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DISPROVING THE DEIFT CONJECTURE: THE LOSS OF ALMOST PERIODICITY

ANDREIA CHAPOUTO, ROWAN KILLIP, AND MONICA VIŞAN

Abstract. In this note we present a result from [CKV24] related to a conjecture of Deift from 2008, who posited that almost periodic initial data leads to almost periodic solutions to the Korteweg-de Vries equation (KdV). We show that this is not always the case. Building on the new observation that the conjecture fails for the Airy equation, we construct almost periodic initial data whose KdV evolution remains bounded, but loses almost periodicity at a later time. This text is based on a Laurent Schwartz seminar given by the first author in November 2024, which in turn is based on joint work with Rowan Killip and Monica Vişan.

1. Introduction

We consider the Korteweg-de Vries equation (KdV) on $\mathbb{R}$:

$$\partial_t u + \partial_x^3 u = 3\partial_x(u^2), \tag{KdV}$$

where u is a real-valued function. This equation was proposed by Boussinesq [B1872] and Korteweg-de Vries [KdV1895] in the 19th century as a model for the propagation of long waves in shallow-water. Since then, it has been applied to describe waves in a variety of physical contexts in fluid dynamics, plasma physics, and acoustics; see [C95]. From an analysis viewpoint, (KdV) is a *dispersive equation*, that is, the velocity of a wave packet depends on its frequency. It is also a *completely integrable system* with a Lax pair formulation (see (2.2)), an infinite number of conservation laws, and an inverse scattering transform.

The literature on the well-posedness problem for (KdV) (existence, uniqueness, and stability) is extensive; see [KV19] for an overview. The family of L^2-based Sobolev spaces has proven to be the canonical choice for work on this problem: $H^s(\mathbb{R})$ when working on the line and $H^s(\mathbb{R}/\mathbb{Z})$ in the circle setting. The circle case is often conflated with that of periodic initial data on the line; indeed, they would be identical but for one key question: Must a solution with periodic initial data remain periodic (with the same period)?

This question leads us to the central theme of [CKV24]: Are structural properties of the initial data, such as periodicity, preserved by the flow? Part of our answer is given by Theorem 1.4 below, which shows that bounded[1] solutions are unique. Thus, we may infer that all bounded solutions with periodic data are themselves periodic, by simply constructing one such periodic solution. The other part of our answer addresses a parallel question of Deift: Is almost-periodicity preserved by the (KdV) flow?

Definition 1.1 (Almost-periodicity, Bohr [B47]). A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *almost periodic* and we write $f \in \mathrm{AP}(\mathbb{R})$, if f is continuous and for every $\varepsilon > 0$ there is an $L_\varepsilon > 0$ so that every interval of length L_ε in $\mathbb{R}$ contains at least one ε *almost period*, that is, a number $\ell \in \mathbb{R}$ such that

$$\big\| f(x+\ell) - f(x) \big\|_{L^\infty(\mathbb{R})} < \varepsilon.$$

[1]Given an open interval $I \subset \mathbb{R}$, we say that $u : I \to L^\infty(\mathbb{R})$ is a *bounded solution* to (KdV) if it is bounded, weak-$*$ continuous, and it solves (KdV) distributionally.

Any continuous *periodic* function is also almost periodic. The simplest example of an almost periodic function that is not periodic is the sum of two continuous periodic functions with incommensurate periods, for example, $\cos(x)+\cos(\sqrt{2}x)$. More generally, the sum of any two almost periodic functions is almost periodic. As we will see in Section 3, this breaks down without the continuity hypothesis imposed by Bohr.

The notion of almost periodicity applies equally well to functions of time (rather than space) and to functions taking values in Banach spaces. For a dynamical system to be almost periodic in time is evidently a very strong form of recurrence. Nevertheless, just such recurrence was observed in numerical simulations of (KdV) by Zabusky–Kruskal [ZK65]. This empirical discovery was made rigorous for smooth solutions by McKean-Trubowitz [MT76], and subsequently extended to $L^2(\mathbb{R}/\mathbb{Z})$ and $H^{-1}(\mathbb{R}/\mathbb{Z})$ in [B93] and [KT06], respectively.

In [D08, D17], Deift conjectured that such almost periodicity in time extends to solutions with initial data that is almost periodic (in space):

Conjecture 1 (Deift [D08, D17])**.** *If u_0 is almost periodic, then the resulting solution to* (KdV) *is almost periodic in spacetime.*

There are various results that support Conjecture 1 for subclasses of almost periodic data [E94, DG16, BDGL18, EVY19, LY20], which exploit the completely integrable structure of (KdV). For example, in [E94], Egorova gave a positive answer for *limit periodic* data, while [BDGL18] considers quasi-periodic data

$$u_0(x)=\sum_{\vec{n}\in\mathbb{Z}^d}\widehat{u_0}(\vec{n})e^{i(\vec{\alpha}\cdot\vec{n})x}$$

with exponentially decaying Fourier coefficients $|\widehat{u_0}(\vec{n})|\leq e^{-c|\vec{n}|}$ and a (quantitative) Diophantine assumption on $\vec{\alpha}$ (analogous to (3.7)).

However, these remarkable results provide only a partial answer to Conjecture 1. In fact, in [DLVY21] the authors outlined an extensive program to build a counter-example to the Deift conjecture. We present an alternative construction, which is guided by the following observation.

Proposition 1.2. *The almost periodicity conjecture fails for the Airy equation*

$$\partial_t u+\partial_x^3 u=0. \tag{Airy}$$

Specifically, there exist $u_0\in \mathrm{AP}(\mathbb{R})$ and $t_0\in\mathbb{R}$ such that the solution $u\in L^\infty(\mathbb{R}\times\mathbb{R})$ to (Airy) *with $u(0)=u_0$ satisfies $u(t_0)\notin \mathrm{AP}(\mathbb{R})$.*

Building on Proposition 1.2, we disprove Conjecture 1:

Theorem 1.3. *There is a bounded solution $u:[-T,T]\times\mathbb{R}\to\mathbb{R}$ of* (KdV) *with almost periodic initial data for which $x\mapsto u(t_0,x)$ is* not *almost periodic at some time $t_0\in[-T,T]$.*

The fundamental building blocks to show Theorem 1.3 are (1) the explicit example constructed to prove Proposition 1.2 and (2) a new *nonlinear smoothing* effect for (KdV). Nonlinear smoothing refers to the phenomenon that the difference between a solution to a nonlinear dispersive PDE and the underlying linear evolution is smoother than each of them individually. In particular, despite the loss of almost periodicity for the linear part of the solution, its nonlinear part does remain almost-periodic. In this way, we see that the failure of Conjecture 1 is a *linear phenomenon* and should occur for many linear and nonlinear PDEs, independently of whether they are completely integrable.

To guarantee that the counter-example in Theorem 1.3 is the only possible evolution arising from u_0, we show uniqueness of bounded solutions, without further assumptions of regularity or spatial asymptotics.

Theorem 1.4. *Let u_1 and u_2 be bounded solutions to* (KdV)*, defined on some open interval $I \subset \mathbb{R}$. If $u_1(t_0) = u_2(t_0)$ in $L^\infty(\mathbb{R})$ for some $t_0 \in I$, then $u_1 \equiv u_2$ on I.*

Theorem 1.4 can be understood as an *unconditional uniqueness* result, as coined by Kato [K95], since it is independent of how solutions are constructed and does not impose auxiliary assumptions. This appears to be the first unconditional uniqueness result for (KdV) where only boundeness is assumed. See also [BIT11, BDGL18, LY20], which obtain uniqueness *within* the classes of periodic or almost-periodic solutions.

The proof of Theorem 1.4 relies on a Gronwall argument based on a new 'distance' function between two (KdV) solutions. This is motivated by the integrable structure of the equation and depends on the diagonal Green's function associated to the Lax structure of (KdV).

We continue this note by presenting a sketch of the uniqueness argument in Section 2. Lastly, in Section 3, we summarize the steps needed to construct the counter-example to the Deift conjecture in Theorem 1.3, including the proof of Proposition 1.2 for (Airy) and the new nonlinear smoothing effect for quasi-periodic functions (Theorem 3.3).

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2. Uniqueness of bounded solutions

We first present the proof of Theorem 1.4 on uniqueness of bounded solutions. The classical strategy to show uniqueness of solutions in $H^s(\mathbb{R})$-spaces is based on a Gronwall argument applied to a suitable distance function between solutions. In the (KdV) setting, the traditional argument is the following: Given two classical solutions u_1, u_2 to (KdV), we have

$$\begin{aligned}\partial_t(u_1-u_2)^2 &= 2\big(\partial_x^3(u_1-u_2)+6(u_1\partial_x u_1-u_2\partial_x u_2)\big)(u_1-u_2)\\ &= 3(u_1-u_2)^2\partial_x(u_1+u_2)-\partial_x^3(u_1-u_2)^2+3\partial_x(\partial_x u_1-\partial_x u_2)^2\\ &\quad+3\partial_x\big[(u_1+u_2)(u_1-u_2)^2\big].\end{aligned} \tag{2.1}$$

Integrating (2.1), we find that

$$\begin{aligned}\partial_t\int(u_1-u_2)^2dx &= 3\int(u_1-u_2)^2\partial_x(u_1+u_2)dx\\ &\le 3\big[\|\partial_x u_1\|_{L^\infty}+\|\partial_x u_2\|_{L^\infty}\big]\int(u_1-u_2)^2dx.\end{aligned}$$

An application of Gronwall's inequality then yields uniqueness of solutions to (KdV) that satisfy $u \in C_tL_x^2$ and $\partial_x u \in L_t^1L_x^\infty$.

Theorem 1.4 goes beyond this classical argument in two ways: it does not require spatial decay of u, nor does it require strong spatial regularity. To achieve this strengthening, we exploit the Lax pair formulation of (KdV) (see (2.2)) to construct a new distance function.

2.1. Diagonal Green's function.

The proof of Theorem 1.4 also relies on a Gronwall-type argument, but based on a more effective notion of 'distance' between solutions, guided by the complete integrability of (KdV). While complete integrability brings with it a multitude of conservation laws for individual solutions, it does not provide general tools for controlling differences of solutions. In truth, there are several competing notions of integrability for infinite dimensional Hamiltonian systems; see [Z91]. The central figure in our development will be the *Lax pair* formulation: Given u sufficiently smooth, consider the pair of operators P and L defined via

$$P := -4\partial_x^3 + 2(\partial_x u + u\partial_x) \quad \text{and} \quad L := -\partial_x^2 + u.$$

Then, we have that

$$u(t) \text{ solves (KdV)} \iff \frac{d}{dt}L(t) = [P(t), L(t)], \tag{2.2}$$

where $[\cdot,\cdot]$ denotes the commutator. The pair (P, L) is called a Lax pair for (KdV). Although a wellposedness theory is necessary to make this rigorous, the equivalence (2.2) shows that $L(t)$ remains in the same conjugacy class for all times.

For $u \in L^\infty(\mathbb{R})$ and $\kappa^2 \geq 4\|u\|_{L^\infty}$, the resolvent

$$R(\kappa) = (L + \kappa^2)^{-1} = (-\partial_x^2 + u + \kappa^2)^{-1}$$

admits a continuous integral kernel $G(x, y)$, which is known as the Green's function. The fundamental object in the proof of Theorem 1.4 is the *diagonal Green's function* $g(x) = G(x, x)$. For $u \in L^\infty(\mathbb{R})$ and $\kappa^2 \geq 4\|u\|_{L^\infty}$, g can be written as the absolutely convergent series

$$\begin{aligned} g(x) = g(x; u, \kappa) := \frac{1}{2\kappa} + \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(2\kappa)^{\ell+1}} \int_{\mathbb{R}^\ell} & e^{-\kappa|x-x_1|} u(x_1) \\ & \times \Big(\prod_{j=1}^{\ell-1} e^{-\kappa|x_j - x_{j+1}|} u(x_j) \Big) e^{-\kappa|x_\ell - x|} u(x_\ell) dx_1 \cdots dx_\ell \end{aligned} \tag{2.3}$$

and satisfies

$$g \in L^\infty(\mathbb{R}), \quad \partial_x g \in L^\infty(\mathbb{R}), \quad \text{and} \quad \partial_x^2 g \in L^\infty(\mathbb{R}). \tag{2.4}$$

While (2.3) shows us how to construct g given u, there is a remarkable identity that allows us to recover u from g, namely,

$$u = \partial_x \Big[\frac{\partial_x g}{2g} \Big] + \Big[\frac{\partial_x g}{2g} \Big]^2 + \frac{1}{4g^2} - \kappa^2; \tag{2.5}$$

see, for example, [KMV20, Lemma 2.14].

It is striking that both g and $\frac{1}{g}$ provide conserved densities for (KdV). Indeed, their dynamical equations[2] take the following conservation law form:

$$\begin{aligned} \frac{d}{dt} g &= \partial_x \Big[-\partial_x^2 g + \frac{3(\partial_x g)^2}{2g} - \frac{3}{2g} - 6\kappa^2 g \Big], \\ \frac{d}{dt} \frac{1}{2g} &= \partial_x \Big[-\partial_x^2 \Big(\frac{1}{2g} \Big) + \frac{3(\partial_x g)^2}{4g^3} + \frac{1}{4g^3} - \frac{3\kappa^2}{g} \Big]. \end{aligned} \tag{2.6}$$

These equations are fundamental for carrying out the Gronwall argument described below.

2.2. **Distance function and Gronwall argument.** Given two bounded solutions u_1, u_2 to (KdV), let $g_1(t, x) = g_1(x; \kappa, u_1(t))$ and $g_2(t, x) = g_2(x; \kappa, u_2(t))$ be their Green's functions as defined in (2.3). The proof of Theorem 1.4 comprises two steps:

- **Step 1:** show uniqueness of the Green's functions g_j by running a Gronwall argument for the following quantity:

$$\int_{\mathbb{R}} \frac{(g_1 - g_2)^2}{2 g_1 g_2} \psi_R dx,$$

where $\psi_R(x) := \operatorname{sech}(\frac{x}{R})$ with $R \geq 1$.

- **Step 2:** use the equation in (2.5) to conclude that $u_1 \equiv u_2$ from the fact that $g_1 \equiv g_2$, as established in Step 1.

[2]The results in [KMV20, KV19] consider Schwartz solutions to (KdV). In [CKV24], we extend these to bounded solutions via a mollification procedure; this leads us to consider a forced version of (KdV). See Remark 2.1 for further details.

Starting with **Step 1**, using (2.6) and considerable rearrangement, we obtain the identity

$$
\begin{aligned}
&\left[\int_{\mathbb{R}} \frac{(g_1-g_2)^2}{2g_1g_2}\psi_R\,dx\right](t_0)\\
&\qquad= \int_0^{t_0}\int_{\mathbb{R}} \frac{(g_1-g_2)^2}{2g_1g_2}\psi_R\Big\{-\frac{\partial_x^3\psi_R}{2\psi_R}+\frac{3}{2}\frac{\partial_x^2\psi_R}{\psi_R}A_2-\frac{3}{2}\frac{\partial_x\psi_R}{\psi_R}A_1+\frac{3}{2}A_0\Big\}\,dx\,dt\\
&\qquad\quad+\frac{3}{2}\int_0^{t_0}\int_{\mathbb{R}}\frac{g_1-g_2}{g_1g_2}\partial_x\psi_R B\,dx\,dt, \qquad (2.7)
\end{aligned}
$$

where the quantities A_0, A_1, A_2, B are given by

$$
\begin{aligned}
A_0 &= -\Big(\frac{\partial_x g_1}{g_1^3}+\frac{\partial_x g_2}{g_2^3}\Big)-\Big(2\kappa^2-\frac{\partial_x g_1\cdot\partial_x g_2}{2g_1g_2}\Big)\Big(\frac{\partial_x g_1}{g_1}+\frac{\partial_x g_2}{g_2}\Big)\\
&\quad -2\Big(\frac{u_1\partial_x g_2}{g_2}+\frac{u_2\partial_x g_1}{g_1}\Big)+\frac{1}{2g_1g_2}\Big(\frac{\partial_x g_1}{g_2}+\frac{\partial_x g_2}{g_1}\Big),\\
A_1 &= \frac{5}{2}\Big(\frac{\partial_x g_1}{g_1}+\frac{\partial_x g_2}{g_2}\Big)^2-7\frac{\partial_x g_1\cdot\partial_x g_2}{g_1g_2}-12\kappa^2+\frac{3}{2}\frac{(\partial_x g_1-\partial_x g_2)^2}{g_1g_2}-2(u_1+u_2)\\
&\quad +\Big(\frac{1}{2}-2\partial_x g_1\cdot\partial_x g_2\Big)\Big(\frac{1}{g_1}-\frac{1}{g_2}\Big)^2+2u_1\frac{g_1}{g_2}+2u_2\frac{g_2}{g_1}+2\kappa^2\Big(\frac{g_1}{g_2}+\frac{g_2}{g_1}\Big),\\
A_2 &= -\Big(\frac{\partial_x g_1}{g_2}+\frac{\partial_x g_2}{g_1}\Big)+2\Big(\frac{\partial_x g_1}{g_1}+\frac{\partial_x g_2}{g_2}\Big),\\
B &= 2u_1g_1-2u_2g_2+\frac{(\partial_x g_1)^2}{2g_1}-\frac{(\partial_x g_2)^2}{2g_2}.
\end{aligned}
$$

Despite the complicated structure of the terms above, for $u_1, u_2 \in L^\infty(\mathbb{R}\times\mathbb{R})$, (2.4) show that these are also in $L^\infty(\mathbb{R}\times\mathbb{R})$! In fact, we have the following estimates

$$
\begin{aligned}
&\|A_0\|_{L^\infty_{t,x}} \lesssim \kappa^3, \qquad \Big\|\tfrac{\partial_x\psi_R}{\psi_R}A_1\Big\|_{L^\infty_{t,x}} \lesssim \tfrac{1}{R}\kappa^2, \qquad \Big\|\tfrac{\partial_x^2\psi_R}{\psi_R}A_2\Big\|_{L^\infty_{t,x}} \lesssim \tfrac{1}{R^2}\kappa,\\
&\|B\|_{L^\infty_{t,x}} \lesssim \kappa^4, \qquad \Big\|\tfrac{\partial_x^3\psi_R}{\psi_R}\Big\|_{L^\infty_{t,x}} \lesssim \tfrac{1}{R^3},
\end{aligned}
\qquad (2.8)
$$

where the implicit constants depend only on $\|\partial_x^\ell g_j\|_{L^\infty_{t,x}}$ and $\|u_j\|_{L^\infty_{t,x}}$, $\ell = 0,1,2$ and $j = 1,2$. The main apparent obstruction for a Gronwall argument is the last contribution in (2.7). This can be overcome by using the decay in R of $|\partial_x\psi_R| \lesssim R^{-1}\psi_R$ and Cauchy's inequality, to obtain

$$
\begin{aligned}
\frac{3}{2}\int_0^{t_0}\int_{\mathbb{R}}\frac{g_1-g_2}{g_1g_2}\partial_x\psi_R B dxdt &\lesssim \frac{1}{\varepsilon R}\int_0^{t_0}\int_{\mathbb{R}}\frac{(g_1-g_2)^2}{2g_1g_2}\psi_R dxdt+\frac{\varepsilon}{R}\|B\|_{L^\infty_{t,x}}\int_0^{t_0}\int_{\mathbb{R}}\psi_R dxdt\\
&\lesssim \frac{1}{\varepsilon R}\int_0^{t_0}\int_{\mathbb{R}}\frac{(g_1-g_2)^2}{2g_1g_2}\psi_R dxdt+\varepsilon T\|B\|_{L^\infty_{t,x}}, \qquad (2.9)
\end{aligned}
$$

for $0<\varepsilon\le 1$ to be chosen sufficiently small.

Combining (2.7), (2.8), and (2.9), we obtain

$$
\int_{\mathbb{R}}\frac{(g_1-g_2)^2}{2g_1g_2}(t_0)\psi_R\,dx \lesssim \Big[\frac{1}{R^3}+\frac{\kappa}{R^2}+\frac{\kappa^2}{R}+\kappa^3+\frac{1}{\varepsilon R}\Big]\int_0^{t_0}\int_{\mathbb{R}}\frac{(g_1-g_2)^2}{2g_1g_2}\psi_R\,dx\,dt+\varepsilon\kappa^4T.
$$

Then, taking $\varepsilon = R^{-\frac{1}{2}}$, and applying Gronwall's inequality, we conclude that

$$
\int_{\mathbb{R}}\frac{(g_1-g_2)^2}{2g_1g_2}(t_0)\psi_R\,dx \le C\frac{\kappa^4T}{\sqrt{R}}e^{CT(1+\kappa^3)},
$$

for some constant $C>0$ independent of R and κ. Taking a limit as $R\to\infty$, gives

$$
g_1(t_0) \equiv g_2(t_0), \qquad (2.10)
$$

for all $0\le t_0\le T$, i.e., uniqueness for the Green's functions (**Step 1**).

To conclude uniqueness for the (KdV) solutions (**Step 2**), from the identity in (2.5) and (2.10), we have

$$\begin{aligned} u_1 - u_2 = &\frac{1}{4}\partial_x^2\Big\{ \underbrace{(g_1-g_2)}_{=0}\Big(\frac{1}{g_1}+\frac{1}{g_2}\Big)\Big\} \\ &+\frac{1}{8}\partial_x\Big\{ \underbrace{(g_1-g_2)}_{=0}\Big[-3\partial_x\Big(\frac{1}{g_1}+\frac{1}{g_2}\Big) - \frac{\partial_x g_1 + \partial_x g_2}{g_1g_2}\Big]\Big\} \\ &+\frac{1}{8}\underbrace{(g_1-g_2)}_{=0}\Big\{-\Big(\frac{1}{g_1}+\frac{1}{g_2}\Big)\Big[2(u_1+u_2)+4\kappa^2-\frac{1}{2}\Big(\frac{1}{g_1}+\frac{1}{g_2}\Big)^2+\frac{3}{g_1g_2}\Big] \\ &\qquad +\frac{3}{2}\Big(\frac{(\partial_x g_1)^2}{g_1^3}+\frac{(\partial_x g_2)^2}{g_2^3}\Big) - \frac{1}{2g_1g_2}\Big(\frac{(\partial_x g_1)^2}{g_1}+\frac{(\partial_x g_2)^2}{g_2}\Big)\Big\} \\ =& \ 0, \end{aligned}$$

from which Theorem 1.4 follows.

Remark 2.1. For simplicity of presentation, we glossed over the fact that the equations in (2.6) as derived in [KMV20, KV19] hold for Schwartz potentials u. In [CKV24], to extend the argument to merely bounded solutions, we employ an approximation argument via mollification and derive analogues of (2.6) for a forced (KdV) equation; this is needed because the mollified solutions do not solve (KdV) exactly. We then establish (2.7) for the mollified solutions u_j^n and their corresponding Green's functions, which after taking limits to remove the mollification justify (2.7) for the bounded solutions u_1, u_2.

3. The Deift conjecture

We now sketch the ideas behind the construction of our counter-example to the Deift conjecture. One of the simplest ways to 'break' almost periodicity is to exhibit discontinuity. In fact, by considering the sum of two periodic but *discontinuous* functions with non-overlapping periods, we obtain a discontinuous function without almost periods! We take as a prototypical example the sum of two periodic square waves:

$$f(x) = \mathrm{sq}(\alpha_1 x) + \mathrm{sq}(\alpha_2 x), \tag{3.1}$$

where $\vec{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ is rationally independent:

$$\vec{\alpha}\cdot\vec{n} \neq 0 \qquad \text{for all} \qquad \vec{n} \in \mathbb{Z}^2_* = \mathbb{Z}^2 \setminus \{0\}, \tag{3.2}$$

and for $\mathrm{sq}(x) = \mathrm{sgn}(\sin(x))$.

We may then construct a counter-example to the Deift conjecture by finding almost periodic initial data u_0 whose KdV evolution at a later time is given by (3.1), which is *not* almost periodic. Given the time-reversibility of (KdV) and (Airy) (that is, $u(t,x) \mapsto u(-t,-x)$ leaves the class of solutions invariant), this is equivalent with constructing a solution with (3.1) as initial data, which exhibits almost periodicity at a later time.

The proof of Theorem 1.3 then splits into the following steps.

- **Step 1:** Construct a counter-example for (Airy) based on the Talbot effect. Namely, find time t_0 for which $e^{-t_0\partial_x^3} f$ is almost periodic, with f as in (3.1).
- **Step 2:** Construct a local-in-time solution for (KdV) with initial data f as in (3.1).
- **Step 3:** Establish *nonlinear smoothing* for (KdV) solutions with quasi-periodic[3] initial data.
- **Step 4:** Combine Steps 1-3 and the uniqueness of bounded solutions from Theorem 1.4.

[3]Here, we understand the notion of quasi-periodicity in more generality, in particular without the assumption of continuity so that it includes data of the type (3.1).

3.1. **Linear dynamics and Talbot effect (Step 1).** The main idea behind Proposition 1.2 on the failure of the Deift conjecture for (Airy) goes back to the work of Oskolkov [O92], which we state in an informal manner:

Lemma 3.1 (Oskolkov [O92]). *Fix $\alpha > 0$ and a $(2\pi/\alpha)$-periodic function f of bounded total variation. Then, the solution $e^{-t\partial_x^3} f$ to the Airy equation* (Airy) *satisfies the following:*
(i) *It is a bounded function.*
(ii) *Given $t_0 \in \mathbb{R}$ with $t_0\alpha/2\pi \in \mathbb{R} \setminus \mathbb{Q}$, the function $x \mapsto [e^{-t\partial_x^3} f](x)$ is* continuous.
(iii) *Given $t_0 \in \mathbb{R}$ with $t_0\alpha/2\pi \in \mathbb{Q}$, $[e^{-t\partial_x^3} f](x)$ is a finite superposition of translates of f.*

From Lemma 3.1, given a periodic discontinuous initial data, we observe a recurrence of its discontinuities at times which are rational multiples of the period of the initial data. This phenomenon was observed by Talbot [T1836] in optical experiments; it is known as the Talbot effect. Mathematically, this phenomenon is colloquially understood as the fact that linear solutions to dispersive equations 'can tell time', and present distinct behaviour at times that are rational and irrational multiples of the period of the initial data (compare Lemma 3.1 (ii) and (iii)). See [BK96, CO14, ET16] for further details on the Talbot effect.

The failure of the Deift conjecture for (Airy) follows from Lemma 3.1. Indeed, let f as the sum of two square waves as in (3.1) with periods $2\pi/\alpha_1, 2\pi/\alpha_2$ and $\vec{\alpha} = (\alpha_1, \alpha_2)$ rationally independent as in (3.2). For example, take $(\alpha_1, \alpha_2) = (1, \sqrt{2})$. Then, the solution $w(t) := e^{-t\partial_x^3} f$ to (Airy) is *bounded* in spacetime and it is *continuous* in x for each time t_0 such that $\alpha_1 t_0/2\pi$ and $\alpha_2 t_0/2\pi$ are both irrational. Taking one such time, $w(t_0)$ is almost periodic (since it is the sum of 2 continuous periodic functions) but $w(0) = f$ is not almost periodic!

3.2. **Quasi-periodic solutions and nonlinear smoothing (Steps 2–3).** Given initial data f in (3.1), we want to construct a (local-in-time) solution to (KdV) which solves the following Duhamel formulation:

$$v(t) = e^{-t\partial_x^3} f + 3\int_0^t e^{-(t-s)\partial_x^3} \partial_x(v^2)(s)\, ds, \tag{3.3}$$

whenever $0 \le t \le T \le 1$, where T is the short time of existence.

Note that the data f in (3.1) falls outside of the more classical well-posedness theory on L^2-based Sobolev spaces H^s. Nevertheless, local well-posedness of (KdV) for a class of quasi-periodic[4] initial data that includes f was shown by Tsugawa in [T12]. Namely, given $\vec{\alpha} \in \mathbb{R}^2$ as in (3.2) and $\theta \in \mathbb{R}$, Tsugawa constructed solutions in the space G^θ of functions satisfying

$$f(x) = \sum_{\vec{n} \in \mathbb{Z}_*^2} \widehat{f}(\vec{n}) e^{i(\vec{\alpha}\cdot\vec{n})x} \quad \text{with} \quad \|f\|_{G^\theta} := \|\widehat{f}\|_{\widehat{G}^\theta} = \left\| \frac{\langle n_1\rangle^\theta \langle n_2\rangle^\theta}{|\vec{\alpha}\cdot\vec{n}|^{1/2}} \widehat{f}(\vec{n}) \right\|_{\ell^2_{\vec{n}}(\mathbb{Z}_*^2)} < \infty, \tag{3.4}$$

where $\vec{n} = (n_1, n_2)$. This well-posedness result resorts to a quasi-periodic version of the Fourier restriction norm method due to Bourgain [B93], based on the following norm:

$$\|v\|_{X^{\theta,\frac{1}{2}}} := \left\| \langle \tau - (\vec{\alpha}\cdot\vec{n})^3 \rangle^{1/2} \mathcal{F}_{t,x} v(\tau, \vec{n}) \right\|_{\widehat{G}^\theta L^2_\tau}. \tag{3.5}$$

Note that the initial data f in (3.1) satisfies

$$f \in G^\theta \text{ for } \theta < 1. \tag{3.6}$$

[4] Again, we do not impose continuity on the notion of quasi-periodicity.

Lemma 3.2 (Tsugawa [T12]). *The KdV equation is locally well-posed in G^θ with $\theta > 1/4$ in the following sense: for each $f \in G^\theta$, there exist $T > 0$ and a unique solution $v \in C\big(\mathbb{R}; G^\theta\big) \cap X^{\theta,\frac{1}{2}}$ of* (3.3).

From Lemma 3.1, the linear contribution in (3.3) gains continuity in space for each time t_0 for which both $(\alpha_1/2\pi)t_0$ and $(\alpha_2/2\pi)t_0$ are irrational. To extend this property to the full solution v to (KdV), we establish a novel *nonlinear smoothing* effect in the quasi-periodic setting. This phenomenon is the property that the nonlinear part $v(t) - e^{-t\partial_x^3} f$ is often smoother than the linear evolution. Here, our notion of smoothness is continuity on the *full interval* $[-T, T]$.

Theorem 3.3. *Let f be as in* (3.4) *with $\vec{\alpha}$ satisfying the following Diophantine condition:*

$$|\vec{\alpha} \cdot \vec{n}| \gtrsim |\vec{n}|^{-\gamma} \quad \text{for all} \quad \vec{n} \in \mathbb{Z}^2_* \quad \text{and some} \quad \gamma > 1. \tag{3.7}$$

Let $v \in C\big(\mathbb{R}; G^\theta\big) \cap X^{\theta,1/2}$ be the corresponding solution of (KdV) *given by Lemma* 3.2. *If* $\max\{\frac{7}{8}, \frac{\gamma}{2}\} < \theta < 1$, *then*

$$\big\|\mathcal{F}_x\big(v(t) - e^{-t\partial_x^3} f\big)(\vec{n})\big\|_{L^\infty_t \ell^1_{\vec{n}}([-T,T]\times\mathbb{Z}^2_*)} < \infty. \tag{3.8}$$

Consequently, for all $t \in [-T, T]$,

$$v(t) - e^{-t\partial_x^3} f \in C(\mathbb{R}) \cap G^\theta \subset \mathrm{AP}(\mathbb{R}).$$

The condition in (3.7) can be understood as a quantitative version of the condition (3.2). Roth's theorem guarantees that this is satisfied when $\vec{\alpha} = (1, \alpha_2)$ and α_2 is an algebraic irrational number.

The proof of Theorem 3.3 relies on the structure of solutions $v \in X^{\theta,1/2}$ and the normal form approach. The latter, based on performing an integration by parts in time to exploit the oscillations due to multilinear dispersion, has been extensively used to establish well-posedness, unconditional uniqueness, and nonlinear smoothing. See [BIT11, GKO13, ET16, K21] and references therein.

Sketch of the proof of Theorem 3.3. Let v solve (3.3) on $[-T, T]$, $\vec{n} \in \mathbb{Z}^2_*$, and $|t| \le T$. From (3.3), we have

$$\mathcal{F}_x\big(v(t) - e^{-t\partial_x^3} f\big)(\vec{n}) = 3i \sum_{\vec{n}=\vec{n}^{(1)}+\vec{n}^{(2)}} e^{i(t-s)(\vec{\alpha}\cdot\vec{n})^3} (\vec{\alpha}\cdot\vec{n}) \widehat{v}(s, \vec{n}^{(1)}) \widehat{v}(s, \vec{n}^{(2)}) ds, \tag{3.9}$$

where the sum is taken over all $\vec{n}^{(j)} = (n_1^{(j)}, n_2^{(j)}) \in \mathbb{Z}^2_*$, $j = 1, 2$. By symmetry, we assume that

$$|\vec{\alpha}\cdot\vec{n}^{(1)}| \ge |\vec{\alpha}\cdot\vec{n}^{(2)}| \qquad \text{and} \qquad |\vec{n}_1^{(1)}| \ge |\vec{n}_2^{(1)}|.$$

Then, the derivative loss from the nonlinearity can be controlled as follows:

$$|\vec{\alpha}\cdot\vec{n}| \lesssim |n_1^{(1)}|. \tag{3.10}$$

We consider 2 cases based on the size relation between the different frequencies. In **Case 1**, we assume $|n_1^{(1)}| \lesssim |n_2^{(1)}| + |n_1^{(2)}| + |n_2^{(2)}|$, which from (3.10) means that the derivative loss $\vec{\alpha}\cdot\vec{n}$ in (3.9) can be split between two frequencies $n_\ell^{(j)}$, $j, \ell \in \{1, 2\}$. Here, we exploit the fact that $v \in X^{s,\theta}$ and use the bound on the time-frequency weights in the norm (3.5),

$$\max\big\{\big|(\vec{\alpha}\cdot\vec{n})^3 - \tau - (\vec{\alpha}\cdot\vec{n}^{(1)})^3\big|, \big|\tau - (\vec{\alpha}\cdot\vec{n}^{(2)})^3\big|\big\} \gtrsim \big|(\vec{\alpha}\cdot\vec{n})(\vec{\alpha}\cdot\vec{n}^{(1)})(\vec{\alpha}\cdot\vec{n}^{(2)})\big|, \tag{3.11}$$

to gain further smoothing in spatial frequency. Then, by the Cauchy-Schwarz inequality we easily get

$$\text{LHS (3.8)} \lesssim \Bigg(\sum_{\substack{\vec{n}\in\mathbb{Z}^2_* \\ \vec{n}=\vec{n}^{(1)}+\vec{n}^{(2)}}} \frac{|\vec{\alpha}\cdot\vec{n}|}{\langle n_1^{(1)}\rangle^{2\theta}\langle n_2^{(1)}\rangle^{2\theta}\langle n_1^{(2)}\rangle^{2\theta}\langle n_2^{(2)}\rangle^{2\theta}} \Bigg)^{1/2} \|v\|^2_{X^{\theta,1/2}} \lesssim \|v\|^2_{X^{\theta,1/2}}, \quad (3.12)$$

since (3.10) and the assumption on frequencies imply that $|\vec{\alpha}\cdot\vec{n}| \lesssim |n_1^{(1)} n_2^{(1)} n_1^{(2)} n_2^{(2)}|^{1/2}$, given that $2\theta - 1/2 > 1 \iff \theta > 3/4$.

In **Case 2**, we assume $|n_1^{(1)}| \gg |n_2^{(1)}| + |n_1^{(2)}| + |n_2^{(2)}|$. Proceeding as in Case 1, the derivative loss in (3.12) is only controlled by $n_1^{(1)}$ due to (3.10), which imposes $2\theta - 1 > 1 \iff \theta > 1$ to get (3.12). From (3.6), we see that the data of interest in (3.1) requires $\theta < 1$, so we need a different strategy in this regime to go beyond the restriction on θ. In particular, we employ the *interaction representation* $w(t) = e^{t\partial_x^3} v(t)$ and a *normal form* reduction (by performing an integration by parts *in time*):

$$\begin{aligned}
&e^{-it(\vec{\alpha}\cdot\vec{n})^3}\widehat{v}(t,\vec{n}) - \widehat{v}(0,\vec{n}) \\
&= 3\int_0^t \sum_{\mathcal{R}} e^{-3is(\vec{\alpha}\cdot\vec{n})(\vec{\alpha}\cdot\vec{n}^{(1)})(\vec{\alpha}\cdot\vec{n}^{(2)})} i(\vec{\alpha}\cdot\vec{n})\widehat{w}_{\vec{n}^{(1)}}(s)\widehat{w}_{\vec{n}^{(2)}}(s)\,ds \\
&= 3\int_0^t \sum_{\mathcal{R}} \frac{d}{ds}\Big(\frac{-e^{-3is(\vec{\alpha}\cdot\vec{n})(\vec{\alpha}\cdot\vec{n}^{(1)})(\vec{\alpha}\cdot\vec{n}^{(2)})}}{3i(\vec{\alpha}\cdot\vec{n})(\vec{\alpha}\cdot\vec{n}^{(1)})(\vec{\alpha}\cdot\vec{n}^{(2)})}\Big) i(\vec{\alpha}\cdot\vec{n})\widehat{w}_{\vec{n}^{(1)}}(s)\widehat{w}_{\vec{n}^{(2)}}(s)\,ds \\
&=: \mathcal{B}(t) - \mathcal{B}(0) + \mathcal{N}_1(t) + \mathcal{N}_2(t),
\end{aligned}$$

where $\mathcal{R}$ denotes the region where $\vec{n} = \vec{n}^{(1)} + \vec{n}^{(2)}$ and includes the assumptions of this case. Explicitly,

$$\begin{aligned}
\mathcal{B}(s) &:= \sum_{\mathcal{R}} e^{-3is(\vec{\alpha}\cdot\vec{n})(\vec{\alpha}\cdot\vec{n}^{(1)})(\vec{\alpha}\cdot\vec{n}^{(2)})} \frac{-1}{(\vec{\alpha}\cdot\vec{n}^{(1)})(\vec{\alpha}\cdot\vec{n}^{(2)})}\,\widehat{w}_{\vec{n}^{(1)}}(s)\widehat{w}_{\vec{n}^{(2)}}(s), \\
\mathcal{N}_1(t) &:= 3i\sum_{\mathcal{R}} \sum_{\vec{n}^{(3)}+\vec{n}^{(4)}=\vec{n}^{(1)}} \int_0^t e^{-is\Phi_{234}} \frac{1}{(\vec{\alpha}\cdot\vec{n}^{(2)})} \widehat{w}_{\vec{n}^{(2)}}(s)\widehat{w}_{\vec{n}^{(3)}}(s)\widehat{w}_{\vec{n}^{(4)}}(s)\,ds, \qquad (3.13)\\
\mathcal{N}_2(t) &:= 3i\int_0^t \sum_{\mathcal{R}} \sum_{\vec{n}^{(3)}+\vec{n}^{(4)}=\vec{n}^{(2)}} e^{-is\Phi_{134}} \frac{1}{(\vec{\alpha}\cdot\vec{n}^{(1)})} \widehat{w}_{\vec{n}^{(1)}}(s)\widehat{w}_{\vec{n}^{(3)}}(s)\widehat{w}_{\vec{n}^{(4)}}(s)\,ds,
\end{aligned}$$

with $\Phi_{jk\ell} = 3[\vec{\alpha}\cdot(\vec{n}^{(j)}+\vec{n}^{(k)})][\vec{\alpha}\cdot(\vec{n}^{(j)}+\vec{n}^{(\ell)})][\vec{\alpha}\cdot(\vec{n}^{(k)}+\vec{n}^{(\ell)})]$. The normal form reduction above allows us to gain the contribution $|(\vec{\alpha}\cdot\vec{n})(\vec{\alpha}\cdot\vec{n}^{(1)})(\vec{\alpha}\cdot\vec{n}^{(2)})|$ in the denominator, which can be seen as a gain in spatial derivatives. In the periodic case, this is always beneficial since the analogue term would be $|nn_1n_2| \geq 1$ for $n, n_1, n_2 \in \mathbb{Z}_*$. In the quasi-periodic setting there is the additional difficulty that $0 < |(\vec{\alpha}\cdot\vec{n}^{(j)})| \ll 1$ can get arbitrarily close to 0. It is to bypass this issue that we impose the Diophantine condition in (3.7), which provided a lower bound on such terms.

Going back to (3.13), the boundary piece $\mathcal{B}(s)$ can be easily estimated in $\ell^1_{\vec{n}}(\mathbb{Z}^2_*)$ for each $|s| \leq T$, by applying Cauchy-Schwarz, the Diophantine condition (3.7) to lower bound $|\vec{\alpha}\cdot\vec{n}^{(2)}|$, and $|\vec{\alpha}\cdot\vec{n}^{(1)}| \sim |n_1^{(1)}|$. A similar argument can be applied to $\mathcal{N}_2(t)$, since $|\vec{\alpha}\cdot\vec{n}^{(1)}| \sim |n_1^{(1)}| \geq 1$.

However, this argument is insufficient to treat $\mathcal{N}_1(t)$ because of the negative power of $|\vec{\alpha}\cdot\vec{n}^{(2)}|$ which can be small. To treat this term, we return to the v-variables:

$$\mathcal{N}_1(t) = 3i\sum_{\mathcal{R}} \sum_{\vec{n}^{(3)}+\vec{n}^{(4)}=\vec{n}^{(1)}} \int_0^t e^{-is(\vec{\alpha}\cdot\vec{n})^3} \frac{1}{(\vec{\alpha}\cdot\vec{n}^{(2)})}\,\widehat{v}(s,\vec{n}^{(2)})\widehat{v}(s,\vec{n}^{(3)})\widehat{v}(s,\vec{n}^{(4)})\,ds,$$

and argue in a similar fashion as in Case 1, using the fact that $v \in X^{\theta,1/2}$ with the following lower bound for the time-frequency weights in (3.5) as a replacement for (3.11):

$$\begin{aligned}\max_{j=2,3,4} |\tau_j - (\vec{\alpha}\cdot\vec{n}^{(j)})^3| &\gtrsim |\tau_2 - (\vec{\alpha}\cdot\vec{n}^{(2)})^3 + \tau_3 - (\vec{\alpha}\cdot\vec{n}^{(3)})^3 + \tau_4 - (\vec{\alpha}\cdot\vec{n}^{(4)})^3| \\ &= |(\vec{\alpha}\cdot\vec{n})^3 - (\vec{\alpha}\cdot\vec{n}^{(2)})^3 - (\vec{\alpha}\cdot\vec{n}^{(3)})^3 - (\vec{\alpha}\cdot\vec{n}^{(4)})^3| \\ &= 3|\vec{\alpha}\cdot(\vec{n}^{(2)}+\vec{n}^{(3)})||\vec{\alpha}\cdot(\vec{n}^{(2)}+\vec{n}^{(4)})||\vec{\alpha}\cdot(\vec{n}^{(3)}+\vec{n}^{(4)})| \\ &= |\Phi_{234}|. \end{aligned}$$

□

3.3. **Counter-example (Step 4).** Finally, we complete the proof of Theorem 1.3. Figure 1 provides a schematic of the distinct ingredients combined in the proof.

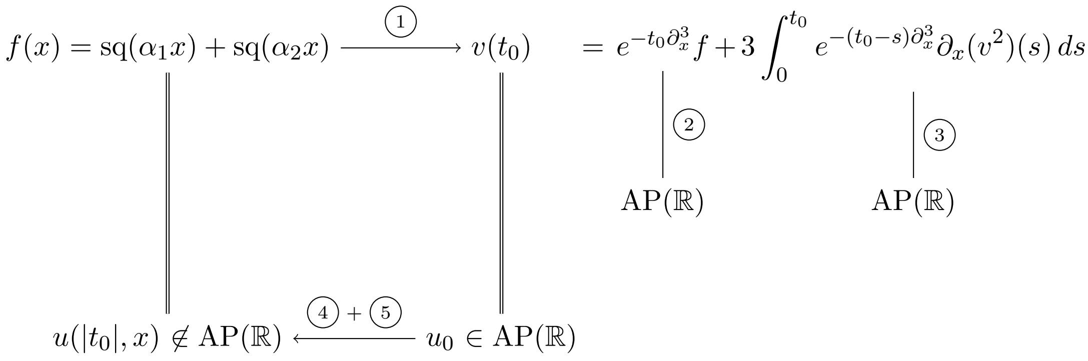

FIGURE 1. Schematic of the proof of Theorem 1.3.

Let $\vec{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ rationally independent as in (3.2) and satisfying the quantitative version in (3.7). Moreover, let f be the sum of square waves as in (3.1), which is not almost periodic.

(1) First, by Lemma 3.2, there exists a solution v to (KdV) with $v(0) = f$ which satisfies the Duhamel formulation (3.3).

(2) From Lemma 3.1, if $t_0\frac{2\pi}{\alpha_1}$ and $t_0\frac{2\pi}{\alpha_2}$ are irrational, then the linear part $e^{-t_0\partial_x^3} f$ is almost periodic in space (as the sum of 2 continuous periodic functions).

(3) Our nonlinear smoothing result, Theorem 3.3, guarantees that the nonlinear part of the solution $v(t_0) - e^{-t_0\partial_x^3} f$ is continuous in space and thus almost periodic (since it is in G^θ by construction).

(4) We then define our initial data $u_0 := v(t_0)$ as the (KdV) solution constructed in (1) evaluated at a time $t_0 < 0$. From time-reversibility of (KdV), there exists a solution $u : t \mapsto v(t + t_0)$ to (KdV) with $u(0) = v(t_0) = u_0 \in \mathrm{AP}(\mathbb{R})$ and $u(|t_0|) = f \notin \mathrm{AP}(\mathbb{R})$, i.e., a solution that has spatial almost periodicity at the initial time but *not* at a later time.

(5) Lemma 3.1 and Theorem 3.3 show that u is a bounded solution to (KdV). Thus, Theorem (1.4) shows that it is the only possible evolution arising from u_0. This completes the proof.

REFERENCES

[BIT11] A. V. Babin, A. A. Ilyin, E. S. Titi, *On the regularization mechanism for the periodic Korteweg-de Vries equation.* Comm. Pure Appl. Math. 64 (2011), no. 5, 591–648.

[BK96] M. V. Berry, S. Klein, *Integer, fractional and fractal Talbot effects.* J. Mod. Optics, 43 (1996), no. 10, 2139–2164.

[BDGL18] I. Binder, D. Damanik, M. Goldstein, M. Lukic, *Almost periodicity in time of solutions of the KdV equation.* Duke Math. J. 167 (2018), no. 14, 2633–2678.

[B47] H. Bohr, *Almost Periodic Functions.* Chelsea Publishing Co., New York, N.Y., 1947.

[B93] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV equation.* Geom. Funct. Anal. 3 (1993), no. 3, 209–262.

[B1872] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond.* J. Math. Pures Appl. (2) 17 (1872), 55–108.

[CKV24] A. Chapouto, R. Killip, M. Vişan, *Bounded solutions of KdV: uniqueness and the loss of almost periodicity.* Duke Math. J. 173 (2024), no. 7, 1227–1267.

[CO14] G. Chen, P. J. Olver, *Numerical simulation of nonlinear dispersive quantization.* Discrete Contin. Dyn. Syst. 34 (2014), no. 3, 991–1008.

[C95] D. G. Crighton, *Applications of KdV.* In "KdV '95 (Amsterdam, 1995)", Acta Appl. Math. 39 (1995), no. 1–3, 39–67.

[DG16] D. Damanik, M. Goldstein, *On the existence and uniqueness of global solutions for the KdV equation with quasi-periodic initial data.* J. Amer. Math. Soc. 29 (2016), no. 3, 825–856.

[DLVY21] D. Damanik, M. Lukić, A. Volberg, P. Yuditskii, *The Deift conjecture: a program to construct a counter-example.* `arXiv:2111.09345`.

[D08] P. Deift, *Some open problems in random matrix theory and the theory of integrable systems.* Contemp. Math., vol. 458, Amer. Math. Soc., Providence, RI, 2008, pp. 419–430.

[D17] P. Deift, *Some open problems in random matrix theory and the theory of integrable systems, II.* SIGMA Symmetry Integrability Geom. Methods Appl. 13 (2017), Paper No. 016, 23 pp.

[E94] I. E. Egorova, *The Cauchy problem for the KdV equation with almost periodic initial data whose spectrum is nowhere dense.* In "Spectral Operator Theory and Related Topics", Adv. Soviet Math. 19, Amer. Math. Soc., Providence, RI, 1994, 181–208.

[EVY19] B. Eichinger, T. VandenBoom, P. Yuditskii, *KdV hierarchy via Abelian coverings and operator identities.* Trans. Amer. Math. Soc. Ser. B 6 (2019), 1–44.

[ET16] M. B. Erdoğan, N. Tzirakis, *Dispersive partial differential equations.* Wellposedness and applications. London Mathematical Society Student Texts, 86. Cambridge University Press, Cambridge, 2016.

[GKO13] Z. Guo, S. Kwon, T. Oh, *Poincaré-Dulac normal form reduction for unconditional well-posedness of the periodic cubic NLS.* Comm. Math. Phys. 322 (2013), no.1, 19–48.

[KT06] T. Kappeler, P. Topalov, *Global wellposedness of KdV in* $H^{-1}(\mathbb{T},\mathbb{R})$. Duke Math. J. 135 (2006), no. 2, 327–360.

[K95] T. Kato, *On nonlinear Schrödinger equations. II.* H^s*-solutions and unconditional well-posedness.* J. Anal. Math. 67 (1995), 281–306.

[KMV20] R. Killip, J. Murphy, M. Vişan, *Invariance of white noise for KdV on the line.* Invent. Math. 222 (2020), no. 1, 203–282.

[KV19] R. Killip, M. Vişan, *KdV is well-posed in* H^{-1}. Ann. of Math. 190 (2019), no. 1, 249–305.

[K21] N. Kishimoto, *Unconditional uniqueness of solutions for nonlinear dispersive equations.* `arXiv:1911.04349`.

[KdV1895] D. J. Korteweg, G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves.* Philos. Mag. (5) 39 (1895), no. 240, 422–443.

[LY20] M. Lukić, G. Young, *Uniqueness of solutions of the KdV-hierarchy via Dubrovin-type flows.* J. Funct. Anal. 279 (2020), no. 7, Paper No. 108705, 30 pp.

[MT76] H. McKean, E. Trubowitz, *Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points.* Comm. Pure. Appl. Math. 29 (1976), no. 2, 143–226.

[O92] K. Oskolkov, *A class of I. M. Vinogradov's series and its applications in harmonic analysis.* In "Progress in approximation theory (Tampa, FL, 1990)", 353–402, Springer Ser. Comput. Math., 19, Springer, New York, 1992.

[T1836] H. F. Talbot, *Facts related to optical science, No. IV.* Philo. Mag. 9 (1836), 401–407.

[T12] K. Tsugawa, *Local well-posedness of the KdV equation with quasi-periodic initial data.* SIAM J. Math. Anal. 44 (2012), no. 5, 3412–3428.

[ZK65] N. J. Zabusky, M. D. Kruskal, *Interaction of "solitons" in a collisionless plasma and the recurrence of initial states.* Phys. Rev. Lett. 15 (1965), no. 15, 240–243.

[Z91] V. E. Zakharov (Ed.), *What is integrability?* Springer Ser. Nonlinear Dynam., Springer-Verlag, Berlin, 1991.

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