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Seung-Yeal Ha

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Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
http://www.ihes.fr/

Centre de mathématiques Laurent Schwartz
CMLS, CNRS, École polytechnique,
Institut Polytechnique de Paris
F-91128 PALAISEAU CEDEX
https://cmls.ip-paris.fr/

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RECENT PROGRESS ON THE MEAN-FIELD LIMIT OF THE CUCKER-SMALE MODEL FOR FLOCKING

SEUNG-YEAL HA

Abstract. In this paper, we give a brief survey on the state-of-the-art results on the mean-field limit of the Cucker-Smale(CS) model for flocking. The CS model is one of well-studied collective dynamics models. Collective motions of self-propelled particles often appear in our nature. Some collective motions are often described by different types of partial differential equations. We discuss that they fall down to the special cases of the universal nonlinear consensus model at the microscopic level. We also discuss how an interacting particle system with a large size can be effectively approximated by the corresponding mean-field model by the rigorous justification of the mean-field limit. In particular, we focus on the uniform-in-time mean-field limit of the CS model for flocking using the uniform-in-time stability estimate and asymptotic flocking estimates under some framework which guarantees the exponential flocking.

Contents

1. Introduction

Collective behaviors of complex systems often appear in our nature and man-made systems, e.g., aggregation of bacteria and information [28, 47], flocking of birds, drones, mobile sensor and robots [12, 13, 14, 35, 41, 42, 48, 50], synchronization of fireflies and pacemaker cells [1, 8, 34, 43, 44] and swarming of fish [5, 14, 21, 46] etc. See survey articles and books [3, 6, 7, 36, 49] for a crash introduction to collective motions. Among them, we are mainly interested in the flocking behaviors of self-propelled particles. The jargon *flocking* denotes some collective motion in which self-propelled particles move with common velocity via communications between particles. Although they are ubiquitous in nature, mathematical modeling of flocking was first done by the computer scientist, C. Reynolds. He proposed a distributed behavior model in almost half a century ago based on three rules

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such as a long range attraction, intermediate velocity alignment and short range repulsion. Reynolds's seminal work [45] was further continued by a group of statistical physicists led by T. Vicsek and his collaborators in another seminal work [50]. They introduced a planar stochastic discrete-time model for particle's heading angles which describes a relaxation to the average angle based on finite-range interaction rules, and they assume that particles move with a unit speed. Moreover, they showed that asymptotic patterns can be classified into four patterns depending on local mass density and strength of noise by numerical simulations. Motivated by this series of works, Felippe Cucker and Steve Smale introduced a second-order Newton-like particle model in [13]. They replaced finite-range interaction with a weighted sum of relative velocities. Here we call the weight as the communication weight, and it depends on the relative distances. To set up the stage, we begin with a brief description of the Cucker-Smale (CS) model.

Let x_i and v_i be the position and velocity of the i-th CS particle in the Euclidean space $\mathbb{R}^d$. Then, their temporal dynamics is governed by the Cauchy problem for the CS model:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \ i \in [N] := \{1, \cdots, N\}, \\ \dfrac{dv_i}{dt} = \dfrac{\kappa}{N} \displaystyle\sum_{j=1}^{N} \phi(\|x_j - x_i\|)(v_j - v_i), \end{cases} \tag{1.1}$$

where κ is the nonnegative coupling strength, $\| \cdot \|$ is the standard ℓ_2-norm in $\mathbb{R}^d$, and $\phi : \mathbb{R}_+ \to \mathbb{R}_+ =: \{x \in \mathbb{R} : x \geq 0\}$ is a nonnegative communication weight function satisfying boundedness, Lipschitz continuity and monotonicity (see (3.2)). The global existence, clustering flocking dynamics of (1.1) have been extensively investigated from diverse perspectives in a series of works [9, 10, 11, 13, 22, 24, 25, 36, 37].

In this paper, we review the state-of-the-art results on the mean-field limit of the particle CS model (1.1), forcing on the results of asymptotic flocking and uniform-in-time stability of (1.1), and finite-in-time mean-field limit [24] and uniform-in-time mean-field limit [23]. As long as there is no confusion, we will use the abbreviated words such as finite-time, uniform-time instead of finite-in-time, uniform-in-time throughout the paper.

The rest of this paper is organized as follows. In Section 2, we discuss a universality hidden in some collective motions (aggregation, flocking and synchronization) via a nonlinear consensus model. It turns out that 3D Keller-Segel's aggregation, CS flocking and Kuramoto synchronization can be integrated as special cases of the proposed nonlinear consensus model. In Section 3, we review asymptotic flocking for the CS model and uniform-in-time stability. In Section 4, we discuss the uniform-time mean-field limit. Finally Section 5 is devoted to a brief summary of presented results and some remaining issues which were not discussed in this paper.

2. PRELIMINARIES

In this section, we briefly introduce three prototype interacting particle systems (particle models) arising from the study of collective dynamics, and discuss the relations between them and explain the hidden universality behind the curtain.

2.1. **Keller-Segel's aggregation model.** Aggregation denotes a collective phenomenon in which relative positions of particles tend to zero, i.e., formation of a Dirac Delta in position variable, hence it can be understood as a consensus in position variable. This aggregate phenomenon has been extensively studied in mathematical biology community in recent years (see [31, 30, 29]) via coupled partial differential equations.

Let $\rho = \rho(t, x)$ and $c = c(t, x)$ be the local mass densities of the bacteria and chemical substance, respectively. Then, one of Keller-Segel type models [32, 33] is given by the coupled parabolic-elliptic system:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \nabla c) = \sigma \Delta \rho, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ -\Delta c = \rho, \end{cases} \tag{2.1}$$

where σ is the nonnegative diffusion coefficient. On the other hand, we return to the particle description of bacteria aggregation. For this, let $x_i = x_i(t)$ be the position process of the i-th bacterium at time t. Then, in three dimensions, the corresponding interacting particle analogue of (2.1) is given by the stochastic interacting particle system:

$$dx_i = \frac{\kappa}{N} \sum_{j \neq i} \frac{x_j - x_i}{\|x_j - x_i\|^3} dt + \sqrt{2\sigma} dB_i, \quad t > 0, \quad i \in [N]. \tag{2.2}$$

When the stochastic noises are turned off by setting $\sigma = 0$, the system (2.2) becomes the deterministic particle model:

$$\frac{dx_i}{dt} = \frac{\kappa}{N} \sum_{j \neq i} \psi_{ks}(x_j - x_i), \quad i \in [N],$$

where ψ_{ks} is the interaction kernel defined by the following relation:

$$\psi_{ks}(x) = \frac{x}{\|x\|^3}, \quad 0 \neq x \in \mathbb{R}^3.$$

2.2. **Cucker-Smale's flocking model.** Flocking represents a collective phenomenon in which relative velocities of particles vanish asymptotically, whereas their relative positions are uniformly bounded. To be more specific, let $x_i = x_i(t)$ and $v_i = v_i(t)$ be the position and velocity of the i-th CS particle. Then, their dynamics is governed by Newton-like system for (x_i, v_i):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \ i \in [N], \\ \dfrac{dv_i}{dt} = \dfrac{\kappa}{N} \displaystyle\sum_{j=1}^{N} \psi_{cs}(x_j - x_i)(v_j - v_i), \end{cases} \tag{2.3}$$

where κ is the nonnegative coupling strength and ψ_{cs} represent a nonnegative communication weight function. For definiteness, we set

$$\psi_{cs}(x) = \frac{1}{(1 + \|x\|^2)^{\frac{\beta}{2}}}, \quad \beta \geq 0, \quad x \in \mathbb{R}^d.$$

Let $F = F(t, x, v)$ be the one-particle distribution function for the CS ensemble at position x with velocity v at time t. Then, the standard BBGKY hierarchy argument provides the corresponding McKean-Vlasov type model as a formal mean-field limit:

$$\begin{cases} \partial_t F + v \cdot \nabla_x F + \nabla_v \cdot (\mathcal{F}_a(F) F) = 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d, \\ \mathcal{F}_a(F)(t, x, v) = -\kappa \displaystyle\int_{\mathbb{R}^{2d}} \psi_{cs}(x - x_*)(v - v_*) F(t, x_*, v_*) dv_* dx_*. \end{cases} \tag{2.4}$$

From the kinetic CS model (2.4), we use the method of velocity moments and mono-kinetic ansatz as a suitable closure assumption to derive the hydrodynamic CS model. More precisely, let $\rho = \rho(t,x), u = u(t,x)$ be the local mass density and bulk velocity of CS flocking ensemble defined as follows:

$$\rho(t,x) := \int_{\mathbb{R}^d} F(t,x,v)dv, \quad \text{local mass density},$$
$$(\rho u)(t,x) := \int_{\mathbb{R}^d} vF(t,x,v)dv, \quad \text{local momentum density}.$$

Then, the hydrodynamic CS model reads as follows:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) = \kappa \rho \displaystyle\int_{\mathbb{R}^d} \psi(|x-y|)(u(t,y) - u(t,x))\rho(t,y)dy. \end{cases}$$

This system corresponds to the pressureless Euler system with a nonlocal source term. As a special case of the system (2.3), we consider the CS model on the real line ($d = 1$) so that

$$x(t),\ v(t) \in \mathbb{R}, \quad \text{for } t \ge 0.$$

In this case, we can introduce the anti-derivative of ψ_{cs}:

$$\Psi_{cs}(x) := \int_0^x \psi_{cs}(y)dy, \quad \text{i.e.,} \quad \Psi_{cs}'(x) = \psi_{cs}(x), \quad x \in \mathbb{R}.$$

Then, the system (2.3) can be rewritten as

$$\frac{d^2 x_i}{dt^2} = \frac{d}{dt}\Big(\frac{\kappa}{N}\sum_{k=1}^N \Psi_{cs}(x_k - x_i)\Big), \quad i \in [N]. \tag{2.5}$$

We integrate (2.5) with respect to t to get

$$\begin{aligned} \frac{dx_i}{dt} &= v_i^0 - \frac{\kappa}{N}\sum_{k=1}^N \Psi_{cs}(x_k^0 - x_i^0) + \frac{\kappa}{N}\sum_{k=1}^N \Psi_{cs}(x_k - x_i) \\ &=: \nu_i(X^0, V^0) + \frac{\kappa}{N}\sum_{k=1}^N \Psi_{cs}(x_k - x_i), \end{aligned}$$

where

$$x_i^0 := x_i(0), \quad v_i^0 = v_i(0), \quad X^0 := (x_1^0, \ldots, x_N^0), \quad V^0 := (v_1^0, \ldots, v_N^0).$$

2.3. **Kuramoto's synchronization model.** Synchronization represents a collective phenomenon in which weakly coupled limit-cycle oscillators adjust their rhythms due to their mutual interactions. In the sequel, we recall a prototype model for synchronization which was introduced by Yoshiki Kuramoto in [34].

Let $\theta_i = \theta_i(t)$ be the phase of the i-th Kuramoto oscillator with a natural frequency $\nu_i \in \mathbb{R}$. Then, the phase dynamics is governed by the Kuramoto model:

$$\frac{d\theta_i}{dt} = \nu_i + \frac{\kappa}{N}\sum_{j=1}^N \psi_k(\theta_j - \theta_i), \quad t > 0, \quad i \in [N],$$

where ψ_k is the Kuramoto interaction kernel defined by

$$\psi_k(\theta) = \sin\theta, \quad \theta \in \mathbb{R}.$$

Next, we consider the kinetic Kuramoto model which is formulated as a scalar conservation law with a nonlocal flux. Let $F = F(t, \theta, \nu)$ be the one-particle distribution function with phase θ and natural frequency ν at time t. Then, the kinetic Kuramoto model reads as follows:

$$\begin{cases} \partial_t F + \partial_\theta(\omega(F)F) = 0, \quad (t, \theta, \nu) \in \mathbb{R}_+ \times [0, 2\pi] \times \mathbb{R}, \\ \omega(F) = \nu - \kappa \displaystyle\int_0^{2\pi} \int_{\mathbb{R}} \sin(\theta_* - \theta) F(t, \theta_*, \nu_*) d\nu_* d\theta. \end{cases} \tag{2.6}$$

Note that the real value ν can be viewed as a real parameter, hence for a fixed ν, equation (2.6) is a hyperbolic conservation law with a nonlocal flux. Therefore, one can view equation (2.6) as an infinite number of hyperbolic conservation laws.

2.4. **Nonlinear consensus model.** In the previous three subsections, we have introduced three prototype models for aggregation, flocking and synchronization. In what follows, we will show that how they can be viewed as special cases of the generalized nonlinear consensus model.

Consider N interacting particle system on some manifold $\mathcal{M}$ embedded in the Euclidean space $\mathbb{R}^d$, and let $q_i = q_i(t)$ be the generalized position of the i-th particle with interaction kernel $K = K(q)$. Then, we propose a nonlinear consensus model on $\mathcal{M}$ whose continuous dynamics is governed by the following first-order interacting particle system:

$$\frac{dq_i}{dt} = \nu_i + \frac{\kappa}{N} \sum_{j=1}^N K(q_j - q_i), \quad q_i \in \mathcal{M}, \quad i \in [N]. \tag{2.7}$$

Note that the tuple $(q, \mathcal{M}, K)$ takes the following forms for Keller-Segel, Cucker-Smale and Kuramoto models in previous subsections:

$$(q, \mathcal{M}, K) = \begin{cases} (x, \mathbb{R}^3, \psi_{ks}), & \text{the Keller-Segel model with } d = 3, \\ (x, \mathbb{R}, \Psi_{cs}), & \text{the CS model with } d = 1, \\ (\theta, \mathbb{T}^1, \psi_k), & \text{the Kuramoto model with } d = 1, \end{cases}$$

where $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$. From this observation, we can see that there should be some hidden universality between aforementioned three collective dynamics. In fact, the Kuramoto model can be derived from the CS model on $\mathbb{T}^1$ (see [17]).

Next, we consider emergent dynamics of (2.7) for $\mathcal{M} = \mathbb{R}$. Let $q_i = q_i(t)$ be a real-valued quantifiable measure of the i-th agent's opinion level at time t whose dynamics is governed by the system (2.7) with $\kappa = 1$. We assume that the coupling function K satisfies the following set of conditions:

$$\begin{cases} K(-q) = -K(q), \quad (K(q) - K(q_*))(q - q_*) \geq 0, \ \forall \ q, q_* \in \mathbb{R}, \\ \lim\limits_{q \to \infty} K(q) = K^\infty > 0, \quad K'(q) \leq 1, \quad K'(0) = 1, \quad K''(q) < 0, \quad q \in \mathbb{R}_+. \end{cases} \tag{2.8}$$

For explicit examples satisfying (2.8), we can consider the following coupling functions:

$$K(q) = \ \tanh(q) \quad \text{or} \quad \int_0^q \frac{1}{(1 + |q_*|)^\beta} dq_*, \quad \beta > 1.$$

Next, we state *eventual well-ordering principle* based on the relative ordering of natural velocities.

Proposition 2.1. [26] *Let $\mathcal{Q} = \{q_i\}$ be a global solution to the system* (2.7) *and* (2.8) *with the initial datum $\mathcal{Q}^0$. Suppose that for $i, j \in [N]$, the following relation holds:*

$$q_i^0 < q_j^0.$$

Then the following trichotomy holds.

(i) If $\nu_i < \nu_j$, then q_i and q_j will never collide in finite time:

$$|\{t_* \in (0, \infty) \ : \ q_i(t_*) = q_j(t_*)\}| = 0,$$

where $|A|$ is the cardinality of the set A.

(ii) If $\nu_i > \nu_j$, q_i and q_j will collide once in finite time:

$$|\{t_* \in (0, \infty) \ : \ q_i(t_*) = q_j(t_*)\}| = 1.$$

(iii) If $\nu_i = \nu_j$, then the relative distance $|q_i - q_j|$ decays to zero exponentially: for $t \geq 0$,

$$e^{-\kappa t} \leq \frac{q_j(t) - q_i(t)}{q_j(0) - q_i(0)} \leq e^{-\frac{\kappa}{N} K'(|q_i^0 - q_j^0|)t}.$$

Next, we recall clustering dynamics for (2.7) - (2.8) in terms of system parameters and initial data.

Theorem 2.1. [26] *Suppose that the natural velocity ν_i is well-ordered:*

$$\nu_1 < \nu_2 < \cdots < \nu_N, \quad \sum_{i=1}^N \nu_i = 0$$

and let $\mathcal{Q} = \{q_i\}$ be a solution to (2.7) - (2.8) *with the initial datum $\mathcal{Q}^0 = \{q_i^0\}$. Then, $\mathcal{Q}$ is completely segregated, i.e.,*

$$\limsup_{t \to +\infty} q_1(t) = -\infty, \quad \liminf_{t \to +\infty} q_N(t) = \infty, \quad \liminf_{t \to +\infty} |q_{i+1}(t) - q_i(t)| = \infty, \quad i \in [N-1],$$

if and only if the coupling strength κ is sufficiently small such that

$$\kappa < \min \left\{ \frac{N\nu_1}{(N-1)K^\infty}, \ \frac{N(\nu_2 - \nu_1)}{2K^\infty}, \dots, \frac{N(\nu_N - \nu_{N-1})}{2K^\infty}, \ \frac{N\nu_N}{(N-1)K^\infty} \right\}.$$

Remark 2.1. (Complete cluster predictability): For each i, system parameters $\{\nu_i\}$, κ and K^∞ determine whether

$$\lim_{t \to \infty} |q_i(t) - q_{i+1}(t)| = \infty \ \text{ or } \ \lim_{t \to \infty} |q_i(t) - q_{i+1}(t)| < \infty, \quad i \in [N-1].$$

The proof for this can be found in [20].

Before we move on to the next section, we briefly summarize the content of this section. As discussed in this section, we show that Keller-Segel's aggregation, Cucker-Smale's flocking and Kuramoto's synchronization can be integrated into a nonlinear consensus with distinct coupling functions. In this sense, there are some hidden universality in some collective motions. If we have enough information on one of aforementioned collective motions, then we can look for similar phenomenon or property in other collective motions. In the following two subsections, we review the recent progress on the uniform-time stability and mean-field limits for the CS model.

3. Uniform-time stability of the CS model

In this section, we discuss emergent flocking dynamics of the CS model and how the uniform-in-time mean field limit can be made using the flocking estimates and finite-in-time mean-field limit.

Recall the Cauchy problem for the CS model:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \ i \in [N], \\ \dfrac{dv_i}{dt} = \dfrac{\kappa}{N} \displaystyle\sum_{j=1}^{N} \phi(\|x_j - x_i\|)(v_j - v_i), \\ (x_i, v_i)\Big|_{t=0+} = (x_i^0, v_i^0), \end{cases} \tag{3.1}$$

where the communication weight function ϕ satisfies nonnegativity, boundedness, Lipschitz continuity and monotonicity conditions:

$$0 \le \phi \le 1, \quad [\phi]_{\mathrm{Lip}} < \infty, \quad (\phi(r_2) - \phi(r_1))(r_2 - r_1) \le 0, \quad \forall\, r_1, r_2 \ge 0. \tag{3.2}$$

and $\|\cdot\|$ is the standard ℓ^2-norm in $\mathbb{R}^d$. For notational simplicity, we set

$$X := (x_1, \cdots, x_N) \quad \text{and} \quad V := (v_1, \cdots, v_N).$$

3.1. **Asymptotic flocking dynamics.** In this subsection, we discuss the flocking dynamics of the Cauchy problem (3.1) for the CS model.

3.1.1. *Preparatory materials.* First, we recall the concept of asymptotic flocking in the following definition.

Definition 3.1. Let (X, V) be a global smooth solution to (3.1). Then, the configuration (X, V) exhibits asymptotic flocking if and only if the following conditions hold.

(1) (Velocity alignment): The relative velocities tend to zero asymptotically:

$$\lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0, \quad \forall\, i, j \in [N].$$

(2) (Spatial cohesion): The relative positions are uniformly bounded in time:

$$\sup_{0\le t<\infty} \|x_i(t) - x_j(t)\| < \infty, \quad \forall\, i, j \in [N].$$

Next, we list basic properties of the CS model as follows.

Proposition 3.1. [13, 24, 25] *Let (X, V) be a global smooth solution to* (3.1)*. Then, the following assertions hold.*

(1) The CS model is Galilean invariant in the sense that it is invariant under the Galilean transformation: for some $c \in \mathbb{R}^d$

$$(x_i, v_i) \ \mapsto \ (x_i + ct, v_i + c).$$

(2) The total momentum is a constant of motion:

$$\frac{d}{dt}\sum_{i=1}^{N} v_i(t) = 0, \quad \forall\, t > 0.$$

(3) The total energy is non-increasing over time:

$$\frac{d}{dt}\sum_{i=1}^{N}\|v_i\|^2 = -\frac{\kappa}{N}\sum_{i,j}\phi(\|x_j - x_i\|)\|v_j - v_i\|^2 \le 0, \quad \forall\ t > 0.$$

Remark 3.1. If we set averages and fluctuations around them:

$$x_c := \frac{1}{N}\sum_{i=1}^{N} x_i, \quad v_c := \frac{1}{N}\sum_{i=1}^{N} v_i, \quad \hat{x}_i := x_i - x_c, \quad \hat{v}_i := v_i - v_c,$$

then it is easy to see that

$$\sum_{i=1}^{N}\hat{x}_i(t) = 0, \qquad \sum_{i=1}^{N}\hat{v}_i(t) = 0, \quad \forall\ t \ge 0,$$

and dynamics for averages and fluctuations are completely decoupled:

$$\begin{cases} \dfrac{dx_c}{dt} = v_c, \quad t > 0, \\[2ex] \dfrac{dv_c}{dt} = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} \dfrac{d\hat{x}_i}{dt} = \hat{v}_i, \quad t > 0,\ i \in [N], \\[2ex] \dfrac{d\hat{v}_i}{dt} = \dfrac{\kappa}{N}\displaystyle\sum_{j=1}^{N}\phi(\|\hat{x}_j - \hat{x}_i\|)(\hat{v}_j - \hat{v}_i). \end{cases}$$

3.1.2. *Nonlinear functional approach.* In this part, without loss of generality, we may assume that

$$\sum_{i=1}^{N} x_i(t) = 0, \quad \sum_{i=1}^{N} v_i(t) = 0, \quad t \ge 0. \tag{3.3}$$

In what follows, we present a nonlinear functional approach leading to exponential flocking of (3.1). For this, we introduce mixed norms as follows:

$$\|X\|_{2,\infty} := \max_{1\le i\le N}\|x_i\|, \quad \|V\|_{2,\infty} := \max_{1\le i\le N}\|v_i\|.$$

Then, $\|X\|_{2,\infty}$ and $\|V\|_{2,\infty}$ are Lipschitz continuous functions in t, hence they are almost everywhere differentiable, and they satisfy the system of dissipative differential inequality (SDDI):

$$\begin{cases} \Big|\dfrac{d}{dt}\|X\|_{2,\infty}\Big| \le \|V\|_{2,\infty}, \quad \text{a.e. } t \in (0,\infty), \\[2ex] \dfrac{d}{dt}\|V\|_{2,\infty} \le -\kappa\phi(\sqrt{2}\|X\|_{2,\infty})\|V\|_{2,\infty}. \end{cases}$$

Now, we introduce nonlinear functionals $\mathcal{L}^{\pm}(t) = \mathcal{L}^{\pm}(X(t), V(t))$:

$$\mathcal{L}^{\pm}(t) := \|V(t)\|_{2,\infty} \pm \frac{\kappa}{\sqrt{2}}\int_0^{\sqrt{2}\|X(t)\|_{2,\infty}}\phi(r)dr, \quad t \ge 0.$$

After some tedious calculations, one can check that for $t \ge 0$,

$$\mathcal{L}^{\pm}(t) \le \mathcal{L}^{\pm}(0), \quad \text{or equivalently} \quad \|V(t)\|_{2,\infty} + \frac{\kappa}{\sqrt{2}}\Big|\int_{\sqrt{2}\|X^0\|_{2,\infty}}^{\sqrt{2}\|X(t)\|_{2,\infty}}\phi(r)dr\Big| \le \|V^0\|_{2,\infty}.$$

where (X^0, V^0) are initial data. This stability estimate yields the following flocking estimate.

Theorem 3.1. [4, 24] *Let* (X, V) *be a global smooth solution to* (3.1) - (3.3) *with initial data* (X^0, V^0) *satisfying*

$$\|X^0\|_{2,\infty} > 0, \quad \|V^0\|_{2,\infty} < \frac{\kappa}{\sqrt{2}} \int_{\sqrt{2}\|X^0\|_{2,\infty}}^{\infty} \phi(r)dr.$$

Then, there exists a positive constant x_M *satisfying*

$$\|X(t)\|_{2,\infty} \leq x_M, \quad \|V(t)\|_{2,\infty} \leq \|V^0\|_{2,\infty} e^{-\kappa\phi(\sqrt{2}x_M)t}, \quad t \geq 0,$$

where x_M *is determined by the following implicit relation:*

$$\|V^0\|_{2,\infty} = \frac{\kappa}{\sqrt{2}} \int_{\sqrt{2}\|X^0\|_{2,\infty}}^{x_M} \phi(r)dr.$$

Remark 3.2. We can also use the same nonlinear functional approach for diameter functionals:

$$\mathcal{D}(X) := \max_{i,j\in[N]} \|x_i - x_j\|, \quad \mathcal{D}(V) := \max_{i,j\in[N]} \|v_i - v_j\|.$$

By the same argument, one can show that these functionals satisfy the system of dissipative differential inequality (SDDI):

$$\begin{cases} \Big|\frac{d}{dt}\mathcal{D}(X)\Big| \leq \mathcal{D}(V), \quad \text{a.e. } t > 0, \\ \frac{d}{dt}\mathcal{D}(V) \leq -\kappa\phi(\mathcal{D}(X))\mathcal{D}(V). \end{cases}$$

Suppose that the coupling strength and initial data satisfy

$$\mathcal{D}(V^0) < \kappa \int_{\mathcal{D}(X^0)}^{\infty} \phi(r)dr,$$

and let $\{(x_i, v_i)\}$ be a global solution to (3.1) - (3.3). Then, there exists a positive constant $\mathcal{D}^\infty$ such that

$$\sup_{0\leq t<\infty} \mathcal{D}(X(t)) \leq \mathcal{D}^\infty \qquad \mathcal{D}(V(t)) \leq \mathcal{D}(V^0)e^{-\kappa\phi(\mathcal{D}^\infty)t}, \quad t > 0,$$

where $\mathcal{D}^\infty$ is uniquely determined by the relation:

$$\int_{\mathcal{D}(X^0)}^{\mathcal{D}^\infty} \phi(s)ds = \frac{\mathcal{D}(V^0)}{\kappa}.$$

We refer to Motsch and Tadmor's paper [37] for a detailed proof.

3.2. **Uniform-in-time stability.** In this subsection, we review the uniform-in-time stability estimate for the Cauchy problem (3.1). We recall the concept of uniform-in-time stability in ℓ^2-norm with respect to initial data as follows.

Definition 3.2. [23] The CS model (3.1) is uniformly ℓ_2-stable with respect to initial data if for any two set of solutions (X, V) and $(\bar{X}, \bar{V})$ corresponding to initial data (X^0, V^0) and $(\bar{X}^0, \bar{V}^0)$, respectively, there exists a nonnegative constant G independent of t such that

$$\sup_{0\leq t<\infty} \Big(\|X(t) - \bar{X}(t)\| + \|V(t) - \bar{V}(t)\|\Big) \leq G\Big(\|X^0 - \bar{X}^0\| + \|V^0 - \bar{V}^0\|\Big),$$

where $\|X - \bar{X}\|$ and $\|V - \bar{V}\|$ are defined as follows:

$$\|X - \bar{X}\| := \Big(\sum_{i=1}^N \|x_i - \bar{x}_i\|^2\Big)^{1/2}, \quad \|V - \bar{V}\| = \Big(\sum_{i=1}^N \|v_i - \bar{v}_i\|^2\Big)^{1/2}.$$

Let (X,V) and $(\bar{X},\bar{V})$ be two global solutions exhibiting flocking dynamics in Remark 3.2: there exists positive constants $\mathcal{D}^\infty$ and $\bar{\mathcal{D}}^\infty$ such that

$$\begin{aligned}
&\sup_{0\le t<\infty}\max_{1\le i,j\le N}\|x_i(t)-x_j(t)\|\le \mathcal{D}^\infty, \qquad \sup_{0\le t<\infty}\max_{1\le i,j\le N}\|\bar{x}_i(t)-\bar{x}_j(t)\|\le \bar{\mathcal{D}}^\infty,\\
&\max_{1\le i,j\le N}\|v_i(t)-v_j(t)\|\le \mathcal{D}(V^0)e^{-\kappa\phi(\mathcal{D}^\infty)t}, \qquad \max_{1\le i,j\le N}\|\bar{v}_i(t)-\bar{v}_j(t)\|\le \mathcal{D}(\bar{V}^0)e^{-\kappa\phi(\bar{\mathcal{D}}^\infty)t},
\end{aligned} \tag{3.4}$$

Then, one can derive a system of two differential inequalities for $\|X-\bar{X}\|$ and $\|V-\bar{V}\|$: there exists positive constants α and C which may depend on ϕ and initial data, but independent of t such that

$$\begin{cases}
\dfrac{d}{dt}\|X-\bar{X}\|\le \|V-\bar{V}\|, \quad \text{a.e., } t>0,\\[2ex]
\dfrac{d}{dt}\|V-\bar{V}\|\le -\kappa\alpha\|V-\bar{V}\|+\kappa Ce^{-\kappa\alpha t}\|X-\bar{X}\|.
\end{cases} \tag{3.5}$$

The first differential inequality in (3.5) is rather obvious. Thus, let us check why the second differential inequality holds. To see this, we consider one-dimensional case $d=1$. Otherwise, we can apply a similar argument for each component.

Consider the equations for v_i and $\bar{v}_i$:

$$\begin{cases}
\dfrac{dv_i}{dt}=\dfrac{\kappa}{N}\displaystyle\sum_{k=1}^N\phi(|x_k-x_i|)(v_k-v_i),\\
\dfrac{d\bar{v}_i}{dt}=\dfrac{\kappa}{N}\displaystyle\sum_{k=1}^N\phi(|\bar{x}_k-\bar{x}_i|)(\bar{v}_k-\bar{v}_i).
\end{cases}$$

These yield

$$\begin{aligned}
\frac{d}{dt}(v_i-\bar{v}_i)&=\frac{\kappa}{N}\sum_{k=1}^N\Big(\phi(|x_k-x_i|)-\phi(|\bar{x}_k-\bar{x}_i|)\Big)(v_k-v_i)\\
&\quad+\frac{\kappa}{N}\sum_{k=1}^N\phi(|\bar{x}_k-\bar{x}_i|)\Big((v_k-\bar{v}_k)-(v_i-\bar{v}_i)\Big).
\end{aligned} \tag{3.6}$$

We multiply $2(v_i-\bar{v}_i)$ to (3.6) and sum up the resulting relations over all $i\in[N]$ to get

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^N|v_i-\bar{v}_i|^2&=\frac{2\kappa}{N}\sum_{i,k}^N\Big(\phi(|x_k-x_i|)-\phi(|\bar{x}_k-\bar{x}_i|)\Big)(v_k-v_i)(v_i-\bar{v}_i)\\
&\quad+\frac{2\kappa}{N}\sum_{i,k}^N\phi(|\bar{x}_k-\bar{x}_i|)\Big((v_k-\bar{v}_k)-(v_i-\bar{v}_i)\Big)(v_i-\bar{v}_i)\\
&=:\mathcal{I}_{11}+\mathcal{I}_{12}.
\end{aligned} \tag{3.7}$$

Below, we estimate the terms $\mathcal{I}_{1i},\ i=1,2$ one by one.

• (Estimate of $\mathcal{I}_{11}$): We use the Lipschitz continuity of ϕ to see

$$\begin{aligned}
&\Big|\phi(|x_k - x_i|) - \phi(|\bar{x}_k - \bar{x}_i|)\Big| \\
&\qquad \le [\phi]_{\text{Lip}}\Big||x_k - x_i| - |\bar{x}_k - \bar{x}_i|\Big| \le [\phi]_{\text{Lip}}\Big|(x_k - \bar{x}_k) - (x_i - \bar{x}_i)\Big| \\
&\qquad \le [\phi]_{\text{Lip}}\Big(|x_k - \bar{x}_k| + |x_i - \bar{x}_i|\Big).
\end{aligned}$$

This yields

$$\begin{aligned}
|\mathcal{I}_{11}| &\le \frac{2\kappa}{N}\sum_{i,k=1}^{N}\Big|\phi(|x_k - x_i|) - \phi(|\bar{x}_k - \bar{x}_i|)\Big||v_k - v_i|\cdot|v_i - \bar{v}_i| \\
&\le \frac{2\kappa[\phi]_{\text{Lip}}}{N}\sum_{i,k=1}^{N}|v_k - v_i|\Big(|x_k - \bar{x}_k||v_i - \bar{v}_i| + |x_i - \bar{x}_i||v_i - \bar{v}_i|\Big) \\
&\le \frac{2\kappa[\phi]_{\text{Lip}}\mathcal{D}(V^0)e^{-\kappa\phi(\mathcal{D}^\infty)t}}{N}\sum_{i,k=1}^{N}\Big(|x_k - \bar{x}_k||v_i - \bar{v}_i| + |x_i - \bar{x}_i||v_i - \bar{v}_i|\Big) \\
&\le 4\kappa[\phi]_{\text{Lip}}\mathcal{D}(V^0)e^{-\kappa\phi(\mathcal{D}^\infty)t}\|X - \bar{X}\|\|V - \bar{V}\|,
\end{aligned} \tag{3.8}$$

where we used flocking estimates (3.4) and the Cauchy-Schwarz inequality in the last inequality.

• (Estimate of $\mathcal{I}_{12}$): We use index exchange map $(i,k) \leftrightarrow (k,i)$ to find

$$\begin{aligned}
\mathcal{I}_{12} &= \frac{2\kappa}{N}\sum_{i,k}^{N}\phi(|\bar{x}_k - \bar{x}_i|)\Big((v_k - \bar{v}_k) - (v_i - \bar{v}_i)\Big)(v_i - \bar{v}_i) \\
&= -\frac{2\kappa}{N}\sum_{i,k}^{N}\phi(|\bar{x}_k - \bar{x}_i|)\Big((v_k - \bar{v}_k) - (v_i - \bar{v}_i)\Big)(v_k - \bar{v}_k) \\
&= -\frac{\kappa}{N}\sum_{i,k}^{N}\phi(|\bar{x}_k - \bar{x}_i|)\Big|(v_k - \bar{v}_k) - (v_i - \bar{v}_i)\Big|^2 \\
&\le -\frac{\kappa\phi(\bar{\mathcal{D}}^\infty)}{N}\sum_{i,k=1}^{N}\Big|(v_k - \bar{v}_k) - (v_i - \bar{v}_i)\Big|^2 \\
&= -2\kappa\phi(\bar{\mathcal{D}}^\infty)\|V - \bar{V}\|^2,
\end{aligned} \tag{3.9}$$

where we used the zero sum condition (3.3) to find

$$\sum_{k=1}^{N}(v_k - \bar{v}_k) = \sum_{k=1}^{N}v_k - \sum_{k=1}^{N}\bar{v}_k = 0$$

and

$$\sum_{i,k=1}^{N}\Big|(v_k-\bar v_k)-(v_i-\bar v_i)\Big|^2$$
$$=\sum_{i,k=1}^{N}\Big(|v_k-\bar v_k|^2+|v_i-\bar v_i|^2-2(v_k-\bar v_k)(v_i-\bar v_i)\Big)=2N\|V-\bar V\|^2.$$

In (3.7), we combine (3.8) and (3.9) to find

$$\frac{d}{dt}\|V-\bar V\|^2$$
$$\le 4\kappa[\phi]_{\mathrm{Lip}}\mathcal{D}(V^0)e^{-\kappa\phi(\mathcal{D}^\infty)t}\|X-\bar X\|\|V-\bar V\|-2\kappa\phi(\bar{\mathcal{D}}^\infty)\|V-\bar V\|^2.$$

This yields $(3.5)_2$:

$$\frac{d}{dt}\|V-\bar V\|\le 2\kappa[\phi]_{\mathrm{Lip}}\mathcal{D}(V^0)e^{-\kappa\phi(\mathcal{D}^\infty)t}\|X-\bar X\|-\kappa\phi(\bar{\mathcal{D}}^\infty)\|V-\bar V\|$$
$$\le 2\kappa[\phi]_{\mathrm{Lip}}\mathcal{D}(V^0)e^{-\kappa\min\{\phi(\mathcal{D}^\infty),\phi(\bar{\mathcal{D}}^\infty)\}t}\|X-\bar X\|-\kappa\min\{\phi(\bar{\mathcal{D}}^\infty),\phi(\mathcal{D}^\infty)\}\|V-\bar V\|,$$

with

$$C=2\kappa[\phi]_{\mathrm{Lip}}\mathcal{D}(V^0),\quad \alpha=\min\Big\{\phi(\mathcal{D}^\infty),\ \phi(\bar{\mathcal{D}}^\infty)\Big\}.$$

Lemma 3.1. [23] *Suppose that two nonnegative Lipschitz functions $\mathcal{X}$ and $\mathcal{V}$ satisfy the coupled differential inequalities:*

$$\Big|\frac{d\mathcal{X}}{dt}\Big|\le\mathcal{V},\qquad \frac{d\mathcal{V}}{dt}\le-\alpha\mathcal{V}+\gamma e^{-\alpha t}\mathcal{X},\quad a.e.\ t>0,$$

where α and γ are positive constants. Then, $\mathcal{X}$ and $\mathcal{V}$ satisfy the uniform bound and decay estimates:

$$\mathcal{X}(t)\le\frac{2M}{\alpha}(\mathcal{X}^0+\mathcal{V}^0),\qquad \mathcal{V}(t)\le M(\mathcal{X}^0+\mathcal{V}^0)e^{-\frac{\alpha t}{2}},\quad t\ge0,$$

where M is given by

$$M:=\max\Big\{1,\frac{2\gamma}{\alpha e}\Big\}+\frac{8\gamma}{\alpha^3e^3}.$$

Now, we apply Lemma 3.1 for (3.5) to derive the uniform-in-time stability in ℓ_2-norm. This can be summarized as follows.

Theorem 3.2. [23] *Suppose that system parameter and initial data satisfy*

$$\sum_{i=1}^{N}v_i^0=\sum_{i=1}^{N}\bar v_i^0=0,\quad \kappa>\max\Big\{\frac{\mathcal{D}(V^0)}{\int_{\mathcal{D}(X^0)}^\infty\phi(s)ds},\ \frac{\mathcal{D}(\bar V^0)}{\int_{\mathcal{D}(\bar X^0)}^\infty\phi(s)ds}\Big\},$$

and let (X,V) and $(\bar X,\bar V)$ be two global smooth solutions to (3.1) - (3.2) *with initial data (X^0,V^0) and $(\bar X^0,\bar V^0)$, respectively. Then, uniform-in-time ℓ_2 stability holds in the sense of Definition 3.2 i.e., there exists a positive constant G independent of t such that*

$$\sup_{0\le t<\infty}\Big(\|X(t)-\bar X(t)\|+\|V(t)-\bar V(t)\|\Big)\le G\Big(\|X^0-\bar X^0\|+\|V^0-\bar V^0\|\Big).$$

4. Uniform-time mean-field limit of the CS model

In this section, we discuss a measure-theoretic formulation of the kinetic CS model and uniform-in-time mean-field limit of the particle CS model (3.1).

4.1. **Preparatory materials.** In this subsection, we recall some measure theoretical results to be used in later part of this section. First, we recall the kinetic CS model:

$$\begin{cases} \partial_t F + v \cdot \nabla_x F + \nabla_v \cdot (\mathcal{F}_a(F)F) = 0, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d, \\ \mathcal{F}_a(F)(t,x,v) = -\kappa \displaystyle\int_{\mathbb{R}^{2d}} \phi(\|x - x_*\|)(v - v_*)F(t,x_*,v_*)dv_* dx_*. \end{cases} \tag{4.1}$$

Next, we briefly summarize a measure-theoretic framework and local-in-time stability result following the work in [24]. In the sequel, we recall the concept of measure-valued solution to (4.1).

Let $\mathcal{P}(\mathbb{R}^{2d})$ be the set of all probability measures on the phase space $\mathbb{R}^{2d}$, which can be understood as normalized nonnegative bounded linear functionals on $\mathcal{C}_0(\mathbb{R}^{2d})$. For a probability measure $\mu \in \mathcal{P}(\mathbb{R}^{2d})$, we use a standard duality relation:

$$\langle \mu, \varphi \rangle := \int_{\mathbb{R}^{2d}} \varphi(x,v)\mu(dx,dv), \quad \varphi \in C_0(\mathbb{R}^{2d}).$$

We first recall a concept of a measure-valued solution to (4.1) as follows.

Definition 4.1. [24] For $T \in [0,\infty)$, $\mu \in L^\infty([0,T);\mathcal{P}(\mathbb{R}^{2d}))$ is a measure-valued solution to (4.1) with initial datum $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d})$ if the following three relations hold:

(1) Total mass is normalized: $\langle \mu_t, 1 \rangle = 1$.

(2) μ is weakly continuous in t:

$$\langle \mu_t, \varphi \rangle \text{ is continuous in } t, \quad \forall\, \varphi \in \mathcal{C}_0^1((0,T) \times \mathbb{R}^{2d}).$$

(3) μ satisfies (4.1) in a weak sense: for any $\varphi \in \mathcal{C}_0^1([0,T) \times \mathbb{R}^{2d})$,

$$\langle \mu_t, \varphi(t,\cdot) \rangle - \langle \mu_0, \varphi(0,\cdot) \rangle = \int_0^t \Big\langle \mu_s, \partial_s \varphi + v \cdot \nabla_x \varphi + \mathcal{F}_a \cdot \nabla_v \varphi \Big\rangle ds.$$

Remark 4.1. Let (X,V) be a global smooth solution to (3.1). Then, the empirical measure

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \otimes \delta_{v_i}$$

is a measure-valued solution in the sense of Definition 4.1.

Definition 4.2. [40, 51]

(1) For $p \in [1,\infty]$, let $\mathcal{P}_p(\mathbb{R}^{2d})$ be a collection of all probability measures with finite p-th moment: for some $z_0 \in \mathbb{R}^{2d}$

$$\langle \mu, \|z - z_0\|^p \rangle < +\infty.$$

Then, p-Wasserstein distance $W_p(\mu,\nu)$ is defined for any $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^{2d})$ as

$$W_p(\mu,\nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \Big(\int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \|z - z^*\|^p d\gamma(z,z^*) \Big)^{1/p},$$

where $\Gamma(\mu,\nu)$ denotes the collection of all probability measures on $\mathbb{R}^{2d} \times \mathbb{R}^{2d}$ with marginals μ and ν.

(2) If $\lim_{p\to\infty} W_p$ exists, then we define W_∞ metric as this limit of $\lim_{p\to\infty} W_p$.

(3) For any $T \in (0,\infty]$, the kinetic equation (4.1) is the mean-field limit from the particle system (3.1) in the time-interval $[0,T)$, if for every solution μ_t of the kinetic equation (4.1) with initial data μ_0, the following condition holds: for some $p \in [1,\infty]$ and $t \in [0,T)$,

$$\lim_{N\to+\infty} W_p(\mu_0^N, \mu_0) = 0 \quad \Longleftrightarrow \quad \lim_{N\to+\infty} W_p(\mu_t^N, \mu_t) = 0,$$

where μ_t^N is a measure valued solution of the particle system (3.1) with initial data μ_0^N.

4.2. **The mean-field limit.** In this subsection, we recall existence of the mean-field limit from (3.1) to (4.1), and local and uniform-time stability estimates.

Theorem 4.1. [24] *The following assertions hold.*

(1) (Local-in-time stability): Let μ and ν be two measure-valued solutions to (4.1) *with initial measures μ_0 and ν_0 with compact supports and finite second moments:*

$$\int_{\mathbb{R}^{2d}} (1+|v|^2)\mu_0(dx,dv) < \infty, \quad \int_{\mathbb{R}^{2d}} (1+|v|^2)\nu_0(dx,dv) < \infty.$$

Then, there exists a nonnegative constant $C = C(T,\mu_0,\nu_0)$ such that

$$W_1(\mu_t,\nu_t) \le C W_1(\mu_0,\nu_0), \quad t \in [0,T). \tag{4.2}$$

(2) Suppose that the initial probability measure $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d})$ is compactly supported, and has finite first two velocity moments:

$$\langle \mu_0, 1\rangle = 1, \quad \langle \mu_0, |v|^2\rangle < +\infty.$$

Then, there exists the unique measure-valued solution $\mu \in L^\infty\big([0,T);\mathcal{P}(\mathbb{R}^{2d})\big)$ to (4.1) *with initial datum μ_0.*

As a direct application of the uniform stability estimate in Theorem 3.2, we obtain the uniform-time mean-field limit in the whole time interval and stability of measure-valued solution in 2-Wasserstein metric. Let $d_x(t)$ and $d_v(t)$ be the diameters of compact support in spatial and velocity variables of μ_t respectively at time t, i.e.

$$d_x(t) := \max_{x,y\in supp_x \mu_t} \|x-y\|, \quad d_v(t) := \max_{v,w \in supp_v \mu_t} \|v-w\|.$$

Theorem 4.2. (Uniform-time mean-field limit) [23] *Suppose that the initial probability measure $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d})$ has a compact support and the first two finite moments:*

$$\int_{\mathbb{R}^{2d}} (1+|v|^2)\mu_0(dx,dv) < \infty, \quad \kappa > \frac{d_v(0)}{\int_{d_x(0)}^\infty \phi(s)ds}. \tag{4.3}$$

Then, the following assertions hold. For $p,q \in [1,\infty]$,

(1) There exists the unique measure-valued solution $\mu_t \in L^\infty\big([0,\infty);\mathcal{P}(\mathbb{R}^{2d})\big)$ to (4.1) *with initial datum μ_0.*

(2) Moreover, if ν_t is the another measure-valued solution to (4.1) *with another initial measure ν_0 with compact support and finite moments* (4.3)*, then there exists nonnegative constant G independent of t such that*

$$W_2(\mu_t,\nu_t) \le G W_2(\mu_0,\nu_0), \quad t \in [0,\infty). \tag{4.4}$$

Remark 4.2. In the sequel, we provide several comments related to the result of Theorem 4.2.

(1) For the construction of measure-valued solution, we have employed so called particle-in-cell method [40] using empirical measure and a priori local-in-time and uniform-in-time stability estimates (see (4.2) and (4.4)) in a suitable p-Wasserstein metric to propagate initial Cauchy approximation in time to construct an approximate sequence of approximate measure-valued solutions via empirical measures. The same methodology has been applied to other collective models, e.g., augmented Kuramoto model [19], thermodynamic Kuramoto model [16], thermodynamic CS model [18], manifold CS model [2].

(2) Recently, S.-Y. Ha, X. Wang and X. Xue removed the compact support assumption of initial measure to construct a measure-value solution to the kinetic CS model (4.1) with non-compact support assumption in [27, 52]. They used the particle-in-cell method and the infinite CS model which corresponds to the infinite counterpart of the CS model. For the flocking estimate of the infinite CS model, we also refer to [53].

(3) Natalini and Paul [39] studied the local-in-time mean-field limit for a generalized CS type model by the coupling method based on the propagation of chaos. Recently, the author and his collaborators applied the coupling method to the derive finite-time mean-field limit of the Motsch-Tadmor model which corresponds to the CS model with a normalized communication weights in [15]. As far as the author knows, the extension of these works to the uniform-time is still an open problem.

(4) Although we restrict our discussions on a regular and bounded communication weights, there is also a parallel results for the CS model with a moderately singular communication weight function by Mucha and Peszek [38].

5. Conclusion

In this note, we have discussed a hidden universality among Keller-Segel model for aggregation, the CS model for flocking and the Kuramoto model for synchronization. By reducing the second-order CS model on the real line to the first-order one, we can unify aforementioned collective dynamics models in the context of a nonlinear consensus model. We also discussed recent results on the uniform-in-time stability of the CS model and as a direct application of this uniform stability, we derived a uniform-in-time mean-field limit which is valid over the whole time interval. Of course, there are several interesting issues that we did not cover in this work. To name a few, Mucha-Peszek's work on the construction of measure-valued solution to the CS model relies on the compact support assumption of the initial measure. Thus, removing this compact support assumption as in the CS model with regular and bounded communication weight will be an interesting problem. In addition to this, extension of the coupling method by Natalini-Paul on the mean-field limit of the CS model to the uniform-in-time counterpart will also be an interesting problem.

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(Seung-Yeal Ha)
Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 08826, Republic of Korea
Email address: syha@snu.ac.kr