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COLLISION OF TWO SOLITARY WAVES OF ZK

FRÉDÉRIC VALET

ABSTRACT. We consider the collision of two co-propagating solitary waves for the Zakharov-Kuznetsov equation, a two-dimensional asymptotic model from plasma physics. Consider a pure two-solitary waves at time $t = -\infty$, with nearly equal velocity and nearly equal ordinate. The emanating solution stays close in H^1 to the sum of two modulated waves on the whole time interval $\mathbb{R}$. This collision is characterized by a minimal distance between the waves and a transfer of mass.

We review in this note the description of the collision phenomenon obtained in [29] from time $-\infty$ to $+\infty$, inspired from the seminal article of Martel and Merle in [17] on the collision of two solitary waves for the quartic Korteweg-de Vries. The result in higher dimension requires substantial modifications to the proof. First, the solution is approximated by the sum of two modulated waves and a non-local term induced by the quadratic non-linearity, with an adjustment in three intrinsic directions related to the translation invariances and scaling. The evolution of the error induced by this approximation and the modulation parameters is controlled by bootstrap techniques, with a new energy functional, a refined argument to control the transverse direction and a non-explicit approximated ODE system to control the height and the distance between the waves.

We also review some asymptotic stability results from [28].

1. The Zakharov-Kuznetsov equation

In this note, we give a review on the recent result obtained with Didier Pilod on the collision of two solitary waves for the Zakharov-Kuznetsov equation in dimensions $d = 2, 3$

$$\text{(ZK)} \qquad \partial_t u + \partial_x \left(\Delta u + u^2\right) = 0, \quad u : (t, \boldsymbol{x}) \in I_t \times \mathbb{R}^d \mapsto u(t, \boldsymbol{x}) \in \mathbb{R},$$

with

- in dimension $d = 2$, $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$ and the laplacian $\Delta = \partial_x^2 + \partial_y^2$,
- in dimension $d = 3$, $\boldsymbol{x} = (x, \boldsymbol{y}) = (x, y_1, y_2) \in \mathbb{R}^3$ and the Laplacian $\Delta = \partial_x^2 + \partial_{y_1}^2 + \partial_{y_2}^2$.

This equation is an asymptotic model for the description of the propagation of ion-acoustic waves in a cold and magnetized plasma. Zakharov and Kuznetsov [33] in 1974 formally derived this equation, and it was later proved, in the context of uniformly magnetized plasma, to be an asymptotic model from the Euler-Poisson equation by Lannes, Linares and Saut [14] and Pu [30] in the long wave limit, and from the Vlasov-Poisson equation by Han Kwan [9] in the cold ions and long wave limit. In particular, these derivations also hold in dimension 1 and one recovers the 1-dimensional Korteweg-de Vries equation

$$\text{(KdV)} \qquad \partial_u + \partial_x \left(\partial_x^2 u + u^2\right) = 0, \quad u : (t, x) \in I_t \times \mathbb{R} \mapsto u(t, x) \in \mathbb{R}.$$

Even if the two former equations are obtained similarly as asymptotic models, the description of the solutions of KdV is eased by the existence of a Lax pair. This equation is thus completely integrable and the Lax pair implies the existence of an infinite number of conserved quantities for KdV. Meanwhile, the higher dimensional model ZK is not known to be integrable. There are still conserved quantities associated to this equation

$$(1.1) \qquad \int_{\mathbb{R}^d} u(t, \boldsymbol{x}) d\boldsymbol{x}, \; M(u) := \int_{\mathbb{R}^d} u^2(t, \boldsymbol{x}) d\boldsymbol{x}, \; E(u) := \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u|^2 - \frac{1}{3} u^3(t, \boldsymbol{x}) d\boldsymbol{x}.$$

Furtheremore, ZK enjoys a scaling operation that keeps the set of solutions invariant. If u is a solution to ZK, then for any positive λ, the function u_λ defined by

$$(1.2) \qquad \forall (t, \boldsymbol{x}), \; u_\lambda(t, \boldsymbol{x}) := \lambda u(\lambda^{3/2} t, \lambda^{1/2} \boldsymbol{x})$$

is also a solution.

For the rest of this note, the results are stated in the 2-dimensional framework for ZK. Similar results also holds the 3-dimensional setting, we refer to [28, 29] for more details.

The two equations KdV and ZK are dispersive in the sense that the displacement of a wave packet depends on its frequency. The dispersion relation encodes the direction of propagation of small data.

For KdV, a wave packet with frequency $k \in \mathbb{R}$ spreads according to the dispersion relation $\omega(k) = -k^3$ and the group velocity is given by $\omega'(k) = -3k^2$. Thus, a small wave packet of KdV spreads in the negative direction along the x-axis. For ZK, the dispersion relation of a wave packet with the frequency $\boldsymbol{k} = (k_x, k_y) \in \mathbb{R}^2$ and the group velocity are given by (see [2], Appendix C)

$$\omega(\boldsymbol{k}) = -k_x(k_x^2 + k_y^2) \quad \text{and} \quad \nabla\omega(\boldsymbol{k}) = \left(-3k_x^2 - k_y^2, -2k_x k_y\right)^T. \tag{1.3}$$

The first coordinate of $\nabla\omega(\boldsymbol{k})$ is negative but the second depends on the value of $\boldsymbol{k}$. The angle $\theta \in [0; 2\pi[$ between the first axis $\mathbf{e}_x$ and the group velocity $\nabla\omega(\boldsymbol{k})$ satisfies, for any value of k_y different from 0,

$$\tan(\theta) = \frac{2k_x k_y}{3k_x^2 + k_y^2} = \frac{2r}{3r^2 + 1} \quad \text{with} \quad r = \frac{k_x}{k_y}. \tag{1.4}$$

Since the odd function $f(r) = \dfrac{2r}{3r^2+1}$ is surjective on $\left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right]$, it implies that θ belongs to $\left[\frac{5}{6}\pi; \frac{7}{6}\pi\right]$. The admissible directions for the group velocity are thus directed along the negative part of the x-axis, symmetric with respect to k_y and the angle formed by the set of admissible directions has a size $\pi/3$. A similar result for the set of group velocities holds in dimension $d = 3$.

In order to describe the set of solutions of a dispersive equation, three natural questions arise.

(1) The first one concerns the well-posedness of the equation and the maximal time of existence of a solution u emanating from an initial condition u_0.
(2) Once one has found a suitable space X for studying the equation, one can consider the asymptotic behavior of the solution in X, thus the behavior next to the maximal time of existence. In particular, if the maximal time of existence T is finite, one needs to investigate the reason why the solution does not belong to the space X at time T; if the T is infinite, one can wonder if any particular structure emanates from the solution.
(3) The third question is the long-time dynamics of the solution. It consists in the description of the solution on the whole time interval of existence.

This short note aims at describing a particular phenomenon for the long-time dynamics. Since this description is based on the answer to the two first questions, we develop in the next paragraphs some partial answers to the well-posedness question and the asymptotic behavior of solutions.

We recall the well-posedness result on ZK. In this note, an equation is referred to be well-posed in a space $H^s(\mathbb{R}^d)$ in the sense of Kato [10] : starting with an initial data u_0 in the Sobolev space $H^s(\mathbb{R}^d)$, there exists a time $T = T(\|u_0\|_{H^s})$ and a resolution space $X_T \hookrightarrow \mathcal{C}((-T,T), H^s)$ such that there exists a unique function $u \in X_T$ satisfying the equation at least in a weak sense; furthermore the flow map data-to-solution $u_0 \mapsto u$ is continuous from a ball of radius R, $B(0,R) \subset H^s$ to $L^\infty((-T_R, T_R), H^s)$, where T_R is uniform in R.[1] The equation is said to be locally well-posed if $T < \infty$ and globally well-posed if the continuity holds for any T arbitrarily large. Faminskiĭ in [6] gave the first result of well-posedness of ZK in 2 dimensions. He proved that the equation is globally well-posed in any space H^m for $m \in \mathbb{N}$ and $m \geqslant 1$ by using a parabolic regularization. Further results whose proofs are based on a fixed point argument in adequate spaces provide local well-posedness results in H^s for lower values of s, see [15, 8, 24], until reaching the value $s = -1/4$ by Kinoshita [11].[2] This result is considered as optimal using a fixed point argument, since as underlined in [11], the flow is not in $\mathcal{C}^2((-T,T), H^s)$ for any value $s < -1/4$. Remark that this value does not correspond to the critical scaling coefficient $s_c = -1$, the coefficient at which the semi-norm of the homogeneous Sobolev space $\dot{H}^{s_c}$ is invariant by the scaling of (1.2). A final result of Kinoshita [11] is the global well-posedness in $L^2(\mathbb{R}^2)$.

2. Solitary waves and multi-solitary waves

On the ZK equation, the well-balance between the dispersion operator $\partial_x\Delta$ and the non-linear term $\partial_x u^2$ allows the existence of non-linear non-dispersive objects. Among those objects, the solitary waves naturally emerge when studying the long time behavior of solutions. The solitary waves consist in a localized profile Q moving without deformation in one direction, with a constant velocity c. The only admissible direction of displacement of a solitary waves, without provoking a contradiction with the

[1] Notice that this definition does not include the unconditional uniqueness, that is the uniqueness of the solution in the space $\mathcal{C}((-T,T), H^s)$.

[2] Even with those results, the unconditional uniqueness of ZK is still an open problem in H^s for any value of s.

Pohozaev identity (see Theorem 1.5 of [2]) is the first direction $\boldsymbol{e}_x$. Thus, if $u(t,\boldsymbol{x}) = Q(\boldsymbol{x} - ct\boldsymbol{e}_x)$ is a solitary wave, the profile satisfies the elliptic equation

$$-cQ + \Delta Q + Q^2 = 0. \tag{2.1}$$

Fix a positive value c. The existence of a particular profile dates back to the seventies-eighties. It has been proved in [31, 1] that there exists a positive profile, spherically invariant about a point, say 0, and radially decreasing with an exponential decay at infinity. Then, the uniqueness, up to translation, of the positive solution to (2.1) was deduced in [13] from the radial symmetry of any positive solution, see [7]. Even if there exist other solutions [22], the ground-states are interesting for their stability, see Section 3. As opposed to the 1-dimensional case, the ground-states do not enjoy an explicit formula. Note also that in order to study (2.1) for any value $c > 0$, one can focus on the case $c = 1$ and find the solutions for other values of c by applying the scaling operation. We thus denote by Q the positive ground-state centered at 0 for $c = 1$ previously exhibited and by $Q_c(\boldsymbol{x}) = cQ(c^{1/2}\boldsymbol{x})$ the ground-state associated to $c > 0$. It is thus natural to define the operator Λ_c which corresponds to the derivative with respect to the scaling direction, and $\Lambda := \Lambda_1$, as

$$\Lambda_c f(\boldsymbol{x}) := \left(\frac{d}{d\widetilde{c}}\left(cf(c^{1/2}\boldsymbol{x})\right)\right)_{|\widetilde{c}=c} = \left(\left(1 + \frac{1}{2}\boldsymbol{x}\cdot\nabla\right)f\right)(c^{1/2}\boldsymbol{x}).$$

By using ODE technique, one obtains a precise asymptotic expansion of Q at infinity. From Lemma B.11 of [29], we obtain, using the modified Bessel function of second kind K_0,

$$\exists \kappa, k > 0,\ \forall |\boldsymbol{x}| > 1,\ |Q(\boldsymbol{x}) - \kappa K_0(|\boldsymbol{x}|)| \leq k|\boldsymbol{x}|^{-1}e^{-2|\boldsymbol{x}|}.$$

In order to give the stability properties of this object, we introduce the linearized operator around Q

$$L := -\Delta + 1 - 2Q.$$

We recall some properties of this operator.

Proposition 2.1 ([29], Proposition 2.1)**.** *The operator $L : H^2(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ satisfies the following properties.*

1. *L is a self-adjoint operator and $\sigma_{\text{ess}}(L) = [1; +\infty)$.*
2. *Non-degeneracy of Q : $\ker(L) = \operatorname{span}\{\partial_x Q, \partial_y Q\}$.*
3. *L has a unique negative eigenvalue $-\lambda_0$, which is simple, associated to a positive radially symmetric eigenfunction χ_0. Without loss of generality, it is chosen to be normalized $\|\chi_0\|_{L^2} = 1$. Furthermore, it holds with other positive constant κ and k that*
$$\exists \kappa_0, k > 0,\ \forall |\boldsymbol{x}| > 1,\ \left|\chi_0(\boldsymbol{x}) - \kappa_0 K_0(\sqrt{1+\lambda_0}|\boldsymbol{x}|)\right| \leq k|\boldsymbol{x}|^{-1}e^{-2\sqrt{1+\lambda_0}|\boldsymbol{x}|}.$$
4. *The operator L is coercive up to three orthogonality conditions : there exists $C > 0$ such that for any $f \in H^2(\mathbb{R}^2)$ orthogonal to $\partial_x Q$, $\partial_y Q$ and Q, it holds*
$$\langle Lf, f\rangle_{L^2} \geqslant C\|f\|_{L^2}^2.$$
5. *It holds $L(\Lambda Q) = -Q$.*

The idea of multi-solitary waves, that are solutions that decompose in large time as a sum of decoupled waves, takes its roots in the soliton resolution conjecture. For ZK, which is globally well-posed in the energy space, the conjecture can be stated as follow: from any initial condition, the emerging solution decomposes in large time as a sum of decoupled non-linear non-dispersive objects plus a radiation term. We now give the definition of multi-solitary waves.

Definition 2.2. A multi-solitary waves (or pure multi-solitary waves, or K-solitary waves) at $-\infty$ is a solution u of ZK having the following asymptotic behavior. There exist K distinct velocities $0 < c_1 < \cdots < c_K$ and some fixed positions $(\boldsymbol{x}_k)_{1\leqslant k\leqslant K}$ such that, by denoting the solitary waves

$$R_k(t,\boldsymbol{x}) := Q_{c_k}\left(\boldsymbol{x} - c_k t\boldsymbol{e}_x\right),$$

it holds

$$u(t) - \sum_{k=1}^{K} R_k(t) \underset{t\to-\infty}{\overset{H^1}{\longrightarrow}} 0. \tag{2.2}$$

The multi-solitary waves can also be defined for positive times from the symmetry $(t,\boldsymbol{x}) \mapsto (-t,-\boldsymbol{x})$. Those particular solutions, without any radiation term appearing at infinity, exist for ZK.

Theorem 2.3 ([32], Theorem 2)**.** *Let K distinct velocities $0 < c_1 < \cdots < c_K$ and K original positions $(\boldsymbol{x}_k)_{1 \leqslant k \leqslant K}$. There exists a multi-solitary waves associated to those parameters. It is unique in the sense of* (2.2)*. Furthermore, the error between the solution and the sum of the decoupled solitary waves decays exponentially:*

$$\exists C, \delta > 0,\ \forall t < 0,\ \left\| u(t) - \sum_{k=1}^{K} R_k(t) \right\|_{H^1} \leqslant C e^{\delta t}.$$

3. Stability

A first notion of stability is the one of orbital stability. A solitary wave is said to be orbitally stable if for any initial condition close to a ground state, the emerging solution stays in the orbit of this ground state. Furthermore, the size of the orbit is proportional to the difference between the initial condition and a ground state. De Bouard in [4] proved using a Grillakis-Shatah-Strauss argument the orbital stability of a solitary wave of ZK. This notion of orbital stability can be generalized for multi-solitary waves as in [20], in the following sense. If an initial condition is close to a sum of K-decoupled solitary waves, well-ordered and decoupled enough, then the emerging solution stays close to a sum of K modulated solitary waves with a control on the modulation parameters, but for positive times only. Côte, Muñoz, Pilod and Simpson in [2] investigated the stability of this structure and obtained the following result, we refer to the article for a precise statement.

Theorem 3.1 ([2], Theorem 1)**.** *The multi-solitary waves are orbitally stable in the following sense. Let $(R_k^0)_{1 \le k \le K}$ be K distinct solitary waves, well-ordered and decoupled enough. There exists $\alpha^\star > 0$ such that the following holds. Let an initial condition u_0 satisfying*

$$\left\| u_0 - \sum_{k=1}^{K} R_k^0 \right\|_{H^1} \leqslant \alpha^\star$$

and $u(t)$ the solution to ZK *emanating from u_0. Then the difference between u and the sum of some modulated solitary waves, as well as the evolution of the modulation parameters, are quantified in terms of α and the initial distance between the solitary waves.*

In order to describe the evolution of the solution after a collision phenomenon, this theorem lacks an explicit dependency of $\alpha^\star$ in term of the difference of sizes. We state in Theorem 5.1 a quantified orbital stability result for two solitary waves with almost same sizes close to 1.

A second notion of stability is the one of asymptotic stability. Many results of asymptotic stability are stated in the following way : in a region of space, that may depend on the equation, the difference between the solution and the sum of the modulated objects tends to 0 as time evolves. Inspired from [20], Côte, Muñoz, Pilod and Simpson in [2] obtained the asymptotic stability of a solitary wave in any half-plane whose minimimal angle with the first axis corresponds to the maximal angle of dispersion in (1.4). A first ingredient was a linear Liouville property around a profile Q_c by using the dual problem. A second ingredient, to deal with the geometry in the second direction, was the monotonicity of local mass and local energy on half-spaces. In particular, those oblique half-spaces are defined by horizontal lines that moves slower than the solitary waves, either parallel to the y-axis or bend with an angle at most $\pi/3$. This angle is once again related to the dispersion relation (1.4).

In the case of two solitary waves, the mass expelled from the first solitary waves may reach the second solitary wave, as stated in [20] for KdV-type equations. Furthermore, the geometry plays an important role: since the dispersion term expelled from the first solitary wave is localized in a cone, one needs to adjust the set of admissible oblique half-spaces around the second solitary wave to deal with the dispersion from the first solitary wave. We give a precise statement of asymptotic stability in Theorem 5.1 from [28].

4. The collision phenomenon

So far we described some asymptotic behaviors of solutions, such as the solitary waves and the multi-solitary waves as well as their stability properties. The next step is to describe the solution on the whole time interval. For instance, starting at time $-\infty$ with a two-solitary waves of ZK with almost same ordinate, the highest solitary wave goes faster than the second solitary wave and the two waves interact. Each wave influences the trajectory and the shape of the other wave. This phenomenon is called a collision.

Depending on the equation, the collision of two solitary waves gives a different phenomenon, see [29] for a complete pictures of phenomena. Eckhaus and Schuur in [5] proved the soliton resolution conjecture for KdV using arguments from inverse scattering theory. In particular, since from an initial condition with enough decay, they obtained the decomposition of the solution as a pure N-solitary waves at $+\infty$, we deduce by symmetry that a pure two-solitary waves at $+\infty$ is also a pure two-solitary waves at $-\infty$ and the shapes after and before the collision are one-to-one. In this sense, the collision is said to be **elastic**. One can notice that after an elastic collision of KdV, the solitary waves are shifted in space and the order of sizes is inverse : the solitary wave on the front is the smallest before the collision, whereas it is the highest after the collision. In contrast with this elactic phenomenon, a collision is said to be **inelastic** if some mass is expelled from this two waves structure along the collision process

This collision phenomenon has been investigated for different equations. We start with equations derived or inspired from fluid dynamics. Mizumachi in [23] studied the generalized subcritical Korteweg-de Vries equations, with a non-linearity $f(u) = |u|^{p-1}u$, with $3 < p < 5$. Starting with an initial datum as the sum of two solitary waves with almost similar velocities before the collision, the highest one on the left, he described the collision and exhibited the repulsive behavior of the collision. In particular, he obtained a bound on the defect for the solution to be a pure two-solitary waves, that will be proved later on to be the size of the mass expelled during the collision term. Craig, Hammack, Guyenne, Handerson and Sulem in [3] observed numerically for the water wave equation the inelasticity of a collision of two solitary waves, that are either co-propagating (same sense) or counter-propagating (different senses). The small term expelled after the collision intuited in [23] has then been quantified and investigated by Martel and Merle [17] for the quartic KdV. They proved the following : if u is at $-\infty$ a pure two-solitary waves with almost same sizes, then u decomposes at $+\infty$ as a sum of two solitary waves plus a small term expelled from the collision. The mass of this term is bounded by below by a positive constant, which proves the inelasticity of the collision for this equation. A similar result has been obtained for the Benjamin-Bono-Mahony equation by Martel, Merle and Mizumachi in [19]. Note that one can also study the collision of two solitary waves with a large difference of sizes. For instance, Martel and Merle in [16] found back this inelasticity result by giving a detailed description of the collision of a large solitary wave with a small one for the quartic KdV equation. Inspired by this result, Muñoz in [26] showed that the collision of a large wave with a small wave is inelastic for a large family of subcritical non-linearities of KdV-type equations, except in three cases. Those cases correspond to the KdV, modified KdV and Gardner equations, which are integrable. With these results in hand, one can observe the relation between integrability of an equation and the elasticity of the collisions.

This inelasticity result has been investigated for other dispersive equations. Perelman in [27] described for some subcritical generalized Schrödinger equations, a collision of a large solitary with a smaller one. The associated solution decomposes after the collision as a solitary wave (the largest one) plus two smaller waves which, up to scaling, evolve according to the flow of the cubic Schrödinger equation. Martel and Merle in [18] established the inelasticity of the collision of two solitary waves for the energy critical wave equation in 5 dimensions. Moutinho in [25] obtained an almost-elasticity result of two kinks for the ϕ^6-model with a lower bound on a large time interval of the difference between the solution and the decoupled kinks.

The Zakharov-Kuznetsov equation is not known to be integrable, one can thus expect an inelastic collision, as observed by Klein, Roudenko and Stoilov in [12]. A first step towards the proof of inelasticity is the description of the collision on the whole time interval.

The goal is to describe the collision of two solitary waves with similar velocities, close to 1 at $-\infty$, but not necessarily to prove the inelasticity of the collision. The two waves are expected to move at any time at a velocity close to 1 at the main order. To focus on this particular behavior, we introduce an adequate change of framework : if u is a solution to ZK, we define the function $v(t, \boldsymbol{x}) := u(t, \boldsymbol{x} + t\boldsymbol{e}_x)$, so that v solves the translated ZK equation

$$\partial_t v + \partial_x \left(\Delta v - v + v^2\right) = 0. \tag{ZK$_t$}$$

In this new framework, a solitary with a profile Q_c for $c>1$ moves from left to right with a velocity $c-1$, whereas for $c<1$, a solitary wave moves from right to left at a velocity $1-c$. Thus, if v is a two-solitary waves at $-\infty$ with profiles associated to $c_1<1<c_2$, it is composed of a small solitary wave on the right with speed c_1-1, and a bigger solitary wave on the left with speed c_2-1. The expected behavior of the solution along the time is depicted in Figure 1 : the first solitary wave R_1 remains on the right, whereas the second wave R_2 remains on the left. The first picture corresponds to the behavior of the solitary waves before the collision. The second picture, where the two solitary waves have almost the same speed, corresponds to the time of minimal distance between the two solitary waves. On the third picture, the two solitary waves after the collision go away one from another.

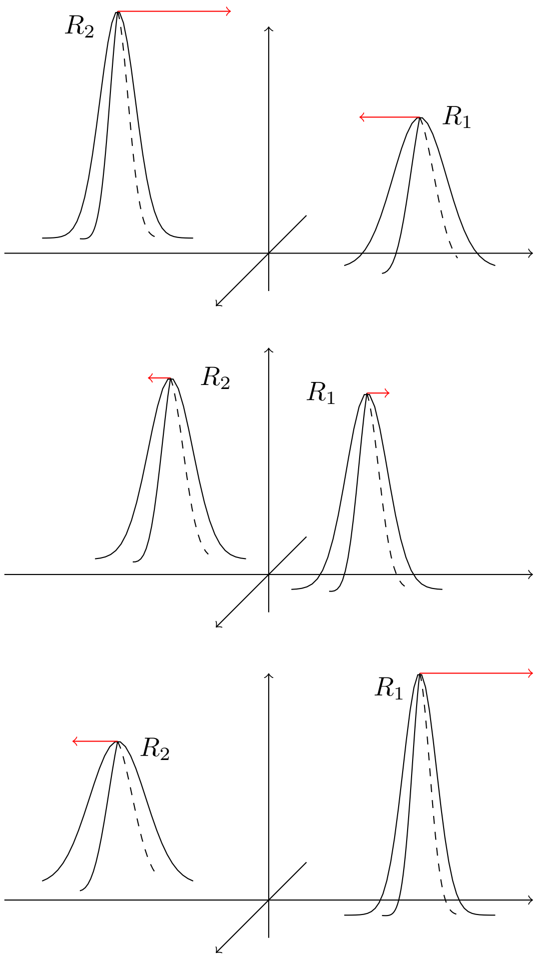

Figure 1. Schematic representations of a collision of two solitary waves with nearly equal speeds.

Six modulation parameters, that are time-dependent, are important to approximate the solution. For each solitary wave R_i with $i\in\{1,2\}$, we use to localize each wave some translations parameters

$$\boldsymbol{z}_i(t)=(z_i(t),\omega_i(t)),$$

as well as a scaling parameter $\mu_i(t)$ for their size. A modulated wave, whose modulation parameters will be chosen later, is defined by

$$R_i(t,\boldsymbol{x}):=Q_{1+\mu_i(t)}\left(\boldsymbol{x}-\boldsymbol{z}_i(t)\right).$$

Along the collision process, two functions are of main importance : the distance between the waves, denoted by

$$\boldsymbol{z}(t):=\boldsymbol{z}_1(t)-\boldsymbol{z}_2(t)=(z(t),\omega(t))$$

and the difference of sizes between the wave

$$\mu(t):=\mu_1(t)-\mu_2(t).$$

Along the collision, the variables $z(t)$ and $\mu(t)$ are approximated by a second order ordinary differential equation (ODE). This ODE takes its roots from the non-linear interaction between the two solitary waves and will be justified during the construction of the approximation. We define the ODE

$$\begin{cases}\ddot{Z}(t)=\dfrac{2}{\langle\Lambda Q,Q\rangle}\displaystyle\int_{\mathbb{R}^2}Q(x+Z(t),y)\partial_x(Q^2)(x,y)dxdy\\ (Z(0),\dot{Z}(0))=(Z_0,Y_0).\end{cases}\tag{4.1}$$

The modulation parameter $(z(t),\mu(t))$ behaves at the main order as the solution $(Z(t),\dot{Z}(t))$ to (4.1) for a particular initial data. Even if the solitary wave Q is not explicit, one can draw the phase portrait of this ODE and obtain Figure 2. In the setting of two solitary waves with scaling parameter $1+\mu_i(t)$ with μ_i small at any time, the trajectories of interest in the phase portrait are situated in the right-hand side having small values of $\dot{Z}$.

The ODE has a conserved Hamiltonian quantity defined by

$$H(Y_0,Y_1)=\frac{1}{2}Y_1^2+\frac{2}{\langle\Lambda Q,Q\rangle}\int_{\mathbb{R}^2}Q(x+Y_0,y)Q^2(x,y)dxdy,$$

so that if $\dot{Z}(0)=0$, $Z(0)>0$, one can relate the value of $\dot{Z}$ at $-\infty$ denoted by $-2\mu_0$ and the initial condition by the relation

$$(2\mu_0)^2=\frac{1}{\langle\Lambda Q,Q\rangle}\int_{\mathbb{R}^2}Q(x+Z_0,y)Q^2(x,y)dxdy.\tag{4.2}$$

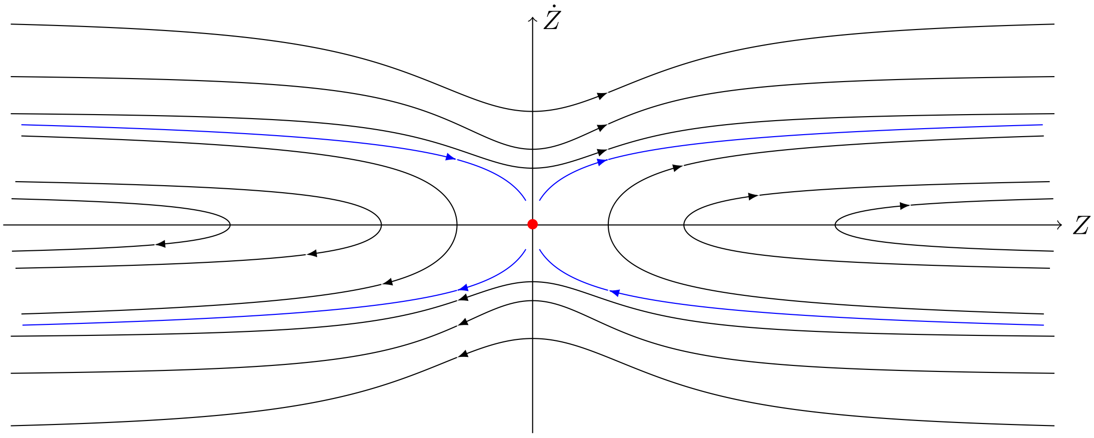

FIGURE 2. Phase portrait associated to (4.1). In red: the constant solution. In blue: the four solutions defined on $\mathbb{R}$ ending at $(0,0)$ when t goes to $-\infty$ or $+\infty$. In black: all the other trajectories.

It implies that among the trajectories that at time 0 have a null velocity $\dot{Z}(0) = 0$ and with $Z(0) > 0$, it is equivalent to choose the value of Z at time 0 and to choose the value of $\dot{Z}$ at $-\infty$. In our case, we thus choose $-2\mu_0$, which corresponds to the difference of sizes of the two solitary waves at $-\infty$, then define the unique $Z_0 > 0$ satisfying (4.2) and finally define the trajectory $(Z, \dot{Z})$ solution to the ODE.

With these tools in hand, we now state the main result.

Theorem 4.1. *There exists $\mu^\star > 0$ and $\kappa > 0$ such that the following holds. Let $\mu_0 \in (0, \mu^\star)$, $Z_0 > 0$ satisfying (4.2) and Z the unique solution to (4.1) with initial condition $(Z, \dot{Z})(0) = (Z_0, 0)$. Let $|\omega| \leqslant \mu_0$. Let v be the unique solution to ZK_t satisfying*

$$\lim_{t\to-\infty} \left\| v(t) - \sum_{i=1}^{2} Q_{1+(-1)^i\mu_0}\left(\cdot + \frac{(-1)^i}{2} Z(t), \cdot + \frac{(-1)^i}{2}\omega_0\right)\right\|_{H^1} = 0. \tag{4.3}$$

Then :

(1) ***Description of the global dynamics**. There exists $(z_i, \omega_i, \mu_i) \in \mathcal{C}^1(\mathbb{R} : \mathbb{R}^2 \times (0; +\infty))$ such that the function*

$$w(t) := v(t) - \sum_{i=1}^{2} Q_{1+\mu_i(t)}\left(\cdot - z_i(t), \cdot - \omega_i(t)\right)$$

satisfies for all $t \in \mathbb{R}$,

$$\|w(t)\|_{H^1} \leqslant \kappa\mu_0^{3/2}, \tag{4.4}$$

$$\sum_{i=1}^{2}\left(|\dot{z}_i(t) - \mu_i(t)| - |\dot{\omega}_i(t)| + \left|\mu_i(t) + \frac{(-1)^i}{2}\dot{Z}(t)\right|\right) \leqslant \kappa\mu_0^{3/2}, \tag{4.5}$$

$$z_1(t) - z_2(t) \geqslant \frac{1}{2} Z_0.$$

(2) ***Accurate description in the collision region.** Fix a small constant $\rho < \frac{1}{32}$ and define the unique positive time T_1 such that $Z(T_1) = \rho^{-1} Z_0$. It holds*

$$\forall t \in (-T_1; T_1),\ \forall i \in \{1, 2\},\ \left|z_i(t) + \frac{(-1)^i}{2} Z(t)\right| + \left|\omega_i(t) + \frac{(-1)^i}{2}\omega_0\right| \leqslant \kappa\mu_0^{1/2}. \tag{4.6}$$

(3) ***Long time behavior as** $t \to +\infty$. There exists μ_1^+, μ_2^+ in $\mathbb{R}_+$ such that $\mu_i(t) \to \mu_i^+$ as $t \to +\infty$. It also holds*

$$\lim_{t\to+\infty} \|w(t)\|_{H^1(x > \frac{99}{100}t)} = 0 \tag{4.7}$$

$$0 \leqslant \mu_1^+ - \mu_0 \leqslant \kappa\mu_0^2 \quad \textit{and} \quad 0 \leqslant -\mu_2^+ - \mu_0 \leqslant \kappa\mu_0^2.$$

Remark 4.2. The assumption at $-\infty$ ensures that v is a pure two-solitary waves as defined in Section 2. Indeed, one can prove that for a fixed value μ_0, there is a constant $l = l(\mu_0)$ such that $Z(t) = -2\mu_0 t + l + o_{t\to-\infty}(1)$. Another two-solitary waves with sizes $1\pm\mu_0$ at infinity with a difference of ordinates bounded by μ_0 falls also into the scope of this theorem by applying space and time translations.

Remark 4.3. Estimate (4.4) is not sharp in terms of powers of μ_0. By constructing an approximation of the solution v, one can observe from the proof that the next order term coming from the interaction is of order $Z_0\mu_0^2$.

Remark 4.4. The time T_1 is large in the sense that the interaction between the two solitary is small on $(-\infty; -T_1]$ and on $[T_1; +\infty)$: on those time intervals, the trajectories (z_i, ω_i) and the sizes μ_i of the waves are influenced at a small order by the interaction between the waves. Moreover, on the time interval $(-T_1; T_1)$ where the interaction has the most influence on the modulation parameters, the evolution of those parameters is quantified in (4.6) by the solution Z to the ODE (4.1). One can finally notice that the influence on the second direction is negligible at this order.

Remark 4.5. Estimate (4.7) comes from the asymptotic stability result as explained in Section 3. In particular, an integration of the estimates on $\dot{z}_i$ and $\dot{\omega}_i$ does not converge as t tends to $+\infty$. This argument is thus not sufficient to get an estimate on the translation parameters. However, the asymptotic stability result as given in Theorem 5.1 provides the convergence of the scaling parameters μ_i. Finally, this result is not sharp enough to conclude the elasticity or the inelasticity of the collision. This question is an on-going project.

We conclude this section with a stability result of the collision in a large region that contains the interesting part of the dynamics.

Theorem 4.6. *Let v and T_1 be as defined in Theorem 4.1. Let $\widetilde{v}_0 \in H^1(\mathbb{R}^2)$ satisfying for some $|\mathcal{T}| \leqslant T_1$,*

$$\|\widetilde{v}_0 - v(\mathcal{T})\|_{H^1} \leqslant Z_0^{-5}\mu_0^{3/2}.$$

The function $\widetilde{v}$ to (ZK$_t$) emanating from $\widetilde{v}(\mathcal{T}) = \widetilde{v}_0$ satisfies

$$\forall t \in [-T_1; T_1],\ \|\widetilde{v}(t) - v(t)\|_{H^1} \leqslant \kappa\mu_0^{3/2}.$$

5. Some elements of the proofs

The proof of the theorems 4.1 and 4.6 are inspired from the roadmap implemented by Martel and Merle in [17] for the collision of two solitary waves for the quartic Korteweg-de Vries equation. However, some new challenges arise for ZK. Let us emphasize two main differences: first, since the non-linearity is quadratic, the interaction between the two solitary waves is stronger than in the quartic case and is not localized around each of the solitary wave. We thus need to take into account the interaction between the waves and introduce an ansatz that is not localized around a particular wave. A second main challenge is that, as opposed to the one-dimensional case, the function Q is not explicit. It has several consequences on the proof : neither the construction of the ansatz nor the solution to the ODE (4.1) are explicit. New ingredients need to be introduced to circumvent these issues and to get a robust construction of the ansatz.

5.1. **Construction of the ansatz.** We first explain how to obtain an adequate ansatz. The goal is to find an approximation $V(t, \boldsymbol{x})$ of the solution v so that the error coming from the flow applied to V is small enough. In other words, we need to cancel at the main order the flow of the approximation defined by

$$\mathcal{E}_V = \partial_t V + \partial_x\left(\Delta V + V - V^2\right).$$

A first approximation of the solution along the whole time interval is given by the sum of the two solitary waves

$$R_1(t, \boldsymbol{x}) + R_2(t, \boldsymbol{x}) = Q_{1+\mu_1(t)}\left(\boldsymbol{x} - \boldsymbol{z}_1(t)\right) + Q_{1+\mu_2(t)}\left(\boldsymbol{x} - \boldsymbol{z}_2(t)\right),$$

where the modulation parameters $\boldsymbol{z}_i$ and μ_i will be fixed later. To get a valid approximation, we suppose that the modulation parameters stay in the regime with high values of z_i and small values of ω_i and μ_i:

$$1 \ll z_1, \quad |\omega_1| + |\mu_1| \ll 1 \text{ and } z_2 \ll -1, \quad |\omega_2| + |\mu_2| \ll 1.$$

By computing the time derivative of each modulated wave, we obtain

$$\partial_t R_i = \left(-\dot{z}_i, -\dot{\omega}_i, \dot{\mu}_i\right)^T \cdot \overrightarrow{MR_i} \text{ with } \overrightarrow{MR_i} = \left(\partial_x R_i, \partial_y R_i, \Lambda R_i\right)^T.$$

One can already notice the importance of three particular directions given by the vector $\overrightarrow{MR_i}$. The evolution of the parameters (z_i, ω_i, μ_i) is given by those intrinsic directions. Notice also that R_i satisfies the equation

$$\Delta R_i - (1+\mu_i)R_i + R_i^2 = 0.$$

Along the flow, this first order approximation gives

$$\mathcal{E}_{R_1+R_2+\varepsilon} = \sum_{i=1}^{2} (-\dot{z}_i + \mu_i, -\dot{\omega}_i, \dot{\mu}_i)^T \cdot \overrightarrow{MR_i} + \partial_t \varepsilon + \partial_x(2R_1R_2) + \partial_x\left(\Delta\varepsilon - \varepsilon + 2(R_1+R_2)\varepsilon + \varepsilon^2\right).$$

The main order error term in this approximation is given by $\partial_x(2R_1R_2)$. A usual technique to compensate this term is to inverse the linearized operator L from Theorem 2.1 around a solitary wave, if some orthogonality conditions are satisfied. However, since the non-linearity is quadratic, the term $-2R_1R_2$ is neither localized around R_1 nor around R_2. Indeed, one can compare with the one-dimensional case where the asymptotic expansion $Q(x) \sim Ke^{-|x|}$ gives, for $z_2 \ll -1$, $1 \ll z_1$ and $x \in [\frac{1}{2}z_2, \frac{1}{2}z_1]$

$$Q(x-z_1)Q(x-z_2) \sim K^2 e^{-(x-z_2)} e^{-(z_1-x)} = K^2 e^{-z_1+z_2}.$$

In other words, if the two solitary waves are far one from another, the quadratic interaction term is almost constant between the two waves and not localized around any of them. In the two-dimensional setting, the interaction is more complex, but it can be considered as being of order $z^{-1/2}e^{-z}$.

Coming back to the two-dimensional setting, instead of inverting the operator L, one can invert the operator $(-\Delta + 1)$. We thus define

$$F := (-\Delta+1)^{-1}(2\widetilde{R}_1\widetilde{R}_2) \text{ with } \widetilde{R}_i(t, \boldsymbol{x}) = Q(x - z_i(t), y).$$

We thus obtain the error along the flow

$$\begin{aligned}\mathcal{E}_{R_1+R_2+F+\varepsilon} = &\sum_{i=1}^{2} (-\dot{z}_i + \mu_i, -\dot{\omega}_i, \dot{\mu}_i)^T \cdot \overrightarrow{MR_i} + \partial_t F + \partial_t \varepsilon \\ &+ \partial_x\left(\Delta\varepsilon - \varepsilon + 2(R_1+R_2)\varepsilon + 2(R_1+R_2)F + (F+\varepsilon)^2\right) + \partial_x\left(2R_1R_2 - 2\widetilde{R}_1\widetilde{R}_2\right).\end{aligned}$$

The last term on the right-hand-side of the former identity is considered small in term of ω_i and μ_i. The new interaction term is now $\partial_x(2(R_1+R_2)F)$. However, even though the size of this term is the same as $2R_1R_2$, thus of order $z^{-1/2}e^{-z}$, it is now localized around R_1 and around R_2. We then adjust the ansatz to get on this interaction term three natural orthogonality conditions, related to the modified energy in Section 5.3.

These adjustments introduce some new terms on the evolution of the modulation parameters. We thus need to focus on the intrinsic directions $\overrightarrow{MR_i}$ and more specifically the approximate form

$$\overrightarrow{M\widetilde{R}_i} := (\partial_x \widetilde{R}_i, \partial_y \widetilde{R}_i, \Lambda \widetilde{R}_i)^T.$$

Inspired by the previous computation using the inversion of the operator $(-\Delta+1)$, we define for $i \in \{1,2\}$, the functions

$$\begin{aligned}&\Lambda\widetilde{R}_i(t,x,y) := \Lambda Q(x - z_i(t), y),\\ &X_i := -(\Delta+1)^{-1}\widetilde{R}_i,\ Y_i := -\partial_x^{-1}(-\Delta+1)^{-1}\partial_y\widetilde{R}_i,\ W_i := -\partial_x^{-1}(-\Delta+1)^{-1}\Lambda\widetilde{R}_i,\\ &\overrightarrow{N\widetilde{R}_i} := (X_i, Y_i, W_i),\end{aligned}$$

where

$$\partial_x^{-1} f(x,y) = -\int_x^{+\infty} f(\widetilde{x}, y)\, d\widetilde{x}.$$

We thus obtain by definition that

$$\partial_x(-\Delta+1)\overrightarrow{N\widetilde{R}_i} = -\overrightarrow{M\widetilde{R}_i}.$$

An important remark is that the functions Y_i and W_i may not vanish as x goes to $-\infty$. We denote the tail of $\overrightarrow{N\widetilde{R}_i}$ as x goes to $-\infty$ by $\overrightarrow{n}(y)$. The final ansatz is thus written in the form

$$V = R_1 + R_2 + F + \overrightarrow{p}_1 \cdot \overrightarrow{N\widetilde{R}_1} + \overrightarrow{p}_2 \cdot \overrightarrow{N\widetilde{R}_2} \tag{5.1}$$

where the functions $\overrightarrow{p}_1$ and $\overrightarrow{p}_2$ only depend on $z = z_1 - z_2$. To obtain those parameters, we compute the flow of the ansatz V and ask for some orthogonality conditions. Since the dispersion goes from right to left, we start around the first solitary wave and ask for the orthogonality conditions

$$2\widetilde{R}_1(F + \overrightarrow{p_1} \cdot \overrightarrow{N\widetilde{R}_1}) \perp \left\{\partial_x \widetilde{R}_1,\ \partial_y \widetilde{R}_1,\ \Lambda \widetilde{R}_1\right\}. \tag{5.2}$$

We obtain the vector $\overrightarrow{p_1}$ whose first coordinate is small, the second vanishes and the third coordinate is

$$\gamma_1(z) = -\frac{1}{\langle \Lambda Q, Q\rangle} \left\langle Q\left(\cdot + z, \cdot\right), \partial_x(Q^2)\right\rangle.$$

We continue with the term around the second solitary wave. Since the tail of $\overrightarrow{N\widetilde{R}_1}$ may not vanish, the orthogonality conditions around R_2 are thus

$$2\widetilde{R}_2(F + \overrightarrow{p_2} \cdot \overrightarrow{N\widetilde{R}_2} + \overrightarrow{p}_1 \cdot \overrightarrow{n}) \perp \left\{\partial_x \widetilde{R}_2,\ \partial_y \widetilde{R}_2,\ \Lambda \widetilde{R}_2\right\}. \tag{5.3}$$

We find in particular that the first coordinate is small, the second coordinate vanishes and the third one is (after computations)

$$\gamma_2(z) = -\frac{1}{\langle \Lambda Q, Q\rangle} \left\langle Q(\cdot - z, \cdot), \partial_x(Q^2)\right\rangle.$$

The flow can now be written as

$$\mathcal{E}_V = \sum_{i=1}^{2} \overrightarrow{m_i} \cdot \overrightarrow{MR_i} + \partial_x S + T \text{ and } \overrightarrow{m_i} = (-\dot{z}_i + \mu_i, -\dot{\omega}_i, \dot{\mu}_i) - \overrightarrow{p}_i,$$

where the term S satisfies approximate orthogonality relations, and T is small.

One can notice that even though we imposed the orthogonality conditions useful to invert the operators L around R_1 and R_2, we do not invert this operator but cancel the error in the three intrinsic directions given by $\overrightarrow{M\widetilde{R}_i}$. These orthogonality conditions will turn out to be crucial when dealing with the error estimates, see Section 5.3.

Finally, even though V in (5.1) is defined from the functions W_i that may non-vanish as x goes to $-\infty$, from the choice of $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$, those tails compensate and V vanishes as $|\boldsymbol{x}|$ tends to $+\infty$.

5.2. **Modulation parameters and ODE system.** Recall that the function V is defined in (5.1). To choose the parameters $\boldsymbol{z}_i$ and μ_i, we use the classical modulation theory. We define a tubular neighborhood $\mathcal{U}$ of two solitary waves with small values of ω_i and μ_i, and large values of z_1 and $-z_2$. For any function $w \in \mathcal{C}(I : H^1(\mathbb{R}^2))$ with values in $\mathcal{U}$, there is a unique function $\Gamma = (z_1, z_2, \omega_1, \omega_2, \mu_1, \mu_2) : t \mapsto \Gamma(t)$ defined on I such that $\varepsilon := w - V(\Gamma)$ satisfies the six orthogonality conditions, with $i = 1, 2$,

$$\int \varepsilon \partial_x R_i = \int \varepsilon \partial_y R_i = \int \varepsilon R_i = 0.$$

Moreover, by computing the time derivatives of the orthogonality conditions, we obtain the modulation estimates for any $t \in I$,

$$\sum_{i=1}^{2} \left(|-\dot{z}_i + \mu_i| + |\dot{\omega}_i|\right) \lesssim \|\partial_x S\|_{L^2} + \|T\|_{L^2} + \|\varepsilon\|_{L^2}, \tag{5.4}$$

$$\sum_{i=1}^{2} |\dot{\mu}_i + \gamma_i(z)| \lesssim \sum_{i=1}^{2} \left|\int S \partial_x R_i\right| + \|T\|_{L^2} + \left(e^{-\frac{15}{16}z} + \|\varepsilon\|_{L^2}\right)\|\varepsilon\|_{L^2}. \tag{5.5}$$

These estimates correspond to the approximate ODE system as given in (4.1). Remark that estimate (5.5) is more precise than the one of (5.4) due to the quadratic order in $\|\varepsilon\|_{L^2}$ and the orthogonality conditions on S in (5.2)–(5.3).

5.3. **Energy estimates.** Inspired by [17] we need to estimate the error along the flow between the solution v and the ansatz V. The error term $\varepsilon(t) = v(t) - V(\Gamma(t))$ satisfies

$$\partial_t \varepsilon + \partial_x \left(\Delta \varepsilon - \varepsilon + (V + \varepsilon)^2 - V^2\right) = -\mathcal{E}_V = -\left(\sum_{i=1}^{2} \overrightarrow{m_i} \cdot \overrightarrow{MR_i} + \partial_x S + T\right).$$

To control the H^1-norm of ε, we make use of two adequate energy functionals. They are defined by

$$\mathcal{F}_+ := \int \left(\frac{|\nabla\varepsilon|^2}{2} + \frac{\varepsilon^2}{2} - \frac{1}{3}\left((V+\varepsilon)^3 - V^3 - 3V^2\varepsilon\right)\right) + \int \frac{\varepsilon^2}{2}\psi_+ - \int S\varepsilon,$$
$$\mathcal{F}_- := \int \left(\frac{|\nabla\varepsilon|^2}{2} + \frac{\varepsilon^2}{2} - \frac{1}{3}\left((V+\varepsilon)^3 - V^3 - 3V^2\varepsilon\right)\right)\psi_{-,e} + \int \frac{\varepsilon^2}{2}\psi_{-,m} - \int S\varepsilon.$$

We use $\mathcal{F}_-(t)$ if the second modulated wave is larger than the first wave, which means $\mu_1(t) < \mu_2(t)$, and $\mathcal{F}_+(t)$ if $\mu_2(t) < \mu_1(t)$.

On one side, by imposing adequate orthogonality conditions on ε by adjusting the modulation parameters, we obtain a coercivity result on the functionals. On the adequate time intervals, we obtain

$$\|\varepsilon(t)\|_{H^1}^2 \lesssim \mathcal{F}_+(t) + \text{l.o.t.}, \quad \|\varepsilon(t)\|_{H^1}^2 \lesssim \mathcal{F}_+(t) + \text{l.o.t.}$$

On the other side, one can control the time derivative of $\mathcal{F}_+$. Indeed, the derivative of the first integral corresponds to the equation satisfied by ε and only two terms remain:

$$I_1 := \int \partial_t(R_1+R_2)\varepsilon^2 \simeq -\sum_{i=1}^{2}\mu_i \int \partial_x R_i \varepsilon^2 \quad \text{and} \quad I_2 := \int \partial_x S(-\Delta + 1 - 2(R_1+R_2))\varepsilon.$$

To compensate the contribution of I_1, we add in $\mathcal{F}_+$ a weighted mass $\int \varepsilon^2\psi_+$. The increasing function ψ_+ is a weight function equal to μ_1 where R_1 is localized, and to μ_2 where R_2 is localized. The other main-order terms of the time derivative of $\int \varepsilon^2\psi_+$ has a good sign to get an upper bound on $\frac{d}{dt}\mathcal{F}_+(t)$.

For the contribution of I_2, we introduce in $\mathcal{F}_+$ the linear term $\int S\varepsilon$. It will first compensate this bad contribution, but also add a limiting source term $\int S\overrightarrow{m_i}\cdot\overrightarrow{MR_i}$ and $\int ST$. In other words, with this linear term in the functional, the limiting term in the bound of $\frac{d}{dt}\mathcal{F}_+(t)$ comes from the approximation V and can only be improved by using a more accurate V. We finally deduce the upper bound on $\mathcal{F}_+$. This argument was not used in the collision setting of [17], we refer to [21] for a similar idea in another context.

Whilst the first solitary wave is smaller than the second $\mu_1(t) < \mu_2(t)$, the computation of the time derivative of $\mathcal{F}_-$ is similar up to a difference. When computing the time derivative of the first integral, we use once again a weighted mass $\int \varepsilon^2\psi_{-,m}$ to compensate the bad term. However, in this case, the weight function $\psi_{-,m}$ equal to μ_2 close to the support of R_2 and to μ_1 close to the support of R_1 is decreasing. The time derivative of $\int \varepsilon^2\psi_{-,m}$ gives some terms with a bad sign, we thus add another weight function $\psi_{-,e}$ to control those terms.

We deduce from the previous discussion some good upper bounds of $\frac{d}{dt}\mathcal{F}_+$ and $\frac{d}{dt}\mathcal{F}_-$.

5.4. **From $-\infty$ to $-T_1$.** When the two solitary waves are far enough one from another, the influence of the two waves is considered small compared to the rest of the collision. We introduce the unique large positive time T_1 such that, for a small constant ρ to fix later,

$$Z(T_1) = \rho^{-1}Z_0.$$

Because of the weak interaction between the waves on the time interval $(-\infty;-T_1]$, we approximate v at time $-T_1$ with the two waves and a small error, taking ρ small enough: for $i=1,2$, there exist $\boldsymbol{z}_i^1 \in \mathbb{R}^2$ and μ_i^1 such that the following holds

$$\left\| v(-T_1) - \sum_{i=1}^{2} Q_{1+\mu_i^1}\left(\cdot - \boldsymbol{z}_i^1\right)\right\|_{H^1} \lesssim \mu_0^{10},$$
$$\sum_{i=1}^{2}\left|\boldsymbol{z}_i^1 + \frac{(-1)^i}{2}(Z(-T_1),\omega_0)\right| + \left|\mu_i^1 - (-1)^i\mu_0\right| \lesssim \mu_0^{10}.$$

A novel tool was introduced to obtain those estimates at time $-T_1$. It relies on a balance on two intervals $(-\infty;-T_0]$ and $[-T_0;-T_1]$, with an adequate T_0. First, we obtain from the exponentially decaying error on the multi-solitary waves in Theorem 2.3 on $(-\infty,T_0]$, that the error at time T_0 is bounded by $Ce^{-\delta T_0}$. Then, on the time interval $[-T_0,-T_1]$, we use a bootstrap argument to integrate this initial error and obtain at time T_1 a bound on the error of order $C|T_1-T_0|e^{-\delta T_0}$. It thus remains to choose T_0 large enough such that the above estimates are satisfied at time $-T_1$.

Note that this argument still holds if the bound in Theorem 2.3 is of order $|t|^{-1^+}$ instead of $e^{\delta t}$.

5.5. **The bootstrap estimates on** $[-T_1;T_1]$**.** The time interval $[-T_1;T_1]$ concentrates the main changes of the waves along the time. Since the error is expected to be small, the modulation theory provides the existence of a function $\Gamma \in \mathcal{C}^1([-T_1;T_1]:\mathbb{R}^6)$ such that the orthogonality conditions on $\varepsilon = v - V$ are satisfied. We use a succession of bootstrap arguments to control the evolution of the error ε and of the modulation parameter Γ.

We split the interval into three intervals $[-T_1;-T_2]$, $[-T_2;T_2]$ and $[T_2;T_1]$. The positive time T_2 is close to 0 and satisfies $Z(T_2) = Z_0 + \eta^2$, with η to fix later. On each interval, we use a bootstrap on the error ε and on Γ. The upper bound is given either in term of the function $Z(t)$ or with the fixed small constant μ_0. For the sake of clarity, we do not provide in this note the bootstraps of Section 4.2 in [29], but give the main ideas to control the different quantities.

5.5.1. *On the time interval* $[-T_1;-T_2]$. On $[-T_1;-T_2]$, from the definition of T_2 and of Z, it holds for any $t < -T_2$, $-\dot{Z}(t) \gtrsim \eta\mu_0$. We control ε by integrating the upper bound of $\frac{d}{dt}\mathcal{F}_-(t)$ and use the coercivity of $\mathcal{F}_-(t)$. Since Z is not explicit, we integrate the upper bound of $\frac{d}{dt}\mathcal{F}_-(t)$ by multiplying by $-\eta^{-1}\mu_0^{-1}\dot{Z}(t)$.

To control the modulation parameters z_1+z_2 and $\mu_1+\mu_2$, we use a direct integration of the modulation equations (5.4)-(5.5).

However, a direct integration of these equations for $z := z_1 - z_2$ and $\mu := \mu_1 - \mu_2$ does not close the estimate. Since (z,μ) satisfies the approximated ODE in (5.4)-(5.5), it is expected to behave as $(Z,\dot{Z})$ and a new perspective is to follow the trajectory in the phase portrait given in Figure 2 (see Appendix D of [29]). Since the ODE system is approximated, we consider the trajectory in two modified phase portraits associated with two hamiltonian H_- and H_+, so that two curves $(Z_-,\dot{Z}_-)$ and $(Z_+,\dot{Z}_+)$ give in their respective phase portraits some lower and upper bounds on the trajectory (z,μ). In other words, we can estimate the trajectories $Z_-^{-1}\circ z$ and $Z_+^{-1}\circ z$ to control the evolution of z along the time by the approximation from the ODE system and the initial condition $z(-T_1)$. This idea is a generalization to the one of [17] where the function Z is explicit.

Finally, it remains to control the evolution of ω_1 and ω_2. An integration of the modulation equations is not enough in that case. To get a more refined bound on the evolution of those parameters, we introduce a functional $\mathcal{K}_i$, adapted from the functional $\mathcal{J}$ in [17] to the transverse direction and defined by

$$\mathcal{K}_i(t) := \int_{\mathbb{R}^2} \varepsilon(t,x,y) \int_{-\infty}^{x} \partial_y R_i(t,\tilde{x},y) d\tilde{x} \chi(\mu_0 x)\, dxdy$$

where χ corresponds to a cut-off function on the right. This functional captures the dispersion influencing the evolution of ω_i. We compute then the time derivatives and get for a non-negative constant c_Q,

$$\frac{d}{dt}\mathcal{K}_1 \simeq c_Q\dot{\omega}_1 \quad \text{and} \quad \frac{d}{dt}\mathcal{K}_2 \simeq c_Q\dot{\omega}_1 + 2c_Q\dot{\omega}_2.$$

By a quantification of the error, we integrate this error to get the estimates of $\omega_1(t)$ and of $\omega_2(t)$.

With this computation we conclude the bootstrap arguments on the time interval $[-T_1;-T_2]$.

5.5.2. *On the time interval* $[-T_2;T_2]$. Since $\dot{Z}$ vanishes on this interval, we cannot integrate the different estimates on this time interval by multiplying by $-\dot{Z}(t)\rho^{-1}\mu_0^{-1}$. We need to adjust the method by using the higher derivative of Z. Since Z is not explicit, we circumvent the issue by giving the lower bound $\ddot{Z}(t) \gtrsim \mu_0^2$ and we multiply the estimates by $\ddot{Z}(t)\mu_0^{-2}$. This allows us to integrate the bounds on the time derivatives of the different terms.

A second challenge is that the "switching time" t_0, such that $\mu_1(t_0) = \mu_2(t_0)$, is in the time interval $[-T_2,T_2]$. We thus control the H^1-norm of the error term ε by using the functional $\mathcal{F}_-$ on $[-T_2,t_0]$ and the functional $\mathcal{F}_+$ on $[t_0;T_2]$.

We finally control the modulation parameters by integrating the estimates on $\frac{d}{dt}\mathcal{K}_i$ or the approximate ODE system (5.4)-(5.5). There is no need to use the phase portrait on this step, we use instead the smallness of the time interval.

Note finally that to close the different bootstraps, we make use in this step of the choice of the small constant η that depends on some universal constants but not on Z_0.

5.5.3. *On the time interval* $[T_2;T_1]$. The control of the estimates on $[T_2;T_1]$ differs from the ones on $[-T_1;-T_2]$ by two main points. First, the control of the error ε is made by using the functional $\mathcal{F}_+$ instead of $\mathcal{F}_-$. The control of z_1+z_2, $\mu_1+\mu_2$, ω_1 and ω_2 involves the same method as before. Then, since the bounds in the bootstraps do not depend on $Z(t)$ anymore but on the constant coefficient μ_0, we need to integrate a constant over the interval $[T_2;T_1]$. We thus split the interval into two other intervals, $[T_2;T_3]$ and $[T_3;T_1]$. On $[T_2;T_3]$, we control the evolution of (z,μ) by following the trajectory in the phase portrait. On $[T_3,T_1]$, we use a direct integration of the ODE approximated system and adjust the time T_3 to fit in the bootstrap's constants.

As a conclusion, the bootstrap's estimates provide at time T_1

$$\begin{aligned}&|z(T_1)-Z(T_1)|+|\omega(T_1)-\omega_0|\lesssim \mu_0^{1/2},\\&\|\varepsilon(T_1)\|_{H^1}\lesssim \mu_0^{3/2}.\end{aligned}$$

5.6. **From T_1 to $+\infty$ and stability results.** We end this section by giving an orbital stability result adequate to our context. From the estimates at time T_1 in hand, we can apply the following quantified stability theorem. Notice that this result relies on a quantification of the smallness of the error and the minimal distance between the waves in terms of the difference of size of the solitary waves.

Theorem 5.1 ([28], Theorem 1.1)**.** *There exist $\mu^\star>0$ and three positive constants k, K and A such that the following holds. Let $\mu_0\in(0;\mu^\star)$ and two positive constants α and Z satisfying*

$$\alpha\leqslant k\mu_0 \quad \textit{and} \quad Z\geqslant K|\ln(\mu_0)|.$$

Let $\boldsymbol{z}_i^0=(z_i^0,\omega_i^0)\in\mathbb{R}^2$ for $i\in\{1,2\}$, $u_0\in H^1(\mathbb{R}^2)$ satisfying

$$\left\|u_0-Q_{1+\mu_0}(\cdot-\boldsymbol{z}_1^0)-Q_{1-\mu_0}(\cdot-\boldsymbol{z}_2^0)\right\|_{H^1}\leqslant\alpha \quad \textit{with} \quad z_1^0-z_2^0>Z,$$

and $u\in\mathcal{C}(\mathbb{R}:H^1(\mathbb{R}^2))$ be the solution to ZK *emanating from u_0.*

There exist some functions $(z_i,\omega_i,\mu_i)\in\mathcal{C}^1(\mathbb{R}_+:\mathbb{R}^2\times(0;+\infty))$, for $i\in\{1,2\}$ such that for all $t\geqslant0$,

$$\begin{aligned}&\left\|u(t)-Q_{1+\mu_1(t)}\left(\cdot-(z_1(t),\omega_1(t))\right)-Q_{1-\mu_2(t)}\left(\cdot-(z_2(t),\omega_2(t))\right)\right\|_{H^1}\leqslant A\left(\alpha+e^{-\frac{1}{32}\sqrt{1-\mu^\star}Z}\right);\\&|\dot z_1(t)-\mu_1(t)|+|\dot z_2(t)-\mu_2(t)|+|\dot\omega_1(t)|+|\dot\omega_2(t)|\leqslant A\left(\alpha+e^{-\frac{1}{32}\sqrt{1-\mu^\star}Z}\right);\\&z(t)=z_1(t)-z_2(t)\geqslant\tfrac12(Z+\sigma t);\\&|\mu_1(t)-\mu_1^0|+|\mu_2(t)-\mu_2^0|\leqslant A\alpha.\end{aligned}$$

Furthermore, the functions μ_1 and μ_2 admit a limit at infinity denoted by μ_1^+ and μ_2^+. It also holds

$$\begin{aligned}&|\mu_1^+-\mu_0|+|\mu_2^++\mu_0|\leqslant A\left(\alpha+e^{-\frac18\sqrt{1-\mu^\star}Z}\right);\\&\lim_{t\to+\infty}\left\|u(t)-Q_{1+\mu_1(t)}(\cdot-\boldsymbol{z}_1(t))-Q_{1+\mu_2(t)}(\cdot-\boldsymbol{z}_2(t))\right\|_{H^1(x>\frac{1}{100}(1-\mu^\star)t)}=0;\\&(\dot z_i(t),\dot\omega_i(t))\underset{t\to+\infty}{\longrightarrow}(\mu_i^+,0),\quad i=1,2.\end{aligned}$$

This result concludes the proof of Theorem 4.1 by choosing $\alpha=\mu_0^{3/2}$.

The proof of the stability of the collision in Theorem 4.6, follows the same line as that for Theorem 4.1. We compare each solution to the profile V. However, this result of stability needs to be restricted to a finite time interval, since the orbital stability result only provides a uniform upper bound on $\dot{\boldsymbol{z}}_i$ and μ_i but not on $\boldsymbol{z}_i$.

We end this section by giving a few ideas on the proof of Theorem 5.1. This theorem is inspired from [20, 2]. First, we give an orbital stability result that is quantified in terms of the error at the initial time. It involves in particular an Abel summation argument as in [20], a typical tool for the study of multi-solitary waves. From this result, we prove the asymptotic stability result by making use of a Liouville type argument obtained in [2] and using the almost-monotonicity of the local mass and local energy. This result differs from the asymptotic stability result for the KdV-type equations in [20] by the dispersive nature of the equation in the transverse direction. To this end, we use some geometric arguments introduced in [2] to control the direction of the dispersion, as explained in (1.3).

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