(I - iV_2UV_1)^{-1}$. To sum up, we have reduced (2.17) to (Cond-1).

Step 2. Reduction to uniform invertibility of Birman-Schwinger operators: via convolution structure and abstract Wiener's theorem. Observe the convolution structure of the space-time operator:

$$I - iV_2UV_1 = (\delta_{t=0} \otimes I_{\dot{H}^\sigma} - iV_2 e^{i(\cdot)\mathring{\mathcal{H}}_b} \chi_{\cdot \geq 0} V_1) *_t,$$

[4] We note that the notation in this note is slightly different from [25]: the $\mathcal{H}_b, \mathring{\mathcal{H}}_b$ here correspond to $\mathcal{H}, \mathcal{H}_b + ibs_c$ in [25].

[5] One can take the Strichartz space as $L_t^2(\mathbb{R}, e^{b(\sigma-s_c)t}\dot{W}_x^{\sigma,2+})$ thanks to the self-similar dispersion, so that we only need V_1, V_2 to be bounded in $\dot{W}_x^{\sigma,2+} \to \dot{H}^\sigma$ and $\dot{H}^\sigma \to \dot{W}_x^{\sigma,2-}$.

we can reformulate (Cond-1) as

$$\delta_{t=0}\otimes I_{\dot H^\sigma} - iV_2 e^{i(\cdot)\mathring{\mathcal{H}}_b}\chi_{\cdot\geq 0}V_1 \text{ is invertible}$$
$$\text{in the Banach algebra } L^1_t(\mathbb{R}, e^{b(\sigma-s_c)t}\mathcal{L}(\dot H^\sigma)). \quad \text{(Cond-2)}$$

From the *abstract Wiener's theorem* [3, Theorem 2.3], the invertibility in such Banach algebra can be reduced to uniform invertibility of its Fourier transform plus some compactness conditions. More specifically, we compute

$$\begin{aligned}&\mathscr{F}_{t\to\lambda}(\delta_{t=0}I_{\dot H^\sigma} - iV_2 e^{it\mathring{\mathcal{H}}_b}\chi_{t\geq 0}V_1)(\lambda)\\ &\qquad = I - iV_2\int_0^\infty e^{it\mathring{\mathcal{H}}_b}e^{-it\lambda}dtV_1 := I + V_2\mathcal{S}_b^-(\lambda)V_1,\end{aligned}$$

where we can further write the resolvent of $\mathring{\mathcal{H}}_b$ using the resolvents of $-\Delta_b$ from (2.6) as $\mathcal{S}_b^-(\lambda) = \mathrm{diag}\{-R_b^+(-1-z), R_{-b}^-(-1+z)\}$. In conclusion, we have reduced (Cond-2) to

$$I + V_2\mathcal{S}_b^-(\lambda)V_1 \text{ is invertible in } \dot H^\sigma, \quad \forall\, \Im\lambda \leq b(\sigma - s_c), \quad \text{(Cond-3)}$$

plus some compactness conditions (see [25, Section 5.1] for more detailed discussion). Lastly, via the resolvent identity [25, (4.17)]

$$(I + V_2\mathcal{S}_b^-(\lambda)V_1)^{-1} = I - V_2 S_{b,V}^-(\lambda)V_1,$$

where $S_{b,V}^-(\lambda)$ is the resolvent of $\mathcal{H}_b$, (Cond-3) can be finally reduced to

$$\sup_{\Im\lambda\leq b(\sigma-s_c)} \|S_{b,V}^-(\lambda)\|_{\mathcal{L}(\dot W^{\sigma,2-}\to \dot W^{\sigma,2+})} < \infty. \quad \text{(Cond-4)}$$

Step 3. Removing discrete spectrum. Clearly, (Cond-4) is not true due to the existence of unstable eigenmodes. To remedy that, we consider instead the projected linear evolution

$$i\partial_t Z + (\mathcal{H}_b P_{\mathrm{ess}} + i\mu P_{\mathrm{disc}})Z = F,$$

with $\mu > b(\sigma - s_c)$, which artificially adds sufficient decay to the eigen directions. Rewriting $\tilde{\mathcal{H}}_b := \mathcal{H}_b P_{\mathrm{ess}} + i\mu P_{\mathrm{disc}} =: \mathcal{H}_b - ibs_c + \tilde V$, we can decompose the nonlocal operator $\tilde V$ and verify (Cond-4) and compactness conditions with respect to $\mathcal{H}_b$. This requires delicate and lengthy analysis of the representation, boundedness and compactness regarding $P_{\mathrm{ess}}, P_{\mathrm{disc}}$ and $\mathcal{H}_b$, recorded in [25, Section 2-4].

Lastly, we conclude the discussion with two remarks.

(1) Not a perturbative argument. Unlike the mode stability analysis which strongly depends on $b(s_c) \ll 1$ for perturbative analysis, the derivation of Strichartz estimate works for any $s_c \in (0,1)$ and Q_b satisfying Conjecture 1.2.

(2) Comparison with Strichartz estimate for $e^{it(\Delta+V)}$. In comparison with the traditional Strichartz estimates, the self-similar Strichartz has a wider range of admissible pairs (2.5) (thanks to the self-similar dispersion estimate for $e^{it\Delta_b}$), requires less decay of the potential (as indicated in the Step 2 above), and does not require absence of embedded eigenvalues or resonances.

For the last feature, we exploit that changing the function space $\dot H^\sigma$ shifts the essential spectrum, so that we can avoid the discrete set of eigenmodes and resonances. This explains the assumption $0 < \sigma - s_c \ll 1$. Besides, to identify that discrete set in different spaces, we use the extended resolvent families to treat them uniformly with analytic Fredholm theory (see [25, Section 4.1]). It is worth mentioning that these arguments also lead to *unconditional finite-codimensional mode* *stability*, namely for any $W_{1,b}, W_{2,b}$ with suitable decay and regularity, P_{disc} related to the matrix operator $\mathcal{H}_b$ from (2.3) has finite-dimensional range, which implies *unconditional finite-codimensional asymptotic stability* as mentioned in Remark 1.10.

References

[1] Yakine Bahri, Yvan Martel, and Pierre Raphaël. Self-similar blow-up profiles for slightly supercritical nonlinear Schrödinger equations. *Ann. Henri Poincaré*, 22(5):1701–1749, 2021.

[2] Árpád Baricz. Turán type inequalities for modified Bessel functions. *Bull. Aust. Math. Soc.*, 82(2):254–264, 2010.

[3] Marius Beceanu. New estimates for a time-dependent Schrödinger equation. *Duke Mathematical Journal*, 159(3):417–477, 2011.

[4] V. S. Buslaev and G. S. Perelman. On the stability of solitary waves for nonlinear Schrödinger equations. In *Nonlinear evolution equations*, volume 164 of *Amer. Math. Soc. Transl. Ser. 2*, pages 75–98. American Mathematical Society, Providence, RI, 1995.

[5] Rémi Carles. Nonlinear Schrödinger equations with repulsive harmonic potential and applications. *SIAM J. Math. Anal.*, 35(4):823–843, 2003.

[6] Thierry Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[7] Shu-Ming Chang, Stephen Gustafson, Kenji Nakanishi, and Tai-Peng Tsai. Spectra of linearized operators for NLS solitary waves. *SIAM Journal on Mathematical Analysis*, 39(4):1070–1111, 2008.

[8] Jiajie Chen and Thomas Y Hou. Stable nearly self-similar blowup of the 2d Boussinesq and 3d Euler equations with smooth data I: Analysis, 2022. `arXiv:2210.07191`

[9] Charles Collot, Tej-Eddine Ghoul, Nader Masmoudi, and Van Tien Nguyen. Spectral analysis for singularity formation of the two dimensional Keller-Segel system. *Ann. PDE*, 8(1):Paper No. 5, 74, 2022.

[10] Charles Collot, Frank Merle, and Pierre Raphaël. Strongly anisotropic type II blow up at an isolated point. *J. Amer. Math. Soc.*, 33(2):527–607, 2020.

[11] Charles Collot, Pierre Raphaël, and Jérémie Szeftel. On the stability of type I blow up for the energy super critical heat equation. *Mem. Amer. Math. Soc.*, 260(1255):v+97, 2019.

[12] Wei Dai and Thomas Duyckaerts. Self-similar solutions of focusing semi-linear wave equations in $\mathbb{R}^N$. *J. Evol. Equ.*, 21(4):4703–4750, 2021.

[13] Roland Donninger. Strichartz estimates in similarity coordinates and stable blowup for the critical wave equation. *Duke Math. J.*, 166(9):1627–1683, 2017.

[14] Roland Donninger and Birgit Schörkhuber. Self-similar blowup for the cubic Schrödinger equation, 2024. `arXiv:2406.16597`

[15] Tarek Elgindi. Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$. *Ann. of Math. (2)*, 194(3):647–727, 2021.

[16] Tarek M. Elgindi and Federico Pasqualotto. From instability to singularity formation in incompressible fluids, 2023. `arXiv:2310.19780`

[17] Gadi Fibich, Nir Gavish, and Xiao-Ping Wang. Singular ring solutions of critical and supercritical nonlinear Schrödinger equations. *Phys. D*, 231(1):55–86, 2007.

[18] Gadi Fibich, Frank Merle, and Pierre Raphaël. Proof of a spectral property related to the singularity formation for the L^2 critical nonlinear Schrödinger equation. *Phys. D*, 220(1):1–13, 2006.

[19] G. Guderley. Starke kugelige und zylindrische Verdichtungsstösse in der Nähe des Kugelmittelpunktes bzw. der Zylinderachse. *Luftfahrtforschung*, 19:302–311, 1942.

[20] Justin Holmer, Galina Perelman, and Svetlana Roudenko. A solution to the focusing 3d NLS that blows up on a contracting sphere. *Trans. Amer. Math. Soc.*, 367(6):3847–3872, 2015.

[21] Herbert Koch. Self-similar solutions to super-critical gKdV. *Nonlinearity*, 28(3):545–575, 2015.

[22] J. Krieger and W. Schlag. Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension. *J. Amer. Math. Soc.*, 19(4):815–920, 2006.

[23] M. J. Landman, G. C. Papanicolaou, C. Sulem, P. L. Sulem, and X. P. Wang. Stability of isotropic singularities for the nonlinear Schrödinger equation. *Physica D: Nonlinear Phenomena*, 47(3):393–415, 1991.

[24] B. J. LeMesurier, G. C. Papanicolaou, C. Sulem, and P. L. Sulem. Focusing and multi-focusing solutions of the nonlinear Schrödinger equation. *Physica D: Nonlinear Phenomena*, 31(1):78–102, 1988.

[25] Zexing Li. On stability of self-similar blowup for mass supercritical NLS, 2023. arXiv:2304.02078v2

[26] Zexing Li. Mode stability for self-similar blowup of slightly supercritical NLS: I. Low-energy spectrum, 2025 arXiv:2507.11248

[27] Zexing Li. Mode stability for self-similar blowup of slightly supercritical NLS: II. High-energy spectrum, 2025. arXiv:2507.11259

[28] Yvan Martel and Frank Merle. A Liouville theorem for the critical generalized Korteweg-de Vries equation. *J. Math. Pures Appl. (9)*, 79(4):339–425, 2000.

[29] Yvan Martel and Frank Merle. Asymptotic stability of solitons for subcritical generalized KdV equations. *Arch. Ration. Mech. Anal.*, 157(3):219–254, 2001.

[30] Jeremy L. Marzuola and Gideon Simpson. Spectral analysis for matrix hamiltonian operators. *Nonlinearity*, 24(2):389, 2010.

[31] David W. McLaughlin, George C. Papanicolaou, Catherine Sulem, and Pierre-Louis Sulem. Focusing singularity of the cubic Schrödinger equation. *Physical Review A*, 34(2):1200, 1986.

[32] Frank Merle and Pierre Raphaël. Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation. *Geom. Funct. Anal.*, 13(3):591–642, 2003.

[33] Frank Merle and Pierre Raphaël. On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation. *Invent. Math.*, 156(3):565–672, 2004.

[34] Frank Merle and Pierre Raphaël. The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. *Ann. of Math. (2)*, 161(1):157–222, 2005.

[35] Frank Merle and Pierre Raphaël. Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation. *Comm. Math. Phys.*, 253(3):675–704, 2005.

[36] Frank Merle and Pierre Raphaël. On a sharp lower bound on the blow-up rate for the L^2 critical nonlinear Schrödinger equation. *J. Amer. Math. Soc.*, 19(1):37–90, 2006.

[37] Frank Merle and Pierre Raphaël. Blow up of the critical norm for some radial L^2 super critical nonlinear Schrödinger equations. *Amer. J. Math.*, 130(4):945–978, 2008.

[38] Frank Merle, Pierre Raphaël, Igor Rodnianski, and Jérémie Szeftel. On the implosion of a compressible fluid I: Smooth self-similar inviscid profiles. *Ann. of Math. (2)*, 196(2):567–778, 2022.

[39] Frank Merle, Pierre Raphaël, and Jérémie Szeftel. Stable self-similar blow-up dynamics for slightly L^2 super-critical NLS equations. *Geom. Funct. Anal.*, 20(4):1028–1071, 2010.

[40] Frank Merle, Pierre Raphaël, and Jérémie Szeftel. On collapsing ring blow-up solutions to the mass supercritical nonlinear Schrödinger equation. *Duke Math. J.*, 163(2):369–431, 2014.

[41] Galina Perelman. On the formation of singularities in solutions of the critical nonlinear Schrödinger equation. *Ann. Henri Poincaré*, 2(4):605–673, 2001.

[42] Pierre Raphaël. Stability of the log-log bound for blow up solutions to the critical non linear Schrödinger equation. *Math. Ann.*, 331(3):577–609, 2005.

[43] Pierre Raphaël. Existence and stability of a solution blowing up on a sphere for an L^2-supercritical nonlinear Schrödinger equation. *Duke Math. J.*, 134(2):199–258, 2006.

[44] Pierre Raphaël. On the singularity formation for the nonlinear Schrödinger equation. *Clay Math. Proc*, 17:269–323, 2013.

[45] Michael Reed and Barry Simon. *Methods of modern mathematical physics. III.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1979. Scattering theory.

[46] W. Schlag. Stable manifolds for an orbitally unstable nonlinear Schrödinger equation. *Ann. of Math. (2)*, 169(1):139–227, 2009.

[47] Catherine Sulem and Pierre-Louis Sulem. Focusing nonlinear Schrödinger equation and wave-packet collapse. In *Proceedings of the Second World Congress of Nonlinear Analysts, Part 2 (Athens, 1996)*, volume 30, pages 833–844, 1997.

[48] William C Troy. Zero energy self-similar solutions describing singularity formation in the nonlinear Schrödinger equation in dimension $n = 3$, 2022. arXiv:2210.05645

[49] Kai Yang, Svetlana Roudenko, and Yanxiang Zhao. Blow-up dynamics and spectral property in the L^2-critical nonlinear Schrödinger equation in high dimensions. *Nonlinearity*, 31(9):4354–4392, 2018.

[50] Kai Yang, Svetlana Roudenko, and Yanxiang Zhao. Blow-up dynamics in the mass super-critical NLS equations. *Phys. D*, 396:47–69, 2019.

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