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SINGULARITIES AND TANGENT CONES FOR AREA-MINIMIZING AND SEMICALIBRATED CURRENTS

ANNA SKOROBOGATOVA

Abstract. The Plateau problem asks: which are the surfaces of least m-dimensional area spanning a given $(m-1)$-dimensional boundary? To guarantee existence of minimizers and desirable compactness properties for sequences of surfaces, one must consider a weak notion of surface, thus allowing for area-minimizing "surfaces" to have singularities. A particularly natural framework for this problem is via integral currents, allowing for surfaces to have integer multiplicities. The Plateau problem has been studied in great depth in this setting since the 1950s, pioneered by works of De Giorgi, Federer & Fleming, Almgren, White and built upon by many others. We present the history of the problem and some recent breakthroughs in the regularity theory, together with the uniqueness of blow-ups, for area-minimizing surfaces in this framework. We additionally demonstrate that semicalibrated integral currents, which are a natural subclass of almost area-minimizers in this framework, exhibit the same regularity and structural properties as area-minimizers. This is based on a series of joint works with Camillo De Lellis and Paul Minter, and a joint work with Paul Minter, Davide Parise and Luca Spolaor.

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The Plateau problem, posed by Lagrange in the 1700s, can be loosely stated as follows.

Problem 1 (Plateau Problem)**.** Given a "boundary" Γ^{m-1} in a smooth Riemannian manifold (M^{m+n}, g), what are the m-dimensional "surfaces" of least surface area "spanning" Γ?

For the purpose of simplicity, we will assume that (M^{m+n}, g) is the Euclidean space $(\mathbb{R}^{m+n}, \delta)$, but all results discussed herein hold true in a Riemannian manifold (M^{m+n}, g) provided that the metric g is sufficiently regular. Indeed, the presence of the ambient manifold (or non-Euclidean metric) merely introduces well-behaved error terms in various estimates, which may be dealt with accordingly. The first fundamental question concerns the natural space in which to make this minimization problem well-posed. Once cannot hope to minimize over all smooth, oriented surfaces with multiplicity one that have a fixed boundary, due to a failure of compactness via phenomena such as disappearance of mass through long, thin "tentacles", and either cancellation or formation of higher multiplicities when sheets of the surface collapse together. Thus, it is natural to relax to a weaker framework in which "surfaces" have higher multiplicities and singularities.

The first such framework was considered by Douglas & Radó in 1930 (see [49, 30]), in which they studied parametric two-dimensional area-minimizing surfaces in $\mathbb{R}^3$ that are topological discs. This approach, however, is very specific to two-dimensional surfaces and constrains the topology of the surface. For surfaces of arbitrary dimension and topological type, there are two main approaches: the set-theoretic framework, originally considered by Reifenberg in [50],[1] and the functional analytic framework that uses *currents*, pioneered by De Giorgi and Federer & Fleming in the 1950s-1960 (see [10, 32]).

We focus herein on the latter framework. An *m-dimensional current* in $\mathbb{R}^{m+n}$ is an element of $(C_c^\infty(\mathbb{R}^{m+n}; \Lambda^m))^*$; namely, dual to the space of smooth compactly supported m-forms on $\mathbb{R}^{m+n}$. In particular, any smooth, oriented submanifold canonically identifies with a current:

Example 2 (Smooth oriented surfaces)**.** Let $\Sigma^m \subset \mathbb{R}^{m+n}$ be a smooth, oriented submanifold. Then one may associate to Σ the m-dimensional current $[\![\Sigma]\!]$ that acts on m-forms via integration:

$$[\![\Sigma]\!](\omega) := \int_\Sigma \omega$$

[1] See also [42], where the now frequently used notion of $\mathcal{C}$-*spanning* is first developed.

The boundary ∂T of an m-dimensional current T is defined by generalizing Stokes' Theorem:

$$\partial T(\omega) := T(d\omega)\,.$$

Since we wish to work with objects that provide a suitable weak generalization of smooth, oriented submanifolds that in particular allows for higher multiplicities, we restrict ourselves to the subclass of *integral currents*.

Definition 3 (Integral currents)**.** An *m-dimensional integral current* T is an m-dimensional current such that[2]

$$T(w) = \sum_{i\in\mathbb{N}} k_i \int_{\Sigma_i} \omega\,, \qquad \Sigma_i \overset{\text{closed}}{\subset} \mathcal{M}_i^m \text{ oriented } C^1 \text{ submanifolds of } \mathbb{R}^{m+n}\,, \tag{1}$$

and

$$\partial T(\sigma) = \sum_{i\in\mathbb{N}} k_i \int_{\Gamma_i} \sigma\,, \qquad \Gamma_i \overset{\text{closed}}{\subset} \mathcal{N}_i^{m-1} \text{ oriented } C^1 \text{ submanifolds of } \mathbb{R}^{m+n}\,.$$

The area (or mass) of an m-dimensional current can thus be naturally defined via duality:

$$\mathbf{M}(T) = \|T\|(\mathbb{R}^{m+n}) := \sup\{|T(\omega)| : \omega \in C_c^\infty(\mathbb{R}^{m+n};\Lambda^m),\ \|\omega\|_c \leqslant 1\}\,,$$

where $\|T\|$ is the mass measure associated to T given by the Riesz Representation Theorem, and $\|\omega\|_c$ denotes the comass norm of ω; see e.g. [11]. Observe that for an integral current T given by (1), we have

$$\mathbf{M}(T) = \sum k_i \mathrm{Vol}_{\mathcal{M}_i}(\Sigma_i)\,,$$

and we implicitly assume that the latter sum is finite as part of our definition. We thus say that an area-minimizing integral current T^m in is *area-minimizing* if[3]

$$\mathbf{M}(T) \leqslant \mathbf{M}(S) \qquad \text{for every integral current } S^m \text{ of the form } S = T + \partial R\,.$$

An important subclass of area-minimizing integral currents are *calibrated currents*:

Definition 4 (Calibrated currents)**.** An integral current T^m is *calibrated* if there exists a C^1 m-form ω with

(i) $d\omega = 0$;
(ii) $\|\omega\|_c \leqslant 1$;
(iii) $\mathbf{M}(T) = T(\omega)$.

A simple consequence of the calibrated condition is that any such T is area-minimizing. Indeed, given any competitor $S = T + \partial R$, we have

$$\mathbf{M}(T) = T(\omega) = (S + \partial R)(\omega) = S(\omega) + \underbrace{R(d\omega)}_{=0} \leqslant \mathbf{M}(S)\,. \tag{2}$$

In practice, area-minimizing currents are hard to find, and the main method one has in order to demonstrate that a particular example is area-minimizing is via demonstrating that it is in fact calibrated. In general, it is not known how prevalent the class of calibrated currents is within the entire class of integral currents, but the work [46] provides promising evidence for the fact that there are many non-calibratable area-minimizing integral currents. More precisely, therein, the author demonstrates that in a general smooth ambient manifold, there exists a non-empty, open set of metrics with respect to which any homologically area-minimizing integral current cannot be calibrated.

Notice that the particularly rigid condition in the calibrated definition is (i). Indeed, any C^1 oriented submanifold satisfies the assumptions (ii) and (iii) with ω taken to be its volume form. With this in mind, a natural way to relax the notion of being calibrated is as follows:

Definition 5 (Semicalibrated currents)**.** An integral current T^m is *semicalibrated* if there exists a C^1 m-form ω with $\|\omega\|_c \leqslant 1$ and $\mathbf{M}(T) = T(\omega)$.

[2] Equivalently, both T and ∂T have rectifiable support (of dimension m and $m-1$ respectively) and the tangent plane that exists at almost-every point comes with integer multiplicity and an orientation, acting on forms via integration.

[3] In ambient Euclidean space, this condition is equivalent to saying that $\partial S = \partial T$.

For a semicalibrated integral current T^m, the mass comparison (2) (localized) becomes

$$\begin{aligned}\|T\|(\mathbf{B}_r(p)) &= (T \llcorner \mathbf{B}_r(p))(\omega) \\ &= (S \llcorner \mathbf{B}_r(p))(\omega) + (R \llcorner \mathbf{B}_r(p))(d\omega) \\ &\leqslant \|S\|(\mathbf{B}_r(p)) + \|d\omega\|_\infty \|R\|(\mathbf{B}_r(p)) \\ &\leqslant \|S\|(\mathbf{B}_r(p)) + C\|d\omega\|_\infty r^{m+1}\,,\end{aligned}$$

where in the last estimate we are using the fact that R is an $(m+1)$-dimensional integral current. In particular, this estimate tells us that a semicalibrated current is *almost area-minimizing* (see e.g. [8] for the latter notion).

Note that area-minimizing does not imply semicalibrated, and vice versa, but calibrated currents are contained in the intersection of the two notions. Calibrated and semicalibrated currents appear naturally in various geometric contexts, particularly related to mirror symmetry in string theory, and more generally in gauge theory; see for instance [43, 38, 57, 40, 39, 44, 56]. The regularity theory discussed herein is thus a frequently exploited tool; a particular instance of this appears in the works [29, 28] relating to genus bounds for pseudoholomorphic curves in almost-complex manifolds, and the characterization of certain invariants in symplectic manifolds. In particular, semicalibrated currents are much more flexible with respect to deformations, unlike calibrated ones, making them more useful to work with in various geometric settings.

Two fundamental questions within the Plateau problem are

(a) Do minimizers exist?
(b) How regular are minimizers in general?

Thanks to the fundamental work [32] of Federer & Fleming, we have existence of minimizers in the class of integral currents, for any given boundary $\Gamma = \partial S$ of an integral current S (of any dimension and codimension).

On the other hand, the question of regularity is much more delicate, with many questions remaining open until recent years. Since our notion of area-minimizing surface is an integral current, singularities are permitted, both in the interior and at the boundary of the current. These are points around which T^m is not locally supported in any smooth m-dimensional submanifold.

Here we focus on the *interior* partial regularity, namely locally around points in $\mathrm{spt}T\backslash\mathrm{spt}(\partial T)$. See [1, 3, 41, 14, 15, 33, 35, 34] for the state of the art on boundary regularity. We will see in addition that the more flexible class of semicalibrated currents enjoy the same regularity properties as area-minimizing integral currents.

Let us begin with some examples, all of which are in fact calibrated.

Example 6 (Simons' cone)**.** The cone

$$C_{3,3} := \{(x,y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 = |y|^2\}$$

is a 7-dimensional calibrated cone in $\mathbb{R}^8$ with an isolated singularity at the origin. $C_{3,3}$ was proven to be area-minimizing by Bombieri, De Giorgi & Giusti in [7] (see also [27] for a simplified proof). One can easily verify that their proof in fact yields the calibrated property.

While Simons' cone yields a codimension one example, we additionally have a whole collection of calibrated examples that are codimension two:

Example 7 (Holomorphic curves)**.** Any holomorphic curve in $\mathbb{C}^2 \cong \mathbb{R}^4$ is calibrated with the suitably normalized Kähler form $\omega = c_0(dx^1 \wedge dy^1 + dx^2 + dy^2)$, where $x_1 + iy_1$, $x_2 + iy_2$ denote the two pairs of complex variables, and c_0 is chosen so that $\|\omega\|_\infty \leqslant 1$. For instance, the current induced by the branched variety

$$\{(w,z) \in \mathbb{C}^2 : w^2 = z^3\}\,, \tag{3}$$

has a branch point singularity at the origin (and no other singularities). In real coordinates, this is a two-dimensional surface in $\mathbb{R}^4$.

Another class of important calibrated examples in higher codimension are special Lagrangians in $\mathbb{C}^n$ (or Calabi-Yau manifolds), which have half the dimension of their ambient space; see [43].

As in many variational problems, an important starting point for developing the regularity theory of area-minimizing currents involves taking *blow-ups*. In this setting, we have a monotonicity formula for mass ratios[4]. For semicalibrated currents (in fact more generally for almost-minimizing ones), this becomes an almost-monotonicity formula, which is still sufficient. Letting $T_{p,r} := \left(\frac{\cdot - p}{r}\right)_\sharp T \llcorner \mathbf{B}_1$ denote the rescalings of an area-minimizing or semicalibrated integral current T, this monotonicity formula tells us that for any fixed center p, any weak-$*$ subsequential limit of the family $\{T_{p,r}\}$ along a sequence $r_k \downarrow 0$ is an *area-minimizing cone C*. Namely, it is invariant under homothetic rescalings about the origin. Note that a priori, the tangent cone depends on its generating sequence of scales, and the question of whether a given tangent cone is the unique one, independent of the sequence of blow-up scales, is deep and generally difficult to answer. In some other contexts, non-uniqueness of blow-ups is in fact possible, see e.g. [6].

However, a major obstruction comes from *flat singular points* (also known as branch points), at which T has a tangent cone supported in an m-dimensional plane.[5] At such points, one cannot hope for an ε-regularity theorem that would allow one to deduce smoothness from the knowledge that T is sufficiently close to a plane. Moreover, since there is a flat tangent cone, performing a tangent cone analysis alone does not allow one to distinguish such points from regular points of the current. The branched holomorphic curve given in (3) provides an example of this: the unique tangent cone at the origin, which is a singular point, is $2[\![\{w=0\}]\!]$. In fact, in this framework, flat singularities necessarily only appear when T has higher codimension,[6] and indeed this is the reason why the regularity theory is significantly simpler for codimension one surfaces.

In spite of the presence of flat singular points, the groundbreaking work of Almgren in the 1980's provided a sharp dimension estimate[7] on the singular set for area-minimizing integral currents:

Theorem 8 (Almgren '80s, [4])**.** *Let T^m be an area-minimizing integral current in $\mathbb{R}^{m+n}$. Then the interior singular set* $\mathrm{Sing}(T)$ *satisfies the dimension bound*

$$\dim_{\mathcal{H}}(\mathrm{Sing}T) \leqslant m-2\,.$$

Almgren's original monograph was significantly longer than 1000 pages, and was poorly understood by many experts in the field until many decades later, when De Lellis & Spadaro revisited Almgren's proof and simplified it in the works [18, 20, 19, 21, 22], making the ideas more transparent to the community. Despite this being the opt

Furthermore, the combined works of White and S.X. Chang yield uniqueness of tangent cones and a sharp structural result on the singular set for two-dimensional area-minimizers:

Theorem 9. *Let T be a two-dimensional area-minimizing integral current in $\mathbb{R}^{2+n}$. Then*

(a) *the tangent cone to T is unique at every interior point p (White '82* [58]*);*
(b) *interior singularities of T are isolated (S.X. Chang* [9]*).*

More recently, Spolaor [55] extended Almgren's dimension bound to semicalibrated currents:

[4]This in fact holds more generally for critical points of the area functional, in the sense of integral varifolds, see e.g. [51].

[5]This planar tangent cone must necessarily come with multiplicity $\mathbb{N} \ni Q \geqslant 2$, in light of Allard's Regularity Theorem [2].

[6]This is a consequence of the identification of boundaryless codimension 1 currents as superpositions of boundaries of Caccioppoli sets, see for example [11].

[7]When the codimension of the surface is one, one may improve this dimension bound to the sharp bound $\dim_{\mathcal{H}}(\mathrm{Sing}T^m) \leqslant m-7$, see [31], which is consistent with Example 6. In fact, in this case a structural result has already been known for some time, thanks to the work [53] of Simon. Such a structure is considered to be sharp, in light of the more recent work [54] the same author, although the latter produces examples of lower dimensional fractal singularities for stable minimal hypersurfaces, which are not known to be necessarily minimizing.

Theorem 10 (Spoalor '15 [55]). *Let T^m be a semicalibrated integral current in $\mathbb{R}^{m+n}$. Then the interior singular set* Sing(T) *satisfies the dimension bound*

$$\dim_{\mathcal{H}}(\mathrm{Sing}T) \leqslant m-2\,.$$

In addition, in the series of works [24, 25, 23, 26], De Lellis, Spadaro & Spolaor again made the work of Chang more transparent, and also extended the conclusions of Theorem 9 to semicalibrated currents:

Theorem 11 (De Lellis-Spadaro-Spolaor [24, 25, 23, 26]). *Let T be a two-dimensional semicalibrated current in $\mathbb{R}^{2+n}$. Then*

(a) *the tangent cone to T is unique at every interior point p;*
(b) *interior singularities of T are isolated.*

In fact, the conclusion (a) holds more generally for almost area-minimizing currents. On the other hand, conclusion (b) crucially relies on the additional structure of the error from minimality given by the semicalibrated condition; see the counterexamples given in [37] that demonstrate dramatic failure of partial regularity for general almost-minimizers. Moreover, the conclusion (b) was additionally extended to the case of spherical links of 3-dimensional area-minimizing cones, which are neither area-minimizing nor semicalibrated in general.

In fact, the conclusion (a) of Theorem 9 and Theorem 11 further yields a classification of possible tangent cones for T: any such tangent cone must be supported in a finite union (possibly just one) of two-dimensional planes meeting at the origin.

Recently, in joint works [16, 17, 13] Camillo De Lellis & Paul Minter and joint work [47] with Paul Minter, Davide Parise & Luca Spolaor, we were able to obtain a generalization of Theorem 9 and Theorem 11 respectively to higher dimensions.

Theorem 12. *Let T^m be an area-minimizing (De Lellis-S. '23, De Lellis-Minter-S.'23* [16, 17, 13]*) or semicalibrated (Minter-Parise-S.-Spolaor '24* [47]*) integral current in $\mathbb{R}^{m+n}$. Then*

(a) *T has a unique tangent cone at $\mathcal{H}^{m-2}$-a.e. interior point p;*
(b) *the interior singular set of T is $(m-2)$-rectifiable, i.e. it is contained, up to a $\mathcal{H}^{m-2}$-negligible set, in a countable union of $(m-2)$-dimensional C^1 submanifolds.*

We additionally obtain the same classification of tangent cones $\mathcal{H}^{m-2}$-a.e. as in the two-dimensional case, after taking a product with the remaining dimensions. This structural result is optimal in full generality, namely in an ambient manifold with a smooth metric (but not real analytic). This is due to the possibility of constructing lower dimensional fractal singularities as done in [45], therefore demonstrating that the $\mathcal{H}^{m-2}$-negligible set is in general non-empty.

The key idea behind Theorem 12 is to use a different monotone quantity in place of the mass ratios in order to blow up around a given flat singular point. Indeed, the major novel idea of Almgren was that locally around flat singularities, the natural quantity to use is the *frequency function* for a suitable graphical approximation of the surface. This detects the radial homogeneity of the graph relative to the given center, at a given radial scale. Note, firstly, that the graphical approximation will in general be *multiple-valued.* For instance, in the example (3), w is parameterized as the two-valued graph $z \mapsto [\![z^{3/2}]\!] + [\![-z^{3/2}]\!]$ relative to the optimal plane $\{w=0\}$. Another complication arises from the fact that we wish the frequency to detect the first *singular* order of collapsing for the sheets of the surface. The following example of a branched holomorphic curve provides an instructive example to illustrate why this requires some additional effort:

Example 13. Consider the variety

$$\{(w,z)\in\mathbb{C}^2 : (w-z^2-z^4)^2 = z^{2025}(1+z^2)^2\}\,.$$

Here, the unique tangent cone at the origin is again $2[\![\{w=0\}]\!]$. Locally around the origin, w may be parameterized as a 2-*valued* graph of z relative to the plane $\{w=0\}$, namely $w = z^2 + z^4 + z^{2025/2} + z^{2027/2}$. The value of the Almgren frequency function (asymptotically as the scale tends to zero) would be 2, namely the degree of the lowest order term in this expansion. However, we wish to detect the degree $\frac{2025}{2}$ of the first branching term.

With this in mind, in [16] we introduce the notion of the *singularity degree*, whose role is precisely to detect this. In general, the singularity degree is defined via a very technical procedure which does not a priori guarantee that its value is providing a number that is dependent only on T and the flat singular point in question. Moreover, in order to "remove all lower order terms in the expansion of T relative to the optimal plane", we must reparameterize T to a *center manifold*, the construction of which was introduced by Almgren. We have to deal with the additional complication that

- we only have graphical approximations for T at those scales where T is sufficiently close to a plane, this may just happen near a given sequence of scales;
- even at the scales where we have an admissible graphical approximation, we may have to keep changing the center manifold and the graphical approximation for the surface with the radial scale;
- the center manifold domain may not necessarily pass through all the singularities of T locally, so they cannot be treated exactly like zeros of the graphical approximation like in the classical application of the frequency function

Nevertheless, we are able to deduce the following important properties:

(1) the singularity degree is indeed a well defined number $\mathrm{I}(T,p) \in [1,\infty)$ at every flat singular point p of T;
(2) at all points p with $\mathrm{I}(T,p) > 1$, the flat tangent cone is unique and the rescalings $T_{p,r}$ decay towards it with a power law rate. For such points, we have a well-defined graphical approximation at all sufficiently small scales (which may be varying with the scale), but nevertheless we may suitably adapt the methods of [48] to obtain the rectifiable structure.[8]
(3) the set of points p with $\mathrm{I}(T,p) = 1$ is $\mathcal{H}^{m-2}$-negligible. To deal with this set, we adapt the methods of [52, 59] to a higher codimension setting in the presence of higher multiplicities.

We have no concrete examples of the kind of singularities described in (3); at such points, we may either have non-uniqueness of tangent cones, or if there is a decay to a unique planar cone then it must be slower than any power. Note that due to the nonlinear structure of the image space for multivalued functions, one cannot easily obtain a classification of possible values for $\mathrm{I}(T,p)$, unlike in the simpler setting of classical harmonic functions and solutions to elliptic PDE (see e.g. [36]). Indeed, this remains an open question in all dimensions $m \geqslant 3$, where it is not even known that there is a discrete spectrum of possible singularity degrees.

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