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Small scale creation in the long time behavior of 2d perfect fluids

Ayman Rimah Said

Abstract

The main issue of this note is the study of the long time behavior of 2d perfect fluids on the 2-torus governed by the incompressible Euler equations and the genericity of small scale creation in said limit. Its aim is twofold: first introduce non specialists to some key conjectures in the field today due to Shnirelman, Šverák and Yudovich respectively as well as some results towards those conjectures. Second we present a recent result by the author [19] on the generic character of small creation for the Lagrangian flow which is build upon a pioneering Lyapunov construction due to Shnirelman [20].

1 Introduction

We study 2d inviscid flows on $\mathbb{T}^2 = [-\frac{1}{2}, \frac{1}{2})^2$ governed by the Euler equations in vorticity form

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{1.1}$$

$$u = \nabla^{\perp} \psi \text{ and } \Delta \psi = \omega. \tag{1.2}$$

Here, the scalar vorticity $\omega : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$ is transported by the velocity field $u : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}^2$ which is uniquely determined at each time $t \in \mathbb{R}$ from ω using the Newtonian potential:

$$u(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \int_{[-1,1)^2} \frac{(x_2 - y_2 - 2n_2, -x_1 + y_1 + 2n_1)}{|x - y - 2n|^2} \omega(y) dy, \text{ thus } \omega = \nabla \times u = \partial_1 u_2 - \partial_2 u_1. \tag{1.3}$$

We adopt the standard notation $v^{\perp} = (-v_2, v_1)$ for $v = (v_1, v_2) \in \mathbb{R}^2$. Without loss of generality we will work with 0 average vorticity, which is a conserved quantity. We define the Lagrangian flow

$$\frac{d}{dt} \Phi_t = u \circ \Phi_t \text{ with } \Phi_0(\cdot) = Id.$$

It is well known that sufficiently smooth solutions of the 2d Euler equation (1.1)-(1.2) retain their smoothness for all finite times. Much less is known in the infinite time limit. Since the Euler equation is fundamentally a (non-linear and non-local) transport equation, there is a strong possibility that despite the plethora of possible initial states, most solutions "relax" in infinite time to simpler states. This a fact often observed experimentally and numerically see Figure 1 below.

We now give 2 natural conjectures that give a rigorous mathematical framework in which one might quantify the previous observations, see [23] and [22] respectively and also the review articles [5, 12]. For this we recall the standard well-posedness theory for 2d Euler equations.

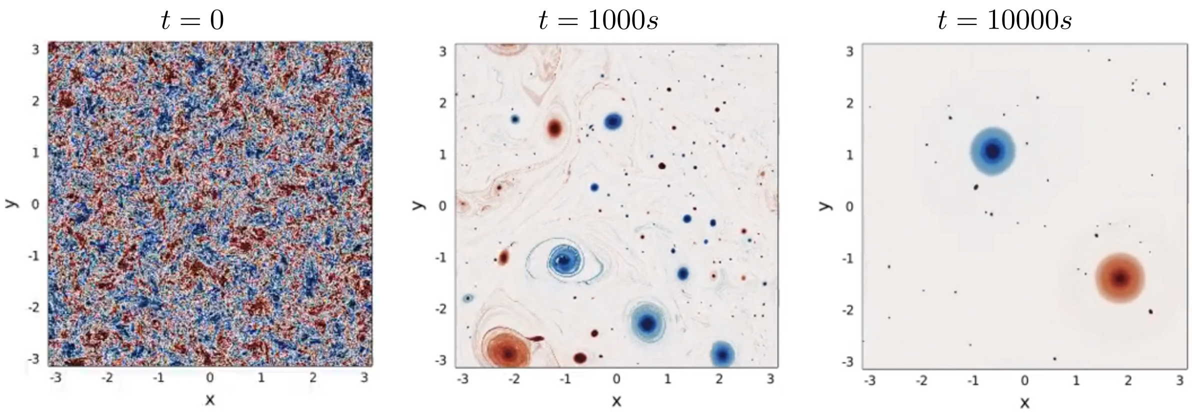

Figure 1: Emergence of a vortex dipole in the long time in a simulation of the Euler equations. For the full simulation by Theo Drivas: `https://www.youtube.com/watch?v=-YdEYumSSJ0`

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Theorem 1.1 (Yudovich [26])**.** *For all initial zero average vorticity $\omega_0 \in L^\infty(\mathbb{T}^2)$ there exists a unique global solution $\omega \in C_*(\mathbb{R}, L^\infty(\mathbb{T}^2))$ to the 2d Euler equations. Furthermore for all $s > 1$, there exists a universal constant C_s such that we have following estimate*

$$\|\omega(t)\|_{H^s} + \|\Phi_t - Id\|_{H^{s+1}} \leq C_s \|\omega_0\|_{H^s} \exp\left(\exp\left(C_s \|\omega_0\|_{L^\infty} t\right)\right).$$

Hence we can define the orbit and omega limit sets for bounded periodic vorticity

$$\mathcal{O}_{\omega_0} = \{\omega(t), t \in \mathbb{R}\} \subset L^\infty(\mathbb{T}^2) \text{ and } \Omega_+(\omega_0) = \cap_{s\geq 0}\overline{\{\ \omega(t), t \geq s\}}^{*}$$

Observe that Ω_{ω_0} is non empty as a decreasing intersection of non empty compact sets (in the weak topology). The conjecture then reads

Conjecture 1.2 (Šverák [23] and Shnirelman [22])**.**

- *Generically for $\omega_0 \in L^\infty(\mathbb{T}^2)$, $\mathcal{O}_{\omega_0}$ is not pre-compact in $L^2(\mathbb{T}^2)$.*
- *Consider $\omega_* \in \Omega_+(\omega_0) \subset L^\infty(\mathbb{T}^2)$ for some $\omega_0 \in L^\infty(\mathbb{T}^2)$ then $\mathcal{O}_{\omega_*}$ is pre-compact in $L^2(\mathbb{T}^2)$.*

This conjecture states that most solutions should, on the one hand, "relax" in infinite time in that they should lose L^2 mass, due to mixing [3]. Note that this is the only way that compactness can be lost on compact domains since the L^2 norm of the vorticity is conserved for all finite times. On the other hand, these limiting states are conjectured to have compact orbits; i.e. they must be very special, such as steady states, time-periodic solutions, etc.

This has been established in perturbative regimes in the ground breaking work of Bedrossian and Masmoudi [1] and later extensions by Ionescu and Jia [9,10] and Masmoudi and Zhao [16]. The conjecture is completely open in the large data setting. Under scaling and symmetry hypothesis the conjecture has been established in the large data setting for scale invariant solutions [7] and logarithmic spirals [11].

From the previous conjecture the long time behavior seems to consistently show some type of small scale creation for smooth solutions [5,12]. From Figure 1, we see that this is necessary for a change of topology of the vorticity streamlines to occur. This can be summarized in the following conjecture by Yudovich.

Conjecture 1.3 (Yudovich (1974), [24,25], quote from [17])**.** *There is a "substantial set" of inviscid incompressible flows whose vorticity gradients grow without bound. At least this set is dense enough to provide the loss of smoothness for some arbitrarily small disturbance of every steady flow.*

The literature towards this conjecture is rich. Of note is the result of Koch [14] in which strong growth of Hölder and Sobolev norms of the vorticity is established near any background solution (stationary or time-dependent) for which the gradient of the flow map is unbounded in time. Yudovich also established (boundary induced) growth results under some mild assumption on the data near the boundary of the domain [25] (see also [17] for an extension of [25]). The conjecture was established within m-fold symmetry for $m \geq 3$ by Elgindi, Murray and the author in [7]. There are also numerous important results on growth of solutions in the neighborhood of stable steady states [4, 5, 13, 18, 27]. In the case of open neighborhoods of shearing stable steady states a finer version of the conjecture including generic fluid aging has been recently established by Drivas, Elgindi and Jeong [6].

The main result we want to present in this note from [19] can be stated informally as follows.

Theorem 1.4. *Consider the 2d Euler equation on $\mathbb{T}^2$. Then for $\omega_0 \in H^s(\mathbb{T}^2) \setminus H^{s+\varepsilon}(\mathbb{T}^2)$ for some $s > 1$ and all $\varepsilon > 0$, then roughly the s+1 derivative of the Lagrangian flow blows up in infinite time at least like $t^{\frac{1}{3}}$.*

1.1 Propagation of exact smoothness in 2d Euler

One of the key quantitative observations in [19] is to describe a *maximal* propagation of smoothness result for the 2d Euler equation. First we set, for $\omega_0 \in L^2$, $\varepsilon \geq 0$

$$dr_{\omega_0}(\varepsilon) = \Bigg(\sum_{|n|+|m| \geq 1/\varepsilon} (\omega_0)_{n,m}^2 \Bigg)^{1/2},$$

where $(\omega_0)_{n,m}$ are the Fourier coefficients of ω_0. We note that dr_{ω_0} is an increasing function of ε with $\lim_{\varepsilon \to 0} dr_{\omega_0}(\varepsilon) = 0$ and $\lim_{\varepsilon \to +\infty} dr_{\omega_0}(\varepsilon) = \|\omega_0\|_{L^2(\mathbb{T}^2)}$. For $f \in C^\infty(\mathbb{T}^2)$ such that dr_f vanishes at 0 at least like dr_{ω_0}, we define

$$\|f\|_{\omega_0} = \sup_{\varepsilon \geq 0} \frac{dr_f(\varepsilon)}{dr_{\omega_0}(\varepsilon)}.$$

Note that by construction $\|\omega_0\|_{\omega_0} = 1$.

Theorem 1.5. *Consider $s > 1$ and $\omega_0 \in H^s(\mathbb{T}^2)$ and $\omega \in C(\mathbb{R}, H^s(\mathbb{T}^2))$ the unique solution of (1.1)–(1.2) with initial data ω_0. Suppose that for all $\lambda < 1$*

$$C_{\omega_0}(\lambda) dr_{\omega_0}(\varepsilon) \leq dr_{\omega_0}(\lambda\varepsilon), \tag{1.4}$$

for some function C_{ω_0} independent of ε. Then there exists a constant C'_{ω_0} such that

$$\|\omega(t)\|_{\omega_0} + \|D\Phi_t - Id\|_{\omega_0} \leq \exp\big(\exp\left(C'_{\omega_0} t\right)\big).$$

The proof of the of the previous theorem, [19], relies essentially on the observation that condition (1.4) is equivalent to the fact that ω_0's Fourier transform decays at most algebraically fast at infinity, in particular the previous theorem applies for all $\omega_0 \in H^{s_1}(\mathbb{T}^2) \setminus H^{s_2}(\mathbb{T}^2)$ for a pair $1 < s_1 < s_2$. This then implies all of the standard calculus holds in the Banach space defined through the norm $\|\cdot\|_{\omega_0}$. We believe that this type of exact smoothness propagation holds more generally for hyperbolic evolution PDEs. For example the proof here works in verbatim to give an analogous result (locally in time) for the SQG equation.

1.2 The para-pull back flow

One of the key observation of Shnirelman in [20] is an explicit identity of the flow of the para-pull backed velocity field. In order to make this statement precise we give a short intuitive summary of the para-differential notions needed for this.

1.2.1 Heuristic representation of paradifferential calculus

• Paraproducts

For the sake of this discussion let us pretend that ∂_x is left-invertible with a choice of ∂_x^{-1} that acts continuously from H^s to H^{s+1}. We follow here analogous ideas to the ones presented by Shnirelman in [21]. One way to define the paraproduct of two functions $f, g \in H^s$ with s sufficiently large is: we differentiate fg k times, using the Leibniz formula, and then restore the function fg by the k-th power of ∂_x^{-1}:

$$\begin{aligned} fg &= \partial_x^{-k}\partial_x^k(fg) \\ &= \partial_x^{-k}\big(g\partial_x^k f + k\partial_x g\partial_x^{k-1} f + \cdots + k\partial_x f\partial_x^{k-1} g + g\partial_x^k f\big) \\ &= T_g f + T_f g + R, \end{aligned}$$

where

$$T_g f = \partial_x^{-k}\big(g\partial_x^k f\big), \quad T_f g = \partial_x^{-k}\big(f\partial_x^k g\big),$$

and R is the sum of all remaining terms. The key observation is that if $s > \frac{1}{2} + k$, then $g \mapsto T_f g$ is a continuous operator in H^s for $f \in H^{s-k}$. The remainder R is a continuous bilinear operator from H^s to H^{s+1}. The operator $T_f g$ is called the paraproduct of g and f and can be interpreted as follows. The term $T_f g$ takes into play high frequencies of g compared to those of f and demands more regularity in $g \in H^s$ than $f \in H^{s-k}$ thus the term $T_f g$ bears the “singularities” brought on by g in the product fg. Symmetrically $T_g f$ bears the ”singularities” brought on by f in the product fg and the remainder R is a smoother function (H^{s+1}) and does not contribute to the main singularities of the product.

• Paradifferential operators

To get a good intuition of a paradifferential operator $T_{p(x,\xi)}$ with symbol $p(x,\xi) \in \Gamma_\rho^\beta(\mathbb{T}^2)$, as a first gross approximation, one can think of $p(x,\xi) \approx f(x)m(\xi)$ and $T_{p(x,\xi)}$ as the composition of a paraproduct T_f with a Fourier multiplier $m(D)$, that is:

$$T_{p(x,\xi)} \approx T_f m(D), \text{ with } f \in W^{\rho,\infty} \text{ and } m \text{ is of order } \beta.$$

Indeed following Coifman and Meyer's symbol reduction given in Proposition 5 of [2], one can show that linear combinations of composition of a paraproduct with a Fourier multiplier are dense in the space of paradifferential operators.

• Paracomposition

We again work with $f \in H^s$ and $g \in C^s$ with s large and consider the composition of two functions $f \circ g$ which bears the singularities of both f and g, and our goal is to separate them. We proceed as before by differentiating $f \circ g$ k times, using the Faá di Bruno's formula, and then restore the function fg by the k-th power of ∂_x^{-1}:

$$\begin{aligned} f \circ g &= \partial_x^{-k} \partial_x^k (f \circ g) \\ &= \partial_x^{-k} \big((\partial_x^k f \circ g) \cdot (\partial_x g)^k + \cdots + (\partial_x f \circ g) \cdot \partial_x^k g \big) \\ &= g^* f + T_{\partial_x f \circ g} g + R, \end{aligned}$$

where

$$g^* f = \partial_x^{-k} \big((\partial_x^k f \circ g) \cdot (\partial_x g)^k \big) \text{ is the paracomposition of } f \text{ by } g$$

and R is the sum of all remaining terms. Again the key observation is that if $s > \frac{1}{2} + k$, then $f \mapsto g^* f$ is a continuous operator in H^s for $g \in C^{s-k}$. Thus this term bears essentially the singularities of f in $f \circ g$. As before $T_{\partial_x f \circ g} g$ bears essentially the singularities of g in $f \circ g$. The remainder R is a continuous bilinear operator from H^s to H^{s+1}. Thus we have separated the singularities of the composition $f \circ g$.

1.2.2 The underlining ODE system

The key algebraic identity observed by Shnirelman is the following (see Section 4.1 of [19])

$$\partial_t \left(T_{[D\Phi_t]^{-1}} \Phi_t \right) = T_{[D\Phi_t]^{-1}} \Phi_t^* u + R,$$

now recall that the pull back of u by Φ_t is given by $[D\Phi_t]^{-1} u \circ \Phi_t$, then the right hand side in the previous identity can be interpreted as a paradifferential version of this pull-back which "selects" the high frequencies of u compared to Φ_t.

Another interpretation of $T_{[D\Phi_t]^{-1}} \Phi_t$ comes from the following observation. Starting from the identity

$$\Phi_t^{-1} \circ \Phi_t = x \Longrightarrow \Phi_t^* \Phi_t^{-1} + T_{[D\Phi_t]^{-1}} \Phi_t + R = x$$

hence

$$T_{[D\Phi_t]^{-1}} \Phi_t = -\Phi_t^* \Phi_t^{-1} + R,$$

geometrically $\Phi_t^* \Phi_t^{-1}$ can be interpreted as the "selection" of the high frequencies introduced by the flow when brought back to a frame in a neighberhood of the identity.

1.3 The forward frequency cascade

We are now in position to state the main theorem summarizing the Lyapunov construction from [19].

Theorem 1.6. *Consider $\chi(\xi) \in C_0^\infty(\mathbb{R}^2 \setminus B(0,1))$ and $\omega_0 \in H^s$ with $s > 1$ verifying (1.4) then there exists a universal constant C and a constant C_{ω_0} such that for $\varepsilon \geq 0$*

$$\begin{aligned} &\frac{d}{dt} \big(\nabla \times T_{[D\Phi_t]^{-1}} \Phi_t, \chi(\varepsilon D) \omega_0 \big)_{L^2} \\ &\qquad = \left\| T_{|\xi| / |[D\Phi_t]^{-1} \xi|} \, \chi(\varepsilon D) \omega_0 \right\|_{L^2}^2 + O\Big(C_\delta e^{C e^{C_{\omega_0} t}} \varepsilon^{\min(s-1-\delta, 1)} dr_{\omega_0}(\varepsilon)^2 \Big), \end{aligned}$$

for all $0 < \delta < s - 1$ and $C_\delta > 0$ is constant depending only on δ.

Remark 1.7. In [20] Shnirelman proved the previous theorem in the case $dr_{\omega_0}(\varepsilon) = O(\varepsilon^s), s > 2$. Theorem 1.6 generalises this result to the exact regularity of ω_0 whatever it is and gives the optimal control in ε.

In particular the leading order decay in ε of $\big(\nabla \times T_{[D\Phi_t]^{-1}}\Phi_t, \chi(\varepsilon D)\omega_0\big)$ is

$$\int_0^t \left\| T_{|\xi|/|[D\Phi_\tau]^{-1}\xi|}\, \chi(\varepsilon D)\omega_0 \right\|_{L^2}^2 d\tau \underset{\varepsilon\to 0}{\sim} c(t) dr_{\omega_0}(\varepsilon)^2$$

where $c(t)$ is an increasing function of time. The explicit estimate on the residual term allows for the following interpretation of the previous result. Fixing an outer frequency region $\{|\xi| \geq R\}$ then there exists $T_R > 0$ increasing in R such that for $|t| \leq T_R$ there is an averaged forward frequency cascade of Φ_t in the signed measure $\mathscr{F}(\omega_0)(\xi)d\xi$ into the region $\{|\xi| \geq R\}$. Thus there is always a positive flux of frequency at "infinity" ($R \to \infty$) and the growth of that rate gives the desired Lyapunov function. Using dr_{ω_0}, it is given explicitly by a re-normalised version of the semi-classical measure first introduced in [8] and independently as the Wigner measure in [15] which is the so called microlocal scalar product introduced by Shnirelman in [20]

$$L_{\chi,\omega_0}(\Phi_t) = \limsup_{\varepsilon\to 0} \frac{\big(\chi(\varepsilon D)\nabla \times T_{[D\Phi_t]^{-1}}\Phi_t, \chi(\varepsilon D)\omega_0\big)_{L^2}}{dr_{\omega_0}(\varepsilon)^2}.$$

2 An explicit example

We give an example where the Lyapunov function can be computed explicitly. We work with odd-odd data and consider the shear steady state on $\mathbb{T}^2 = [-\pi, \pi)^2$, $\omega(y) = \sin(y)^\alpha$, for $y > 0$ $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then the stream function solves

$$\partial_y^2 \psi(y) = \sin(y)^\alpha \text{ with } \psi(0) = \psi(\pi) = 0,$$

thus

$$u(x,y) = \begin{pmatrix} -\partial_y\psi(y) \\ 0 \end{pmatrix} \Longrightarrow \Phi_t = \begin{pmatrix} x - t\partial_y\psi(y) \\ y \end{pmatrix} \Longrightarrow D\Phi_t = \begin{pmatrix} 1 & -t\partial_y^2\psi(y) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -t\omega(y) \\ 0 & 1 \end{pmatrix},$$

hence

$$[D\Phi_t]^{-1} = \begin{pmatrix} 1 & t\omega(y) \\ 0 & 1 \end{pmatrix} \Longrightarrow T_{[D\Phi_t]^{-1}} = \begin{pmatrix} T_1 & tT_{\omega(y)} \\ 0 & T_1 \end{pmatrix}.$$

Thus we compute explicitly

$$T_{[D\Phi_t]^{-1}}\Phi_t = \begin{pmatrix} T_1 x - tT_1\partial_y\psi(y) + tT_{\omega(y)}y \\ T_1 y \end{pmatrix} = \begin{pmatrix} -t\partial_y\psi(y) \\ 0 \end{pmatrix} + R,\ R \in C^\omega(\mathbb{T}^2),$$

hence

$$\nabla \times T_{[D\Phi_t]^{-1}}\Phi_t = t\omega + R', R' \in C^\omega(\mathbb{T}^2).$$

Consider Δ_k the k-th Littlewood-Paley projector then

$$(\nabla \times T_{[D\Phi_t]^{-1}}\Phi_t, \Delta_k\omega)_{L^2} = 2t\,\|\Delta_k\omega\|_{L^2}^2 + (R, \Delta_k\omega)_{L^2},$$

and now we see that the hypothesis $\alpha \notin \mathbb{N}$ (i.e finite smoothness) is crucial to guarantee

$$|(R, \Delta_k\omega)_{L^2}| \ll t\,\|\Delta_k\omega\|_{L^2}^2 \Longrightarrow (\nabla \times T_{[D\Phi_t]^{-1}}\Phi_t, \Delta_k\omega)_{L^2} \gtrsim t\,\|\Delta_k\omega\|_{L^2}^2.$$

References

[1] Jacob Bedrossian and Nader Masmoudi. Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. *Publications mathématiques de l'IHÉS*, 122(1):195–300, 2015.

[2] Ronald R Coifman and Yves Meyer. *Au delà des opérateurs pseudo-différentiels*, volume 57 of *Astérisque*. Société mathématique de France, 1978.

[3] Peter Constantin. On the Euler equations of incompressible fluids. *Bulletin of the American Mathematical Society*, 44(4):603–621, 2007.

[4] Sergey A Denisov. Infinite superlinear growth of the gradient for the two-dimensional Euler equation, 2009. `arXiv:0908.3466`.

[5] Theodore D Drivas and Tarek M Elgindi. Singularity formation in the incompressible Euler equation in finite and infinite time, 2022. `arXiv:2203.17221`.

[6] Theodore D Drivas, Tarek M Elgindi, and In-Jee Jeong. Twisting in Hamiltonian flows and perfect fluids, 2023. `arXiv:2305.09582`.

[7] Tarek Elgindi, Ryan Murray, and Ayman Said. On the long-time behavior of scale-invariant solutions to the 2d Euler equation and applications, 2022. `arXiv:2211.08418`.

[8] Patrick Gérard. Mesures semi-classiques et ondes de Bloch. In *Séminaire Équations aux dérivées partielles (Polytechnique) dit aussi" Séminaire Goulaouic-Schwartz"*, pages 1–19. Éditions de l'École polytechnique, 1991.

[9] Alexandru Ionescu and Hao Jia. Axi-symmetrization near point vortex solutions for the 2D Euler equation. *Communications on Pure and Applied Mathematics*, 75(4):818–891, 2022.

[10] Alexandru D Ionescu and Hao Jia. Nonlinear inviscid damping near monotonic shear flows. *Acta Math*, 230:321–399, 2023.

[11] In-Jee Jeong and Ayman R Said. Logarithmic spirals in 2d perfect fluids, 2023. `arXiv:2302.09447`.

[12] Boris Khesin, Gerard Misiołek, and Alexander Shnirelman. Geometric hydrodynamics in open problems. *Archive for Rational Mechanics and Analysis*, 247(2):15, 2023.

[13] Alexander Kiselev and Vladimir Šverák. Small scale creation for solutions of the incompressible two-dimensional Euler equation. *Annals of mathematics*, 180(3):1205–1220, 2014.

[14] Herbert Koch. Transport and instability for perfect fluids. *Mathematische Annalen*, 323(3):491–523, 2002.

[15] Pierre-Louis Lions and Thierry Paul. Sur les mesures de Wigner. *Revista matemática iberoamericana*, 9(3):553–618, 1993.

[16] Nader Masmoudi and Weiren Zhao. Nonlinear inviscid damping for a class of monotone shear flows in finite channel, 2020. `arXiv:2001.08564`.

[17] Andrey Morgulis, Alexander Shnirelman, and Victor Yudovich. Loss of smoothness and inherent instability of 2D inviscid fluid flows. *Communications in Partial Differential Equations*, 33(6):943–968, 2008.

[18] Nikolai Semenovich Nadirashvili. Wandering solutions of Euler's D-2 equation. *Functional Analysis and Its Applications*, 25(3):220–221, 1991.

[19] Ayman Rimah Said. Small scale creation of the Lagrangian flow in 2d perfect fluids, 2024. `arXiv:2401.06476`.

[20] Alexander Shnirelman. Evolution of singularities, generalized Liapunov function and generalized integral for an ideal incompressible fluid. *American Journal of Mathematics*, 119(3):579–608, 1997.

[21] Alexander Shnirelman. Microglobal analysis of the Euler equations. *Journal of mathematical fluid mechanics*, 7:S387–S396, 2005.

[22] Alexander Shnirelman. On the long time behavior of fluid flows. *Procedia IUTAM*, 7:151–160, 2013.

[23] Vladimir Šverák. Course notes.
`https://www-users.cse.umn.edu/~sverak/course-notes2011.pdf`, 2011/2012.

[24] V. Yudovich. The loss of smoothness of the solutions of Euler equations with time. *Dinamika Splosn. Sredy Vyp*, 16:71–78, 1974.

[25] V. Yudovich. On the loss of smoothness of the solutions of the Euler equations and the inherent instability of flows of an ideal fluid. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 10(3):705–719, 2000.

[26] Victor I Yudovich. Non-stationary flow of an ideal incompressible liquid. *USSR Computational Mathematics and Mathematical Physics*, 3(6):1407–1456, 1963.

[27] Andrej Zlatoš. Exponential growth of the vorticity gradient for the Euler equation on the torus. *Advances in Mathematics*, 268:396–403, 2015.