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On unique continuation for the Schrödinger equation

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Abstract

Motivated by applications to approximate and exact controllability, we are interested in the following unique continuation question: assume the solution of the linear Schrödinger equation on a domain vanishes on a very small open set for a very short time interval, then is this solution identically zero? In the situation where the Schrödinger operator includes a potential, the answer to this question depends on the regularity of the latter. We present a result proved in [FLL24] which assumes that the potential is Gevrey 2 in time and bounded in space, relaxing in this context the analyticity assumption of the Tataru-Robbiano-Zuily-Hörmander theorem. We also give a sketch of proof.

Keywords

Unique continuation, Schrödinger operators, Gevrey regularity, Carleman estimates.

2010 Mathematics Subject Classification: 35B60, 35Q41, 47F05, 93B07, 93C20, 93C73.

1 A motivation from control theory

Before entering the heart of the question of unique continuation, we start by discussing one of its applications coming from control theory.

Given $T > 0$, $\Omega \subset \mathbb{R}^d$ an open set with boundary $\partial\Omega$, we consider the control problem

$$\begin{cases} i\partial_t v + \Delta_g v + \mathfrak{q} v = \mathbb{1}_\omega f, & \text{in } (0,T)\times\Omega, \\ v = 0, & \text{on } (0,T)\times\partial\Omega, \\ v(0,\cdot) = 0, & \text{in } \Omega, \end{cases} \tag{1.1}$$

starting from rest at initial time. Here, g is a locally Lipschitz-continuous metric on $\overline{\Omega}$, that is to say $g = (g^{jk})_{1\le j,k\le d}$ with $g^{jk} \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$ and for all $x \in \overline{\Omega}$,

$$g^{jk}(x) = g^{kj}(x) \text{ and there is } c_0 > 0 \text{ such that } |\xi|_g^2 \ge c_0|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^d, \tag{1.2}$$

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where we have written

$$\langle \xi, \eta \rangle_g := \sum_{j,k=1}^{d} g^{jk}(x)\xi_j \eta_k, \quad |\xi|_g^2 := \langle \xi, \xi \rangle_g = \sum_{j,k=1}^{d} g^{jk}(x)\xi_j \xi_k.$$

The operator Δ_g is the symmetric operator defined by

$$\Delta_g u := \sum_{j,k=1}^{d} \partial_{x_j}\left(g^{jk}(x)\partial_{x_k} u\right). \tag{1.3}$$

The function $\mathsf{q} \in L^\infty((0,T)\times\Omega;\mathbb{C})$ is a bounded potential and $\omega \subset \Omega$ an open set (the control domain). The function f is a control force acting on the system on the small open set ω and one would like to control the state v of the equation to a given target state at time T.

Before discussing controllability questions regarding Equation (1.1), we briefly digress to explain well-posedness of the Cauchy problem (1.1) in general (a difficulty arising in case Ω is unbounded). We first let $H_0^1(\Omega)$ be the completion of $C_c^1(\Omega)$ for the norm

$$\|u\|^2_{\tilde{H}_g^1(\Omega)} := \int_\Omega \left(|\nabla u|_g^2 + |u|^2\right) dx. \tag{1.4}$$

Note that $C_c^1(\Omega)$ being dense in $L^2(\Omega)$, we have a continuous embedding $H_0^1(\Omega) \subset L^2(\Omega)$. Second, we take the Friedrichs extension on $L^2(\Omega)$ of $-\Delta_g$ defined on $C_c^\infty(\Omega)$, which we denote by $-\Delta_{g,\mathrm{F}}$. It is defined by

$$\begin{aligned} D(-\Delta_{g,\mathrm{F}}) := \Big\{ & u \in H_0^1(\Omega), \text{ there exists } h \in L^2(\Omega), \\ & \int_\Omega \langle \nabla u, \nabla \varphi \rangle_g^2 + u\varphi \, dx = \int_\Omega h\varphi \, dx \quad \text{for all } \varphi \in H_0^1(\Omega)\Big\}. \end{aligned} \tag{1.5}$$

For $u \in D(-\Delta_{g,\mathrm{F}})$, there is a unique h satisfying (1.5), and we set $(-\Delta_{g,\mathrm{F}} + \mathrm{Id})u := h$. Third, for $\mathsf{q} \in L^\infty((0,T)\times\Omega;\mathbb{C})$, the solution to (1.1) is defined via the Duhamel formula for the unitary group $\left(e^{it\Delta_{g,\mathrm{F}}}\right)_{t\in\mathbb{R}}$ and is a solution of the first equation of (1.1) in the sense of distributions on $(0,T)\times\Omega$. Note that if we assume that $\partial\Omega$ is bounded, then $H_0^1(\Omega) = \{u \in \tilde{H}_g^1(\Omega), \mathrm{Tr}(u) = 0\}$, where $\tilde{H}_g^1(\Omega)$ is defined as the completion of $C^1(\Omega)$ functions with finite $\tilde{H}_g^1$ norm for this norm (Definition (1.4)) and $\mathrm{Tr} : \tilde{H}_g^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator. This remark justifies the formal writing of the Cauchy problem as an initial-boundary value problem in (1.1). If we further assume that Ω itself is bounded, then $\tilde{H}_g^1(\Omega) = H^1(\Omega)$ and $H_0^1(\Omega)$ are the usual Sobolev spaces.

We may now come back to controllability issues. The following two questions are of crucial importance in control theory:

1. (Exact controllability) Do we have: for all target $v_T \in L^2(\Omega)$, the existence of a control force $f \in L^2((0,T)\times\omega)$ such that the associated solution v of (1.1) satisfies $v(T,\cdot) = v_T$?

2. (Approximate controllability) Do we have: for all target $v_T \in L^2(\Omega)$ and all precision $\varepsilon > 0$, the existence of a control force $f = f_\varepsilon \in L^2((0,T)\times\omega)$ such that the associated solution v of (1.1) satisfies $\|v(T,\cdot) - v_T\|_{L^2(\Omega)} \leq \varepsilon$?

If we introduce the "final time evaluation" map

$$\begin{array}{rcl} R : L^2((0,T)\times\omega) & \to & L^2(\Omega) \\ f & \mapsto & v(T,\cdot), \end{array}$$

where v denotes the solution of (1.1) associated to f, then the above two questions rephrase as

1. (Exact controllability) $\mathrm{range}(R) = L^2(\Omega)$, i.e. surjectivity,
2. (Approximate controllability) $\overline{\mathrm{range}(R)} = L^2(\Omega)$, i.e. dense image.

These "surjectivity type" properties are often hard to tackle directly, and it is therefore customary to reformulate them as "injectivity type" properties for a dual problem. In the present setting, the latter is the following free backward Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta_g u + \mathsf{q}u = 0 & \text{in } (0,T)\times\Omega, \\ u = 0, & \text{on } (0,T)\times\partial\Omega, \\ u(T,\cdot) = u_T, & \text{in } \Omega, \end{cases} \tag{1.6}$$

dual to the control problem (1.1) if $\mathsf{q} = \overline{\mathsf{q}}$, what we assume in the present section. Pairing the first line of (1.1) with u yields

$$(i\partial_t v + \Delta_g v + \mathsf{q}v, u)_{L^2((0,T)\times\Omega)} = (\mathbb{1}_\omega f, u)_{L^2((0,T)\times\Omega)}\,.$$

Integrating by parts in time and space in the left-hand side and using that u solves (1.6) and v has $v(0,\cdot) = 0$ then implies

$$(iv(T,\cdot), u_T)_{L^2(\Omega)} = (f, \mathbb{1}_\omega u)_{L^2((0,T)\times\omega)}\,. \tag{1.7}$$

If we now define the observation operator as

$$\begin{array}{rcl} \mathrm{Obs} : L^2(\Omega) & \to & L^2((0,T)\times\omega), \\ u_T & \mapsto & \mathbb{1}_\omega u, \end{array}$$

where u denotes the solution to (1.6), Identity (1.7) rewrites

$$(iR(f), u_T)_{L^2(\Omega)} = (f, \mathrm{Obs}(u_T))_{L^2((0,T)\times\omega)}\,,$$

that is to say, the operator Obs is the adjoint to the operator iR:

$$(iR)^* = \mathrm{Obs}\,. \tag{1.8}$$

This duality together with standard functional analysis (e.g. $\ker(R^*) = \mathrm{range}(R)^\perp$, whence R has dense image if and only if R^* is injective) provides with the central reformulation of control problems into observation problems, see Dolecki and Russell [DR77].

Lemma 1.1 (Duality)**.** *Given $T > 0$ and $\omega \subset \Omega$ open, the following two statements hold:*

1. (1.1) *is exactly controllable in time T if and only if the following observability inequality holds:*

$$\begin{aligned} &\text{there is } C > 0 \text{ s.t. for all } u_T \in L^2(\Omega) \text{ and } u \text{ solution to } (1.6), \\ &\qquad\qquad \|u_T\|_{L^2(\Omega)} \leq C\|u\|_{L^2((0,T)\times\omega)}. \end{aligned} \tag{1.9}$$

2. (1.1) *is approximately controllable in time T if and only if the following unique continuation statement holds:*

$$\big(u \text{ solution to } (1.6), u = 0 \text{ on } (0,T)\times\omega\big) \quad \Longrightarrow \quad u_T = 0. \tag{1.10}$$

In view of Lemma 1.1, the unique continuation property (1.10) is clearly of primary importance for its application to approximate controllability.

It is actually also of primary importance in the analysis of the exact controllability question, on account to the classical compactness-uniqueness principle [RT74, BLR92].

Lemma 1.2 (Compactness uniqueness)**.** *Equation* (1.1) *is exactly controllable (resp. the observability inequality* (1.9) *holds) if and only if*

1. *unique continuation* (1.10) *holds, and*
2. *there is a* compact *operator* K *on* $L^2(\Omega)$ *and* $C > 0$ *such that*

$$\text{for all } u_T \in L^2(\Omega) \text{ and } u \text{ solution to (1.6)},\\ \|u_T\|_{L^2(\Omega)} \leq C\|u\|_{L^2((0,T)\times\omega)} + C\|\mathsf{K}u_T\|_{L^2(\Omega)}. \tag{1.11}$$

Lemma 1.2 (proved by a rather classical contradiction argument) allows to decouple what happens at high-frequency and space-infinity (in case Ω is unbounded), encoded in the inequality (1.11), from what happens for bounded frequencies and on bounded sets, encoded in the unique continuation statement (1.10).

2 Unique continuation results

In all of our results, Gevrey regularity takes a central place.

Definition 2.1. Given $\mathsf{d} \in \mathbb{N}^*$, $U \subset \mathbb{R}^{\mathsf{d}}$ an open set, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ a Banach space and $s > 0$, we say that f is a s-Gevrey function valued in $\mathcal{B}$, denoted $f \in \mathcal{G}^s(U;\mathcal{B})$, if $f \in C^\infty(U;\mathcal{B})$ is such that for every compact set $K \subset U$, there are constants $C, R > 0$ such that for all $\alpha \in \mathbb{N}^{\mathsf{d}}$

$$\max_{t\in K} \|\partial^\alpha f(t)\|_{\mathcal{B}} \leq CR^{|\alpha|}\alpha!^s.$$

These spaces were introduced by Gevrey [Gev18] to investigate regularity properties for solutions of the heat equation between real-analyticity and C^∞ regularity. Recall that for $s = 1$, $\mathcal{G}^1(U;\mathcal{B}) = C^\omega(U;\mathcal{B})$ is the space of real-analytic $\mathcal{B}$-valued functions. However, for $s > 1$, $\mathcal{G}^s(U;\mathcal{B})$ contains nontrivial compactly supported functions. The paradigmatic example of such a function is, for $\alpha > 0$, $t \mapsto \mathbb{1}_{\mathbb{R}^+}(t)e^{-1/t^\alpha}$, which belongs to $G^{1+1/\alpha}(\mathbb{R};\mathbb{R})$. In what follows, we mostly consider the case $\mathsf{d} = 1$, t being the time variable.

We start by stating our global unique continuation result (which is a particular case of [FLL24]).

Theorem 2.2 (Global unique continuation)**.** *Let* $T > 0$ *and* $\Omega \subset \mathbb{R}^d$ *be a connected open set. Suppose that* $g \in W^{1,\infty}_{\text{loc}}(\Omega)$ *is a Riemannian metric on* Ω*, in the sense that* (1.2) *is satisfied for all* $x \in \Omega$*, and recall* Δ_g *is defined in* (1.3)*. Assume that* $\mathsf{q} \in \mathcal{G}^2((0,T); L^\infty_{\text{loc}}(\Omega;\mathbb{C}))$*, and consider the differential operator*

$$P_{\mathsf{q}} := i\partial_t + \Delta_g + \mathsf{q}(t,x) = i\partial_t + \sum_{j,k=1}^d \partial_{x_j} g^{jk}(x)\partial_{x_k} + \mathsf{q}(t,x), \tag{2.1}$$

Then given ω *a nonempty open set of* Ω*, we have*

$$\begin{cases} P_{\mathsf{q}}u = 0 \text{ in } (0,T)\times\Omega \\ u \in L^2_{\text{loc}}((0,T)\times\Omega) \\ u = 0 \text{ in } (0,T)\times\omega \end{cases} \implies u = 0 \text{ in } (0,T)\times\Omega.$$

Note that by $\mathsf{q} \in \mathcal{G}^2((0,T); L^\infty_{\text{loc}}(\Omega;\mathbb{C}))$, we mean $\mathsf{q} \in \mathcal{G}^2((0,T); L^\infty(K;\mathbb{C}))$ for all compact subsets K of Ω. Note also that under the assumptions of Theorem 2.2, the Cauchy problem $P_{\mathsf{q}}u = 0, u(0,\cdot) = u_0$ is not well-posed in general. However the divergence form of the principal part of P_{q} together with the respective regularity assumptions on g^{jk}, q and u allow to make sense of $P_{\mathsf{q}}u$ in $\mathcal{D}'(\Omega)$.

The global result of Theorem 2.2 is a consequence of the following local result, near a point $(t_0, x_0) \in (0,T)\times\Omega$.

Theorem 2.3 (Local unique continuation)**.** *Assume $X = I \times V$ where $I \subset \mathbb{R}$ is an open interval and $V \subset \mathbb{R}^d$ an open set, and let $(t_0, x_0) \in X$. Assume $g^{jk} \in W^{1,\infty}(V)$ satisfies* (1.2) *for $x \in V$, that $\mathsf{q} \in \mathcal{G}^2(I; L^\infty(V;\mathbb{C}))$. Let $\Psi \in C^1(X;\mathbb{R})$ such that $(\nabla_x \Psi)(t_0, x_0) \neq 0$. Then, there is a neighborhood W of (t_0, x_0) such that*

$$\big(P_{\mathsf{q}} u = 0 \quad \text{in } X, \quad u \in L^2(X), \quad u = 0 \text{ in } \{\Psi > 0\}\big) \implies u = 0 \text{ in } W.$$

The global result of Theorem 2.2 follows after successive applications of Theorem 2.3 through a family of well-chosen non-characteristic hypersurfaces (see [LL19, Proof of Theorem 6.7 p. 100] for the geometric construction in the proof). Before commenting this local result in Sections 4 and 5 below, we now briefly come back to control theory as presented in Section 1 and discuss applications of Theorem 2.2.

3 Application to controllability

The unique continuation result of Theorem 2.2 combined with Lemma 1.1 Item 2 and well-posedness of Equation (1.1) yields the following corollary.

Corollary 3.1. *Let $T > 0$, assume $\Omega \subset \mathbb{R}^d$ is a connected open set, that $g^{jk}(x) \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$ satisfies* (1.2) *for all $x \in \overline{\Omega}$, and $\mathsf{q} \in L^\infty((0,T)\times\Omega;\mathbb{C}) \cap \mathcal{G}^2((0,T); L^\infty_{\mathrm{loc}}(\Omega;\mathbb{C}))$. Then, for any nonempty open set $\omega \subset \Omega$, Equation* (1.1) *is approximately controllable from ω in time T.*

Note that we actually only need to assume $\mathsf{q} \in \mathcal{G}^2(I; L^\infty_{\mathrm{loc}}(\Omega;\mathbb{C}))$ for some nonempty open set $I \subset (0,T)$.

To illustrate the use of Theorem 2.2 for exact controllability purposes, we furnish now a single example of geometry (Ω, g), in which the high-frequency statement is available and directly usable in the literature.

Theorem 3.2. *Assume that Ω is the open unit Euclidean disk in $\mathbb{R}^2$ and $g = \mathrm{Id}$ is the Euclidean metric. Let $T > 0$ and suppose that $\mathsf{q} \in C^\infty([0,T] \times \overline{\Omega};\mathbb{R}) \cap \mathcal{G}^2((0,T); L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}))$ is real valued and ω is any nonempty open set of Ω such that $\overline{\omega} \cap \partial\Omega$ contains an open set of $\partial\Omega$. Then Equation* (1.1) *is exactly controllable from ω in time T.*

Our contribution in Theorem 3.2 is to include more general time-dependent potentials q, using Theorem 2.2 for the "low frequency" part of the proof. Theorem 3.2 is a direct combination of [ALM16, Theorem 1.2] (in which a compactness-uniqueness argument as Lemma 1.2 is used) and Theorem 2.2. Note that the C^∞ regularity of q can be relaxed, see [ALM16, Remark 1.6].

4 Comparison with the literature

The Schrödinger operator (2.1) under study here is a second order differential operator, with "principal symbol" (with its usual definition) given by

$$p_2(t, x, \xi_t, \xi_x) = -\sum_{j,k} g^{jk}(x)\xi_{x_j}\xi_{x_k}, \tag{4.1}$$

where ξ_t is the dual variable to t and ξ_{x_j} the dual variable to x_j.

A few unique continuation results are available in this situation. First, the Holmgren-John theorem [Hör63, Theorem 5.3.1] applies to P_{q} and yields unique continuation assuming all of its coefficients (i.e. all g and q) are real-analytic (with respect to all variables (t, x)), and the hypersurface S is *non-characteristic*, that is to say

$$p_2(t_0, x_0, d\Psi(t_0, x_0)) \neq 0, \quad \text{where } S = \{\Psi = 0\}. \tag{4.2}$$

Note that in view of the expression of p_2 in (4.1), the latter non-characteristicity assumption (4.2) is equivalent to the condition $(\nabla_x \Psi)(t_0, x_0) \neq 0$ required in Theorem 2.3. From the geometric point of view, we will see below that the non-characteristicity assumption is essentially optimal. From the point of view of regularity requirements, however, analyticity is of course very demanding.

Second, the classical Hörmander theorem [Hör94, Theorem 28.3.4] is empty in this situation. Taking advantage of the anisotropic (or quasi-homogeneous) nature of the Schrödinger operator, Lascar and Zuily proved in [LZ82] that the Hörmander theorem can be generalized to the anisotropic case with an appropriate modification of the symbol classes and Poisson bracket. See also [Deh84], [Isa93] and [Tat97] for later results in this direction. In the context of (2.1), this result applies for coefficients $g^{jk} \in C^1$ and $\mathsf{q} \in L^\infty$, under a pseudoconvexity condition on the hypersurface. The latter is a very strong *local* geometric assumption on the (oriented) surface for local unique continuation to hold, which necessarily leads to a very strong *global* geometric assumption of the observation set ω in an associated global unique continuation statement of the form of Theorem 2.2. For applications to control or inverse problems, related global Carleman estimates for Schrödinger operators have been proved for instance in [BP02] (constant leading order coefficients) and in [TX07, Lau10] (Riemannian manifolds or varying coefficients). A weak pseudoconvexity condition has also been proved sufficient in [MOR08] for a flat metric and in [Lau10] with varying metrics. Yet, in all of these references, a form of pseudoconvexity related to that of [LZ82] is required and global statements hold under strong geometric assumptions.

Third, the Tataru-Robbiano-Zuily-Hörmander [Tat95, RZ98, Hör97, Tat99] theorem (following earlier results by Robbiano [Rob91] for hyperbolic operators, subsequently improved in [Hör92]) also applies to the Schrödinger operator (2.1). In that case, it implies local unique continuation assuming that the surface S is *non-characteristic*, i.e. (4.2), that g^{ij} is either C^∞ in [RZ98, Hör97, Tat99] or C^1 in [Tat95], and that q is *real-analytic* with respect to the time variable t (only). This is to be compared to the Holmgren theorem where real-analyticity is assumed w.r.t. all variables (t, x). Note finally that T'joën [T'j00] proved a quasi-homogeneous variant of the Tataru-Hörmander-Robbiano-Zuily theorem in a general setting and Masuda [Mas67] proved a global uniqueness result in the case of C^2 principal coefficients and time independent coefficients.

In the present work, with respect to the Tataru-Robbiano-Zuily-Hörmander theorem for the Schrödinger operator (2.1), we relax the analyticity-in-time assumption for the lower order terms to a Gevrey 2 condition. We also relax the regularity of the main coefficients (assumed either C^∞ in [RZ98, Hör97, Tat99] or C^1 in [Tat95]), replaced here by Lipschitz regularity. We now discuss optimality issues for the assumptions of Theorem 2.3.

5 Optimality of the assumptions

Note first that our result is of no interest in space dimension $d = 1$, for unique continuation applies to $L^\infty(I \times V)$ lower order coefficients (without any Gevrey assumption; the appropriate pseudoconvexity condition being satisfied in $1D$), see e.g. [Isa93, Corollary 6.1.]. Let us thus only discuss optimality of the assumptions in higher dimension $d \geq 2$.

The geometric assumption. From the geometric point of view, the non-characteristicity assumption is optimal. Notice that it excludes only surfaces tangent to $\{t = t_0\}$, for which we know that local unique continuation may fail (this would otherwise imply finite speed of propagation for Schrödinger equations). Let us provide with an example illustrating this point. Consider the operator P_{q} in (2.1) on $\Omega = \mathbb{R}^d$ with $g = \mathrm{Id}$ and $\mathsf{q} = 0$. Given $w_0 \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} w_0 = \overline{B}_{\mathbb{R}^d}(0,1)$ the closed unit ball in $\mathbb{R}^d$, the function

$$w(t,x) := \frac{e^{-id\ \mathrm{sgn}(t)\pi/4}}{(4\pi|t|)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} w_0(y)dy \tag{5.1}$$

solves $(i\partial_t + \Delta)w = 0$ on $\mathbb{R}^{1+d}$ and $w(0,\cdot) = w_0$ on $\mathbb{R}^d$. Moreover, given the assumptions on w_0, we also have $w \in C^\infty(\mathbb{R}^{1+d})$ and $\partial_t^k w|_{t=0} = 0$ on $\mathbb{R}^d \setminus \overline{B}_{\mathbb{R}^d}(0,1)$ for all $k \in \mathbb{N}$. As a consequence, the function $u(t,x) := \mathbb{1}_{\mathbb{R}^+} w(t,x)$ satisfies $(i\partial_t + \Delta)u = 0$ in $B_{\mathbb{R}^{1+d}}((0,x_0),1)$ for any $x_0 \in \mathbb{R}^d$ such that $|x_0| > 2$. However, we notice that the explicit expression (5.1) implies that $u(t,\cdot)$ extends as a holomorphic function for any $t > 0$, namely

$$\mathbb{C}^d \ni z \mapsto u(t,z) = w(t,z) = \frac{e^{-id\pi/4}}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{i(z-y)^2/4t} w_0(y)dy, \qquad \text{for } t > 0.$$

As a consequence, for any $t > 0$ we have $\operatorname{supp}(u(t,\cdot)) = \mathbb{R}^d$ (if $u(t,\cdot) = w(t,\cdot)$ would vanish on a nonempty open set, it would vanish on the whole $\mathbb{R}^d$ from analytic continuation, and solving the Cauchy problem backwards would imply $w(0,\cdot) = w_0 = 0$). We finally deduce that u satisfies, with $B_0 := B_{\mathbb{R}^{1+d}}((0,x_0),1)$

$$(i\partial_t + \Delta)u = 0 \text{ in } B_0, \quad u \in C^\infty(B_0), \quad \operatorname{supp} u \cap B_0 = \{t \geq 0\} \cap B_0.$$

Hence, the operator $i\partial_t + \Delta$ does not satisfy local unique continuation near $(0,x_0)$ across $S = \{t = 0\}$.

The Lipschitz regularity of g. As already mentioned, we replace the C^∞ regularity (in [RZ98, Hör97, Tat99]) or C^1 regularity (in [Tat95]) of the metric g by Lipschitz regularity. In the elliptic context, this is essentially the minimal regularity in dimension $d \geq 3$ for local uniqueness to hold. See [Pli63] and [Mil74] for $C^{0,\alpha}$ counterexamples for all $\alpha < 1$, for operators in divergence forms or not. Since solutions to the elliptic equation yield stationary solutions to the Schrödinger equation, we deduce that Lipschitz regularity is essentially optimal in dimension $d \geq 3$.

The Gevrey regularity of q. The main improvement of Theorem 2.3 as compared to the general Tataru-Robbiano-Zuily-Hörmander concerns the regularity of the potential q (and more generally of lower order terms): we replace the analyticity-in-time assumption by a Gevrey 2 condition. Let us now discuss the optimality of this assumption.

First, it was proved in [LZ82, Théorèmes 1.4 & 1.6] that a quasihomogeneous version of pseudoconvexity is actually *necessary* for unique continuation to hold for general C^∞ lower order terms. As an illustration, [LZ82, Théorème 1.6] proves that if $d \geq 2$, there exist $u, \mathsf{q} \in C^\infty(B_{\mathbb{R}^{1+d}}(0,1);\mathbb{C})$ such that for $g = \mathrm{Id}$,

$$P_{\mathsf{q}} u = 0, \quad \text{in } B_{\mathbb{R}^{1+d}}(0,1), \qquad u = 0 \text{ on } x_1 > 0, \quad \text{and} \quad 0 \in \operatorname{supp}(u),$$

whence unique continuation *does not hold* across the non-characteristic surface $\{x_1 = 0\}$. Hence the statement of Theorem 2.3 is false without the Gevrey-in-time regularity assumption of q. A semiglobal version of this counterexample was constructed by Takase in [Tak21], where the author constructs $u, \mathsf{q} \in C^\infty(\mathbb{R}^{1+2})$ solving, for $g = \mathrm{Id}$, $P_\mathsf{q} u = 0$ in $\mathbb{R}^{1+2}$ together with $\operatorname{supp}(u) = \mathbb{R} \times \big(\mathbb{R}^2 \setminus B(0,1)\big)$.

Second, in the case of the wave equation, the classical counterexamples of Alinhac-Baouendi [AB79, Ali83, AB95], as refined by Hörmander [Hör00], prove the following statement. For any $s > 1$ and $d \geq 2$, there exist $u, \mathsf{q} \in \mathcal{G}^s(B_{\mathbb{R}^{1+d}}(0,1); \mathbb{C})$ so that

$$\partial_t^2 u - \Delta u + \mathsf{q} u = 0, \quad \text{in } B_{\mathbb{R}^{1+d}}(0,1), \text{ and } \operatorname{supp}(u) = \{(t, x_1, \ldots, x_d) \in B_{\mathbb{R}^{1+d}}(0,1), x_1 \leq 0\}.$$

This shows that, without any further assumptions, the analyticity in time of q is essentially optimal (within the class of Gevrey spaces; note that Hörmander's statement is even stronger) in geometrical situations where the strong pseudoconvexity of the hypersurface is not satisfied. Theorem 2.3 shows that in the context of the Schrödinger equation, Gevrey $1+\varepsilon$ counterexamples do not exist. It would be interesting to know if a similar counterexample can be proved for Schrödinger type equations with Gevrey $2+\varepsilon$ coefficients, that is to say, whether the Gevrey 2 regularity in time is the critical one.

6 Additional remarks

Before sketching the proof of Theorem 2.3, we briefly present in this section some of its variants or generalizations. We refer the possible interested reader to [FLL24] for more details on these topics.

Regularity of the solution. One can further lower the regularity of the solution u by assuming additional regularity of the coefficients g^{ij}, q. For instance, assuming (in addition to the assumptions of Theorem 2.3) that $g^{ij} \in C^\infty(V), \mathsf{q} \in C^\infty(X; \mathbb{C})$, then we have unique continuation for distributions:

$$(P_\mathsf{q} u = 0 \quad \text{in } X, \quad u \in \mathcal{D}'(X), \quad u = 0 \text{ in } \{\Psi > 0\}) \implies u = 0 \text{ in } W.$$

Divergence form of the operator. In Theorems 2.3 and 2.2, as already noticed, it is important for the elliptic part of the operator to be in divergence form in order to make sense of $\mathcal{P}_\mathsf{q} u$ in the sense of distributions. Nevertheless, the principal term $\Delta_g = \partial_{x_j} g^{jk} \partial_{x_k}$ in these two statements may be replaced by *any* operator of the form

$$\Delta_{g,\varphi} := \operatorname{div}_\varphi \nabla_g,$$

where g is a Lipschitz continuous Riemannian metric, φ is a Lipschitz continuous nowhere vanishing density and $\operatorname{div}_\varphi$ and ∇_g denote respectively the associated divergence and gradient:

$$\operatorname{div}_\varphi(X) = \sum_{j=1}^d \frac{1}{\varphi} \partial_{x_j} (\varphi X_j), \qquad \nabla_g u = \sum_{j,k=1}^d g^{jk} (\partial_{x_j} u) \frac{\partial}{\partial_{x_k}}.$$

The results of Theorem 2.3 and Theorem 2.2 actually depend on the density chosen (i.e. the result for one density cannot be deduced from that for another density). They are however valid for any locally Lipschitz nonvanishing density.

More general lower order terms. Finally, first order terms of the form $\sum_{j=1}^d \mathsf{b}_j(t,x) \partial_{x_j}$ can also be included in our results, provided that the coefficients b_j belong to $\mathcal{G}^2((0,T); L^\infty_{\mathrm{loc}}(\Omega))$. This however requires the solution to be of regularity $L^2_{\mathrm{loc}}((0,T); H^1_{\mathrm{loc}}(\Omega))$, for the equation to make sense. One may also include $\mathbb{C}$–antilinear lower order terms (with coefficients having the same regularity) in the unique continuation statements.

7 Sketch of the proof

Since the pioneering work of [Car39], Carleman inequalities are one of the main tools for proving unique continuation results. Carleman estimates are weighted inequalities of the form

$$\left\|e^{\tau\phi} P u\right\|_{L^2} \gtrsim \tau^{3/2}\left\|e^{\tau\phi} u\right\|_{L^2}, \quad \tau \geq \tau_0, \tag{7.1}$$

which are uniform in the large parameter τ and apply to compactly supported functions u. The weight $e^{\tau\phi}$ allows to propagate uniqueness from large to low level sets of ϕ by letting $\tau \to \infty$. The presence of the parameter $\tau^{3/2}$ in (7.1) allows to absorb perturbations of order zero (with L^∞ regularity) of the operator P. However, in case $P = i\partial_t + \Delta_g$ under interest here, (7.1) essentially never holds unless a condition related to the strong pseudoconvexity assumption of [LZ82] is satisfied.

The key additional idea in [Tat95] (following the introduction in this problem of the FBI transform in time in [Rob91]) is to make use of the nonlocal Fourier multiplier in time $e^{-\varepsilon|D_t|^2/2\tau}$, and replace (7.1) by

$$\left\|e^{-\varepsilon|D_t|^2/2\tau} e^{\tau\phi} P u\right\|_{L^2} + e^{-\mathsf{d}\tau}\left\|e^{\tau\phi} u\right\|_{L^2} \gtrsim \tau^{3/2}\left\|e^{-\varepsilon|D_t|^2/2\tau} e^{\tau\phi} u\right\|_{L^2}, \quad \tau \geq \tau_0. \tag{7.2}$$

A key feature of this approach is that, although (7.2) carries less information on $e^{\tau\phi}u$, it is still enough to prove unique continuation. And the advantage of (7.2) with respect to (7.1) is that the operator and the function are localized in a low frequency regime with respect to the variable t. Hence (7.2) holds if we only assume the classical pseudoconvexity assumption in a smaller subset of the phase space, namely where $\xi_t = 0$ (here, ξ_t is the dual variable to t). See [Tat95, RZ98, Hör97, Tat99] for the original proofs and [LL23] for introductory lecture notes on this topics in the case of the wave operator.

In the setting of the wave operator $P = -\partial_t^2 + \sum \partial_{x_j} g^{jk}(x)\partial_{x_k}$, the principal symbol $p_2 = \xi_t^2 - \sum g^{jk}(x)\xi_{x_j}\xi_{x_k}$ is homogeneous of degree two in all cotangent variables (ξ_t, ξ_x). When proving Carleman estimates like (7.1) or (7.2), the large parameter τ plays the role of a derivative, which, naturally results in $D_t \sim D_x \sim \tau$. In this scaling, the Fourier multiplier $\varepsilon|D_t|^2/2\tau$ appearing in (7.2) is "of order one", and large frequencies $|D_t| \geq c_0\tau$ only contribute to admissible remainders of size $e^{-\varepsilon(c_0^2/2)\tau}$.

The first main idea for the proof of Theorem 2.3 is to prove a Carleman estimate adapted to the anisotropy of the Schrödinger operator (2.1) in case $\mathsf{q} = 0$. In this setting, we want to consider that D_t is homogeneous to $D_x^2 \sim \tau^2$. With this new definition of homogeneity/order/scaling, the natural "first order" Fourier multiplier in time is $|D_t|^2/\tau^3$. Therefore, the first step of the proof of Theorem 2.3 is a Carleman estimate of the form

$$\left\|e^{-\mu|D_t|^2/2\tau^3} e^{\tau\phi} P u\right\|_{L^2} + e^{-\mathsf{d}\tau}\left\|e^{\tau\phi} u\right\|_{L^2} \gtrsim \tau^{3/2}\left\|e^{-\mu|D_t|^2/2\tau^3} e^{\tau\phi} u\right\|_{L^2}, \quad \tau \geq \tau_0, \tag{7.3}$$

for the unperturbed Schrödinger operator $P = i\partial_t + \sum \partial_{x_j} g^{jk}(x)\partial_{x_k}$. Note that as compared to (7.2), frequencies $|D_t| \geq c_0\tau^2$ contribute to admissible remainders of size $e^{-\mu(c_0^2/2)\tau}$. In other words, (7.3) carries information on time-frequencies $|D_t| \lesssim \tau^2$ of the function $e^{\tau\phi}u$ whereas the usual estimate (7.2) only contains information on time-frequencies $|D_t| \lesssim \tau$. This is also clearly seen in the proof of [LL19] of the optimal quantitative version of the Tataru-Hörmander-Robbiano-Zuily theorem. In [LL19], the Carleman estimate (7.3) allows to propagate low frequency information of the solution in the sense $|D_t| \lesssim \tau$; whereas the Carleman estimate (7.3) will allow to propagate low frequency information of order $|D_t| \lesssim \tau^2$. This indicates that the new weight allows to "propagate more information".

The key step in the proof of the Carleman inequality (7.3) is a subelliptic estimate for the conjugated operator $P_{\phi,\mu}$ defined by

$$e^{-\mu|D_t|^2/2\tau^3}e^{\tau\phi}P = P_{\phi,\mu}e^{-\mu|D_t|^2/2\tau^3}e^{\tau\phi}, \tag{7.4}$$

where the time independence of the coefficients of P is crucial for the computation of $P_{\phi,\mu}$. The latter takes the form

$$\|P_{\phi,\mu}v\|_{L^2} + \tau^{-1/2}\|D_t v\|_{L^2} \gtrsim \tau^{3/2}\|v\|_{L^2}, \quad \tau \geq \tau_0. \tag{7.5}$$

That the subelliptic estimate (7.5), applied to $v = e^{-\mu|D_t|^2/2\tau^3}e^{\tau\phi}u$, implies the Carleman inequality (7.3) follows from the fact that $e^{-\mu|D_t|^2/2\tau^3}$ localizes exponentially close to $D_t = 0$. Hence the term $\|D_t v\|$ mostly contributes to the exponentially small remainder in (7.3) plus a small term that one can absorb in the right hand-side of (7.3) . The proof of (7.5) relies on two steps. We first perform the computations in the case $\mu = 0$, that is to say, as for a traditional Carleman estimate of the form (7.1), with the difference that all terms involving $\|D_t v\|$ can be considered as remainder terms. This essentially reduces this step to a usual Carleman estimate for elliptic operators with only Lipschitz regularity (plus remainder terms involving time derivatives), for which we rely on [LL21, Appendix A]. Then the second step consists in considering the general case $\mu > 0$ as a perturbation of the previous step plus admissible remainder terms. A related (although different) perturbation argument is used in the proofs of [Tat95, Hör97, RZ98, Tat99], see e.g. [LL23, Section 3.3]. A remarkable difference is that we prove (7.3) for all $\mu > 0$, whereas (7.2) only holds for *small* $\varepsilon > 0$.

The second main step for the proof of Theorem 2.3 is to prove that (7.3) still holds for general q having Gevrey 2 time-regularity. To this aim, we perform again a perturbation argument and essentially need to prove that

$$\left\|e^{-\mu|D_t|^2/2\tau^3}(\mathsf{q}w)\right\|_{L^2} \lesssim \left\|e^{-\mu|D_t|^2/2\tau^3}w\right\|_{L^2} + e^{-\mathsf{d}\tau}\|w\|_{L^2}, \tag{7.6}$$

which is an admissible remainder in (7.3) (i.e. may be absorbed in the right-hand side for τ large). The proof of (7.6) relies on a conjugation result of the form (7.4) but for the multiplication by a function, say q, depending on t. The issue of defining a conjugate of a multiplication operator q by $e^{-\varepsilon|D_t|^2/2\tau}$ is one of the main difficulties in [Tat95, RZ98, Hör97, Tat99]. Even if the function q is real analytic with respect to t a conjugate operator with respect to $e^{-\varepsilon|D_t|^2/2\tau}$ does not necessarily exist. However, one can define an approximate conjugate up to an error of the form $e^{-d\tau}\|u\|_{L^2}$, which is admissible in view of (7.6) and (7.2). In the present setting and if typically $\mathsf{q} \in \mathcal{G}^2(\mathbb{R};\mathbb{C})$ depends only on t, our conjugation result writes (say, in case $\mu = 1$ for readability)

$$e^{-|D_t|^2/2\tau^3}\mathsf{q} = \mathrm{op}^w\left(\tilde{\mathsf{q}}_\tau(t,\xi_t)\right)e^{-|D_t|^2/2\tau^3} + O\left(e^{-\delta\tau}\right)_{\mathcal{L}(L^2(\mathbb{R}))}, \quad \tau \to +\infty, \tag{7.7}$$

where $\mathrm{op}^w\left(\tilde{\mathsf{q}}_\tau(t,\xi_t)\right)$ is the Weyl quantization of a symbol $\tilde{\mathsf{q}}_\tau(t,\xi_t)$ constructed from q. Here, $(t,\xi_t) \in \mathbb{R}\times\mathbb{R}$, with the second variable being the dual variable to t, that is to say such that $\mathrm{op}^w(\xi_t) = D_t$. More precisely, in this expression, the symbol $\tilde{\mathsf{q}}_\tau(t,\xi_t)$ of the approximate conjugated operator is given by

$$\tilde{\mathsf{q}}_\tau(t,\xi_t) = \eta\left(\xi_t/\tau^2\right)\tilde{\mathsf{q}}\left(t + i\xi_t/\tau^3\right), \quad \text{for } (t,\xi_t) \in \mathbb{R}\times\mathbb{R},$$

where

1. $\tilde{\mathsf{q}}$ is an almost analytic extension of q to $\mathbb{C}$ (in the sense that $\partial_{\bar{z}}\tilde{\mathsf{q}}$ vanishes at any order on the real line), well-suited to the $\mathcal{G}^2$ regularity of q (in the sense that it satisfies $\tilde{\mathsf{q}} \in \mathcal{G}^2(\mathbb{C};\mathbb{C})$). For $\mathsf{q} \in \mathcal{G}^s(\mathbb{R};\mathbb{C})$ such a well-chosen almost analytic extension $\tilde{\mathsf{q}}(z)$ satisfies
$$\|\partial_{\bar{z}}\tilde{\mathsf{q}}(z)\| \leq C \exp\left(-\frac{1}{C_0|\operatorname{Im}(z)|^{1/(s-1)}}\right);$$
2. $\eta \in C_c^\infty(\mathbb{R})$ satisfies $\eta = 1$ in a neighborhood of zero. In particular, η cuts-off high frequencies $|D_t| \gtrsim \tau^2$, which, as already mentioned, is the right scale in the present setting.

Our proof of the conjugation result (7.7) is inspired by the strategy of Tataru [Tat99], with particular attention paid to the different scalings and to the fact that the functions involved are not analytic. It proceeds with a deformation of contour on the support of $\eta\left(\xi_t/\tau^2\right)$, that is to say in an $O(\tau^{-2})$ neighborhood of the real axis, where the almost analytic extension $\tilde{\mathsf{q}}$ satisfies
$$\left\|\left(\partial_{\bar{z}}\tilde{\mathsf{q}}\right)\left(t + i\xi_t/\tau^3\right)\right\| \lesssim \exp\left(-\tau^3/C_0|\xi_t|\right) \lesssim \exp\left(-\tau/C_0'\right), \quad \text{on} \quad \operatorname{supp}\eta\left(\xi_t/\tau^2\right).$$

Owing to the fact that $\operatorname{op}^w\left(\tilde{\mathsf{q}}_\tau(t,\xi_t)\right)$ is uniformly bounded on $L^2(\mathbb{R})$, the conjugation result (7.7) provides a proof of (7.6) and eventually of (7.3) for the perturbed operator P_{q}.

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