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Leading-order term expansion for the Teukolsky equation on subextremal Kerr black holes

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Abstract

In this note, we present a result based on [47] providing the large time leading-order term for initially localized solutions of the Teukolsky equation on a general subextremal Kerr black hole. This equation is important in the study of Maxwell's equations and Einstein's equations on a Kerr background. The method used is based on a careful analysis of the resolvent operator permitted by recent advances in microlocal analysis including non elliptic Fredholm theory, radial point estimates and semiclassical analysis near the trapped set together with a result about the absence of modes for the Teukolsky equation.

1 Introduction

1.1 Einstein's equations

Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold with signature $(+,-,-,-)$ and let ∇ be the associated Levi-Civita connection. The Riemann tensor of the metric is defined by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where X, Y, Z are any smooth vector fields on $\mathcal{M}$. Since for fixed vector fields Y and Z, $R(\cdot, Y)Z$ is a smooth section of $End(T\mathcal{M})$ we can define its trace:

$$\mathrm{Ric}(Y,Z) = \mathrm{tr}(R(\cdot, Y)Z)$$

which is called the Ricci tensor. Taking again the trace with respect to the metric g, we obtain the scalar curvature S. Einstein's equations are then written as:

$$\mathrm{Ric} - \frac{1}{2} S g - \Lambda g = T$$

where Λ is a constant called the cosmological constant, T is (up to a constant[1]) the stress-energy tensor which describes the matter content of the spacetime. We focus here on the simplest case of Einstein *vacuum* equations with vanishing cosmological constant:

$$\mathrm{Ric} = 0. \tag{1}$$

[1]For simplicity, we choose the physical units such that the speed of light is equal to 1 and the gravitational constant G is equal to $1/8\pi$ so that the constant in the Einstein equation is exactly 1.

1.2 Black hole solutions

The simplest solution to (1) is the Minkowski spacetime:

$$\mathcal{M} = \mathbb{R}_t \times (\mathbb{R}^3)_x$$
$$g = (\mathrm{d}t)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2$$

which has a vanishing curvature.

More interesting solutions are given by the family of subextremal Kerr black holes. The family is indexed by two parameters $M > 0$ (mass of the black hole) and $a \in \mathbb{R}$ (angular momentum per unit of mass) with $|a| < M$. The exterior region is described by the manifold $\mathcal{M} := \mathbb{R}_t \times (r_+, +\infty)_r \times \mathbb{S}^2$ with $r_+ := M + \sqrt{M^2 - a^2}$ and the metric:

$$g_{M,a} := \frac{\Delta_r - a^2 \sin^2\theta}{\rho^2}\mathrm{d}t^2 + \frac{4Mar\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\phi - \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 \tag{2}$$
$$- \rho^2\mathrm{d}\theta^2 - \frac{\sin^2\theta}{\rho^2}((a^2 + r^2)^2 - a^2\Delta_r \sin^2\theta)\mathrm{d}\phi^2 \tag{3}$$
$$\Delta_r := a^2 + r^2 - 2Mr$$
$$\rho^2 := r^2 + a^2\cos^2\theta.$$

The value r_+ is the largest root of Δ_r.

In the case $a = 0$, we obtained the Schwarzschild solution:

$$g_M := \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right).$$

We record some important geometric observations about the Kerr solution:

- The metric $g_{M,a}$ is stationary ($T := \partial_t$ is a Killing vector field) and invariant by rotation of angle ϕ (∂_ϕ is a Killing vector field).
- The vector field ∂_t is timelike for large r but becomes spacelike[2] inside the region

  $$\mathcal{E} := \left\{\Delta_r - a^2\sin^2\theta < 0\right\}.$$

  Moreover, there is no global choice of a Killing vector field which is globally timelike on $\mathcal{M}$.
- The metric $g_{M,a}$ is asymptotically flat, meaning that as $r \to +\infty$:

  $$g_{M,a} = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + O(r^{-1})_{t,r,\theta,\phi}.$$
- There exists a non-empty set of null[3] geodesics $\mathcal{K}$ such that for all $\gamma \in \mathcal{K}$, $r(\gamma)$ is contained in a compact subset of $(r_+, +\infty)$. We say that $\mathcal{K}$ is the set of trapped null geodesics.

[2] A vector is called spacelike if it has negative Lorentz norm.
[3] A vector is called null if it has zero Lorentz norm.

Based on (2), there is a coordinate singularity at $r = r_+$. The metric can in fact be extended to a strictly larger spacetime if we make the change of coordinate:

$$t_* = t + T(r)$$
$$\phi_* = \phi + A(r)$$

with

$$T'(r) = \frac{a^2 + r^2}{\Delta_r}$$
$$A'(r) = \frac{a}{\Delta_r}.$$

In these new coordinates, $g_{M,a}$ extends analytically to a solution of the Einstein vacuum equations on the slightly larger manifold $\mathcal{M}^{ext} := \mathbb{R}_{t_*} \times (M, +\infty)_r \times \mathbb{S}^2_{\theta,\phi_*}$. In particular, this region includes the null hypersurface $\mathcal{H}^+ := \{r = r_+\}$ which is called the future event horizon of the black hole.

We finally add a hypersurface $\mathscr{I}^+$ diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$ called null infinity corresponding to the point at infinity of outgoing null geodesics. More concretely, we define the manifold with boundaries $\tilde{\mathcal{M}} := \mathcal{M}^{ext} \bigsqcup \mathbb{R} \times \mathbb{S}^2$ using the local chart defined on $\mathcal{M} \bigsqcup \mathbb{R} \times \mathbb{S}^2$ (which is a neighborhood of the boundary):

$$\phi : \begin{cases} x \in \mathcal{M} \mapsto (t(x) - T(r(x)), 1/r(x), \omega(x)) \\ (u, \omega) \in R \times \mathbb{S}^2 \mapsto (u, 0, \omega) \end{cases}$$

To be able to set up a Cauchy problem on $\mathcal{M}$, we define the initial hypersurface Σ_0 to be a spacelike hypersurface of $\mathcal{M}^{ext}$ equal to the zero level set of t for large r and transverse to $\mathcal{H}^+$. Finally, we introduce a function $\mathfrak{t}$ whose level sets are transverse both to $\mathfrak{H}^+$ and to $\mathcal{I}^+$ (see Figure 1). It is not necessary that $\mathfrak{t}$ is timelike but we require $\mathfrak{t} = t_* + h(r)$ for some smooth function h. In practice, we interpolate between $\mathfrak{t} = t_*$ near the horizon and $\mathfrak{t} = t - T(r) + C$ (where C is some constant) near infinity but other choices are possible.

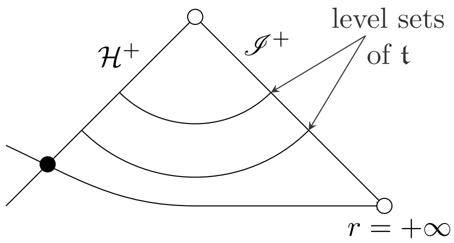

Figure 1: Penrose diagram of the future exterior region of a Kerr black hole.

1.3 Black hole stability problem

Let $(\mathcal{M}, g)$ be a Lorentzian metric and let $\Sigma_0 \subset \mathcal{M}$ be a spacelike hypersurface with unit normal n. We say that Σ_0 is a Cauchy surface of $(\mathcal{M}, g)$ if every causal[4] curve intersects Σ_0 exactly once. We call Cauchy data of g on Σ_0 the couple (h, k) of symmetric 2-tensors on Σ_0 such that

[4]timelike or null

$h = g_{|T\Sigma_0 \otimes T\Sigma_0}$ is the induced metric on Σ_0 and k is the second fundamental form of Σ_0. If (x^i) are local coordinates on Σ_0, we have $k_{i,j} = -g(\nabla_{\partial_i} n, \partial_j)$.

The Cauchy problem for the Einstein vacuum equations is as follows: Given (Σ_0, h) a (negative definite) Riemannian manifold and a symmetric 2-tensor k on Σ_0, find a solution of the Einstein vacuum equations $(\mathcal{M}, g)$ endowed with an inclusion $\Sigma_0 \subset \mathcal{M}$ such that Σ_0 is a Cauchy hypersurface of $\mathcal{M}$ and the Cauchy data of g on Σ_0 is precisely (h, k). The equation $Ric(g) = 0$ imposes constraints on the Cauchy data (h, k). This constraints are therefore necessary conditions for the existence of a solution.

The fact that the constraint equations are also sufficient for the existence of a solution has been proved in the foundational work of Choquet-Bruhat [15]

Theorem 1.1. *The Cauchy problem with initial data (Σ_0, h, k) has a solution if and only if (h, k) satisfy the constraint equations.*

The question of uniqueness is more delicate since the Einstein equation are invariant under diffeomorphisms: $Ric(\phi^* g) = \phi^* Ric(g)$ and of course there exists a lot of non trivial diffeomorphisms from $\mathcal{M}$ to $\mathcal{M}$ which are the identity near Σ_0. Therefore, g is not unique but all other solutions obtained by this process are isometric to one another. The other problem is that from our definition of solution there is no requirement for a solution to be the largest possible, we can restrict g to any subset of $\mathcal{M}$ containing Σ_0 as a Cauchy hypersurface and still get a solution. The situation is clarified by Choquet-Bruhat-Geroch [8]:

Theorem 1.2. *If (Σ_0, h, k) satisfy the constraint equations, there exists a solution $(\mathcal{M}, g)$ (unique up to isometry) called the maximal globally hyperbolic development (MGHD) of (Σ_0, h, k) such that every other solution can be isometrically embedded into $(\mathcal{M}, g)$.*

We can now state an informal version of the Kerr black hole stability conjecture. Let (Σ_0, h, k) be Cauchy data of a black hole solution $(\mathcal{M}, g_{M,a})$.

Conjecture 1.3. If $(\tilde{h}, \tilde{k})$ is sufficiently close to (h, k) (in some suitable functional norm) and satisfy the constraint equations, then denoting by $(\tilde{\mathcal{M}}, \tilde{g})$ the solution of the Cauchy problem with data $(\Sigma_0, \tilde{h}, \tilde{k})$, there exists a diffeomorphism $\phi : \mathcal{M} \to \tilde{\mathcal{M}}$ and black hole parameters (M', a') close to (M, a) such that $\phi^* \tilde{g}$ is close to $g_{M',a'}$ and $\left|\phi^* \tilde{g} - g_{M',a'}\right| \square \mathfrak{t}^{-\alpha}$ for $\alpha > 0$.

The first fully nonlinear result about the stability of slowly rotating Kerr black holes has been obtained by Hintz-Vasy [25] in the case of positive cosmological constant $\Lambda > 0$ and $|a| \ll M, \Lambda$ (see also [23] for the extension to slowly rotating charged black hole). For $\Lambda = 0$, Klainerman-Szeftel [29] proved the stability of a Schwarzschild solution for a particular class of initial data (polarized perturbations) for which $a' = 0$. The stability of Schwarzschild for the largest set of initial data imposing $a' = 0$ (which is codimension 3 in the set of all admissible initial data) was established by Dafermos-Holzegel-Rodnianski-Taylor in [10]. Finally, the stability of the slowly rotating Kerr family ($|a| \ll M$) was obtained in a series of works [30, 16, 54]. The problem remains open for a not slowly rotating subextremal black hole.

A necessary step to address the black hole stability problem is to understand the linearized problem at a fixed subextremal Kerr solution $g_{M,a}$. This yields to the linearized Einstein vacuum equations:

$$D_{g_{M,a}} Ric(\dot{g}) = 0 \tag{4}$$

Where $\dot{g}$ is a symmetric two tensor on $\mathcal{M}$ representing the linear metric perturbation. The gauge freedom arising from the diffeomorphism invariance of the Einstein vacuum equations provides an infinite dimensional space of solutions. In particular, for any smooth compactly supported vector field X on $\mathcal{M}$, $\dot{g}_X := \mathcal{L}_X g_{M,a}$ is a solution of (4).

1.4 The Teukolsky equation

We now define the Teukolsky scalars which are quantities built from $\dot{g}$ which decouple from the rest of the system. We need some additional geometric definitions first. Let $W(g)$ denote the Weyl tensor of g which is the trace-free part of the Riemann curvature tensor. In local coordinates:

$$W(g)_{\alpha\beta\mu\nu} := R_{\alpha\beta\mu\nu} - \frac{1}{2}\left(Ric_{\beta\nu}g_{\alpha\mu} - Ric_{\alpha\nu}g_{\beta\mu} + Ric_{\alpha\mu}g_{\beta\nu} - Ric_{\beta\mu}g_{\alpha\nu}\right) + \frac{1}{6}S\left(g_{\alpha\mu}g_{\beta\nu} - g_{\beta\mu}g_{\alpha\nu}\right)$$

Note that for g a solution of (1), $W(g) = R(g)$ and at the linear level, if $\dot{g}$ is solution of (4), $D_{g_{M,a}}W(\dot{g}) = D_{g_{M,a}}R(\dot{g})$. As a consequence, we can equivalently use the Riemann curvature tensor to define the Teukolsky scalars in this case.

Definition 1.4. A null vector vector $v \in T_x\mathcal{M}$ is said to be a principal null vector if for all $v_1, v_2 \in v^{\perp}$:

$$W(g)(v, v_1, v, v_2) = 0$$

The vector v is said to be a double principal null vector if the same is true for a general $v_2 \in T_x\mathcal{M}$.

The key geometric feature of the Kerr spacetime which allows to partially decouple (4) is stated in the following lemma (see for example [50] for a proof):

Lemma 1.5. *There exist l and n smooth null vector fields such that, for all $x \in \mathcal{M}$, $l(x)$ and $n(x)$ are independent double principal null vectors of the background metric $g_{M,a}$.*

Given such (l, n), we can assume that $g(l, n) = 1$ and construct locally a tetrad $(l, n, m, \overline{m})$ such that m is a complex vector field on $\mathcal{M}$ satisfying:

$$g(m, \overline{m}) = -1$$
$$g(m, m) = g(l, m) = g(n, m) = 0.$$

Definition 1.6 (Teukolsky scalars)**.** We define the following scalars:[5]

$$\Psi_2 = D_{g_{M,a}}W(\dot{g})(l, m, l, m)$$
$$\Psi_{-2} = (r - ia\cos\theta)^4 D_{g_{M,a}}W(\dot{g})(n, \overline{m}, n, \overline{m})$$

The key result of [58] is:

Theorem 1.7 (Teukolsky)**.** *There exists two scalar second order linear differential operators T_2 and T_{-2} with principal symbol $\rho^2 g^{\mu\nu}\xi_\mu\xi_\nu$ (where we used the Einstein summation convention for repeated indices) such that:*

$$T_2\Psi_2 = 0$$
$$T_{-2}\Psi_{-2} = 0$$

[5]For completeness, we mention that the above tetrad cannot be defined globally on the Kerr exterior. Therefore, Teukolsky scalars are defined globally as section of a nontrivial complex line bundle over $\mathcal{M}$. For keeping the exposition simple, we do not develop this point in this note, see [46] for details.

The following proposition provides a second motivation for considering Teukolsky scalars as significant quantities.

Proposition 1.8 (Gauge invariance)**.** *The Teukolsky scalar are gauge invariant, namely for any smooth compactly supported vector field X:*

$$\Psi_2(\dot{g}) = \Psi_2(\dot{g} + \mathcal{L}_X g_{M,a})$$
$$\Psi_{-2}(\dot{g}) = \Psi_{-2}(\dot{g} + \mathcal{L}_X g_{M,a})$$

Proof. We need to prove:

$$D_{g_{M,a}} W(\mathcal{L}_X g_{M,a})(l, m, l, m) = 0$$
$$D_{g_{M,a}} W(\mathcal{L}_X g_{M,a})(n, \overline{m}, n, \overline{m}) = 0$$

We focus on the first equality, the second one being similar. We define ϕ^s the flow at time s of X.

$$\begin{aligned} D_{g_{M,a}} W(\mathcal{L}_X g_{M,a}) &= \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} W((\phi^s)^* g_{M,a}) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} (\phi^s)^* W(g_{M,a}) \\ &= \mathcal{L}_X W(g_{M,a}) \end{aligned}$$

We deduce:

$$\begin{aligned} &D_{g_{M,a}} W(\mathcal{L}_X g_{M,a})(l, m, l, m) \\ &\qquad = X(W(g_{M,a})(l, m, l, m)) - 2W(g_{M,a})([X, l], m, l, m) - 2W(g_{M,a})(l, [X, m], l, m) \end{aligned}$$

Using that l is double principal null, we get:

$$D_{g_{M,a}} W(\mathcal{L}_X g_{M,a})(l, m, l, m) = -2W(g_{M,a})([X, l], m, l, m)$$

Since the tetrad is normalized, we have:

$$[X, l] = g_{M,a}(n, [X, l])l + g_{M,a}(l, [X, l])n - g_{M,a}(\overline{m}, [X, l])m - g_{M,a}(m, [X, l])\overline{m}$$
$$D_{g_{M,a}} W(\mathcal{L}_X g_{M,a})(l, m, l, m) = g_{M,a}(l, [X, l])R(g_{M,a})(n, m, l, m) - g(m, [X, l])R(g_{M,a})(\overline{m}, m, l, m).$$

On the other hand we have:

$$0 = Ric(l, m) = R(n, l, l, m) + R(l, l, n, m) - R(m, l, \overline{m}, m) - R(\overline{m}, l, m, m)$$
$$0 = Ric(m, m) = R(n, m, l, m) + R(l, m, n, m) - R(m, m, \overline{m}, m) - R(\overline{m}, m, m, m)$$

leading to

$$R(n, m, l, m) = R(\overline{m}, m, l, m) = 0.$$

□

A remarkable fact is that a similar decoupling also occurs in the Maxwell system. Let F denote a 2-form on $\mathcal{M}$ (the electromagnetic tensor) solution to the Maxwell vacuum equations:

$$\mathrm{d}F = 0$$
$$\mathrm{d} \star F = 0$$

where $\star$ is the Hodge operator associated to $g_{M,a}$.

Definition 1.9 (Teukolsky scalars for Maxwell)**.**

$$\Psi_1 := F(l,m)$$
$$\Psi_{-1} := (r - ia\cos\theta)^2 F(\overline{m}, n)$$

Theorem 1.10 (Teukolsky)**.** *There exists two scalar second order linear differential operators T_1 and T_{-1} with principal symbol $\rho^2 g^{\mu\nu}\xi_\mu\xi_\nu$ (where we used the Einstein summation convention for repeated indices) such that:*

$$T_1\Psi_1 = 0$$
$$T_{-1}\Psi_{-1} = 0$$

In fact, it turns out that the Teukolsky operator T_s can be defined for every $s \in \frac{1}{2}\mathbb{Z}$ and the operators $T_{\pm\frac{1}{2}}$ are obtained when decoupling the massless neutrino equation (see [58] for details). In Boyer-Lindquist coordinates and in the tetrad:[6]

$$\begin{aligned} l &:= (r^2+a^2)\partial_t + \Delta_r\partial_r + a\partial_\phi \\ n &:= \frac{r^2+a^2}{2\rho^2\Delta_r}\partial_t - \frac{1}{2\rho^2}\partial_r + \frac{a}{2\rho^2\Delta_r}\partial_\phi \\ m &:= \frac{ia\sin\theta}{\sqrt{2}(r+ia\cos\theta)}\partial_t + \frac{1}{\sqrt{2}(r+ia\cos\theta)}\partial_\theta + \frac{i}{\sqrt{2}(r+ia\cos\theta)\sin\theta}\partial_\phi \end{aligned}$$

the Teukolsky operator is[7]

$$\begin{aligned} T_s = \rho^2\Box_{g_{M,a}} &+ 2s(r-M)\partial_r - 2s\left(\frac{a(r-M)}{\Delta_r} + \frac{i\cos\theta}{\sin^2\theta}\right)\partial_\phi \\ &- 2s\left(\frac{M(r^2-a^2)}{\Delta_r} - r - ia\cos\theta\right)\partial_t + \left(s^2\cot^2\theta + s\right) \end{aligned}$$

2 Main result

2.1 Related works about polynomial tails for waves on black hole spacetimes

The question of characterizing the precise polynomial rate of decay for wave equations on black hole spacetimes was first addressed in the physics literature. The optimal rate for linear perturbations on a Schwarzschild spacetime was conjectured by Price [51, 52], later clarified by Price-Burko [53] and generalized to Kerr spacetimes in [26, 17]. In the mathematical literature, the wave equation $s = 0$ has been the focus of extensive study starting with the pioneer contributions by Wald and Kay-Wald [62, 28]. The question of optimal decay on subextremal Kerr black holes in this case is now well understood. Tataru [57] (see also [44] for a generalization) obtained the optimal decay rate for a family of stationary spacetimes including the subextremal Kerr family assuming a local integrated energy estimate (this estimate holds for the *full subextremal range* of parameters as proved by Dafermos–Rodnianski–Shlapentokh-Rothman [11]). Using a different approach Donninger-Schlag-Soffer [12, 13] obtained the optimal local decay rate on Schwarzschild black holes.

[6]It is a renormalized version of the so called Kinnersley tetrad.

[7]Despite the apparent singularity of T_s on the rotation axis, it is smooth as a differential operator acting on the suitable complex line bundle as indicated in a previous footnote.

The global optimal decay (by global we mean uniform up to null infinity) was obtained for spherically symmetric backgrounds by Angelopoulos-Aretakis-Gajic [3, 2] where they also compute the leading order term. The global optimal decay and leading order term are obtained by Hintz [24] for subextremal Kerr black holes using spectral methods. A similar expansion including the case of initial data with non zero Newman-Penrose charges was later obtained by Angelopoulos-Aretakis-Gajic [4] using a physical space approach.

For higher spin, Donninger-Schlag-Soffer [13] (later refined by Metcalfe-Tataru-Tohaneanu [45] in the case $s = \pm 1$ under an integrated local energy decay assumption) obtained an explicit but sub optimal polynomial decay rate for Teukolsky solutions with spin $\pm 1, \pm 2$ on a Schwarzschild background. On a slowly rotating Kerr black hole, integrated local energy decay was proved for the Teukolsky equation for spin $s = \pm 1$ by Ma [36] and for spin $s = \pm 2$ by Ma [37] and independently by Dafermos-Holzegel-Rodnianski [9]. On the Schwarzschild background, Price's law was obtained by Ma-Zhang for spin $s = \pm\frac{1}{2}, \pm 1, \pm 2$ in [39, 40]. Ma-Zhang [38] further generalized their result to the Price's law (with computation of the leading order term) for spin $s = \pm 1, \pm 2$ on the exterior region of a slowly rotating Kerr black hole $|a| \ll M$ (and under an integrated local energy decay assumption in the case $|a| < M$). The result of [47] uses a different approach to make the leading order development unconditional for $|a| < M$ and it also extends to all spins $s \in \frac{1}{2}\mathbb{Z}$. Independently, a boundedness of the energy and integrated local energy decay result in the full subextremal range has been obtained by Shlapentokh-Rothman-Teixeira da Costa [55, 56].

On more general asymptotically flat background, the recent works by Morgan and Morgan-Wunsch [48, 49] which establish a connection between the rate of convergence of a stationary Lorentzian metric towards the Minkowski metric, the regularity of the resolvent near zero energy and the pointwise decay rate of solutions of the wave equation using techniques closely related to the ones we present here. For a different approach on this problem including the non-stationary case, see also the work of Looi [33]. Finally, we highlight the new framework developed by Luk-Oh [35] to obtain leading-order expansions from weaker a priori bounds in a very general context (including for dynamical backgrounds and non linear equations) and the work Looi-Xiong [34] which employs a generalized low resolvent expansion to obtain the leading order term for semilinear wave equations on an asymptotically flat background building upon previous works by Looi [32, 31].

2.2 Large time asymptotics for solutions of the Teukolsky equation

The main result of [47] is the following:

Theorem 2.1. *We consider a* subextremal *Kerr spacetime ($|a| < M$). We fix $s \in \frac{1}{2}\mathbb{Z}$. Let u_0, u_1* smooth and compactly supported *on Σ_0 and let u be the solution of the Cauchy problem*

$$\begin{cases} T_s u = 0 \\ u_{|\Sigma_0} = u_0 \\ \frac{\partial}{\partial t} u_{|\Sigma_0} = u_1 \end{cases}$$

Then there exists $\varepsilon > 0$ such that:

$$|u(r, \mathfrak{t}, \theta, \phi) - \mathfrak{p}_{u_0,u_1}(r, \mathfrak{t}, \theta, \phi)| \leq C r^{-1+} \mathfrak{t}^{-2-|s|+s-\varepsilon} \left(\frac{\mathfrak{t}}{r} + 1\right)^{-1-s-|s|}$$

with

$$\mathfrak{p} = \mathfrak{t}^{-3-2|s|} \frac{(2\,|s|+2)\left(\frac{\mathfrak{t}}{r}\right)^{2+|s|+s} + 2(|s|-s+1)\left(\frac{\mathfrak{t}}{r}\right)^{1+|s|+s}}{\left(\frac{\mathfrak{t}}{r}+2\right)^{2+|s|+s}} F_{u_0,u_1}$$

where $F_{u_0,u_1}(r,\theta,\phi)$ can be expressed with hypergeometric functions and spin weighted spherical harmonics.

Remark 2.2. Instead of smoothness, we can require a finite (high) order of Sobolev regularity. Denoting by $\not{\partial}$ any smooth vector field on $\mathbb{S}^2$, we can consider the following decay assumption instead of compact support:

$$\begin{aligned} r^{1+\alpha}(r\partial_r)^{k_1}\not{\partial}^{k_2}u_0 \in L^2\left((M,+\infty)\times\mathbb{S}^2, \frac{\mathrm{d}r}{r}\mathrm{dvol}_{\mathbb{S}^2}\right) \\ r^{2+\alpha}(r\partial_r)^{k_1}\not{\partial}^{k_2}u_1 \in L^2\left((M,+\infty)\times\mathbb{S}^2, \frac{\mathrm{d}r}{r}\mathrm{dvol}_{\mathbb{S}^2}\right) \end{aligned}$$

for all $k_1, k_2 \geq 0$ with $k_1 + k_2 \leq N$ where N is sufficiently large and $\alpha \in (0,1)$. In this case, the result is a polynomial decay for the solution namely, for any $\varepsilon > 0$:

$$|u| \square \frac{1}{r^{1-\varepsilon}\mathfrak{t}^{\alpha-\varepsilon}}. \tag{5}$$

We refer the reader to [47] for a more precise statement (including a better time decay in (5) at the cost of a worst r weight) and bounds on derivatives of u.

3 Overview of the proof

3.1 General strategy

The proof relies on microlocal and spectral methods (and in particular on works by Vasy [59, 60, 61], Wunsch-Zworski [64] and Dyatlov [14]). It uses the b and scattering pseudodifferential algebras introduced by Melrose [43, 42] and Vasy's formulation of radial point estimates first introduced by Melrose [41]. A crucial point in the proof of our result is the analysis of the low energy limit of the resolvent which has been initiated in the Euclidean context by the work of Jensen-Kato [27]. Although we adopt Vasy's point of view [61] for the low energy analysis, we mention [6, 5, 7, 18, 19, 20] for a different perspective on this problem. The spectral and microlocal methods have recently led to a linear stability result for Einstein's equations on Kerr black holes with small angular momentum by Häfner-Hintz-Vasy [21], a linear stability result for weakly charged and slowly rotating Kerr-Newman black hole by He [22] and to the sharp asymptotic description of scalar waves by Hintz [24] for $|a| < M$.

The first step is to translate the Cauchy problem to a forcing problem. Let $\mathfrak{t}_0 \in \mathbb{R}$, χ be a smooth function such that $\mathrm{supp}(\chi) \subset [\mathfrak{t}_0, +\infty)$ and $\chi = 1$ on $[\mathfrak{t}_0 + \eta, +\infty)$ with $\eta > 0$. Choosing $\mathfrak{t}_0$ and η correctly, we can assume that $\mathrm{supp}(u) \cap \{\mathfrak{t}_0 \leq \mathfrak{t} \leq \mathfrak{t}_0 + \eta\}$ is compact. We define:

$$\begin{aligned} &v := \chi(\mathfrak{t})u \\ &f := T_s v = [T_s, \chi(\mathfrak{t})]u \in C_c^\infty(\mathcal{M}) \end{aligned}$$

Since $u = v$ for large $\mathfrak{t}$, we are reduced to study the forward forcing problem:

$$T_s v = f.$$

Since T_s commutes with $\mathfrak{t}$-translation, we take the Fourier-Laplace transform with respect to $\mathfrak{t}$:

$$\begin{aligned}\hat{f}(\sigma) &:= \int e^{i\sigma\mathfrak{t}} f(\mathfrak{t})\mathrm{d}\mathfrak{t} \\ \hat{v}(\sigma) &:= \int e^{i\sigma\mathfrak{t}} v(\mathfrak{t})\mathrm{d}\mathfrak{t} \\ \hat{T}_s(\sigma) &:= e^{i\sigma\mathfrak{t}} T_s e^{-i\sigma\mathfrak{t}} \\ \hat{T}_s(\sigma)\hat{v} &= \hat{f}\end{aligned}$$

For this to make sense, we take $\Im(\sigma) > C$ where C is the constant in some crude Gronwall exponential bound that we have on v. A crucial step is then to prove that the resolvent operator $R(\sigma)$ exists in order to write the following integral representation formula for v:

$$v(\mathfrak{t}) = \int_{\mathbb{R}+i(C+1)} e^{-i\sigma\mathfrak{t}} R(\sigma)\hat{f}(\sigma)\mathrm{d}\sigma$$

The next goal is to prove that the family $R(\sigma)$ is in fact holomorphic on $\{\Im(\sigma) > 0\}$ and continuous up to the real axis. A contour deformation argument will then lead to:

$$v(\mathfrak{t}) = \int_{\mathbb{R}} e^{-i\sigma\mathfrak{t}} R(\sigma)\hat{f}(\sigma)\mathrm{d}\sigma.$$

Formal integration by part provides:

$$\|v(\mathfrak{t})\|_{\mathcal{X}} \leq \mathfrak{t}^{-k} \int_{\mathbb{R}} \left\| \partial_\sigma^k R(\sigma)\hat{f}(\sigma) \right\|_{\mathcal{X}} \mathrm{d}\sigma$$

where $\mathcal{X}$ is a weighted Sobolev space on $(M, +\infty)_r \times \mathbb{S}^2$. In view of this informal argument, we need to understand the regularity of $R(\sigma)$ on the real axis to quantify the decay of v. The contour deformation argument and the polynomial bound in time also require decay estimates for $R(\sigma)\hat{f}(\sigma)$ and $\partial_\sigma R(\sigma)\hat{f}(\sigma)$ as $|\Re(\sigma)| \to +\infty$ (high frequency estimate). The leading-order expansion of v for large $\mathfrak{t}$ then corresponds to the inverse Fourier transform of the most singular part of $R(\sigma)\hat{f}(\sigma)$.

3.2 Existence of the resolvent

To obtain the existence of the resolvent, we prove that $\hat{T}_s(\sigma)$ is Fredholm of index zero between suitable weighted Sobolev spaces and that it has a trivial kernel. The Fredholm property is proved through Fredholm estimates. A possible general statement is given in the following lemma (see [59, Section 2.6] or [47, Lemma 6.23] for a proof):

Lemma 3.1. *Let $X_0 \subset X_1 \subset X_2$ and $Y_0 \subset Y_1 \subset Y_2$ be Banach spaces (with continuous dense inclusions). Let $P : X_1 \to Y_2$ be a bounded operator such that $P_{|X_0}$ is bounded from X_0 to Y_1. We assume that both inclusions $X_1 \subset X_2$ and $Y_0 \subset Y_1$ are compact and that there exists $C > 0$ such that for all $u \in X_1$ and all $v \in Y_1^*$:*

$$\|u\|_{X_1} \leq C \left(\|Pu\|_{Y_1} + \|u\|_{X_2} \right) \tag{6}$$

$$\|v\|_{Y_1^*} \leq C \left(\|P^* v\|_{X_1^*} + \|v\|_{Y_0^*} \right). \tag{7}$$

where the right hand side may be infinite (in which case the estimate is empty). Under these assumptions, P is Fredholm as an operator between the Banach spaces $\mathfrak{X} := \{u \in X_1 : Pu \in Y_1\}$ (endowed with the norm $\|u\|_{\mathfrak{X}}^2 = \|u\|_{X_1}^2 + \|Pu\|_{Y_1}^2$) and Y_1.

In the setting of weighted Sobolev spaces, the inclusion $X_1 \subset X_2$ is compact if the norm on X_1 is stronger than the norm on X_2 both in terms of regularity and in terms of decay and similarly for the inclusion $Y_0 \subset Y_1$. The usual principal symbol provides some indication about the behavior of a differential operator with respect to regularity and there is an analog for decay that we call here the principal scattering symbol near infinity.

Definition 3.2. Let $\rho = \frac{1}{r}$ be the boundary defining function of infinity (the structure of manifold with boundary being defined by $\overline{X} := (M, +\infty)_r \times \mathbb{S}^2 \bigsqcup [0,1)_\rho \times \mathbb{S}^2/(r,\omega) \sim (\rho^{-1}, \omega))$.

The scattering differential algebra is the $C^\infty(\overline{X})$-algebra generated by 1, $\rho^2\partial_\rho$ and $\rho\not{\partial}$ for all smooth vector fields $\not{\partial}$ on $\mathbb{S}^2$.

The key property of this algebra allowing us to define a notion of principal symbol is the commutator inclusion:

$$[\mathrm{Diff}_{\mathrm{sc}}, \mathrm{Diff}_{\mathrm{sc}}] \subset \rho \mathrm{Diff}_{\mathrm{sc}}$$

In other words, $\mathrm{Diff}_{\mathrm{sc}}$ is commutative at first order in ρ. Abstractly, the principal symbol map can be defined as the projection in the quotient $\mathrm{Diff}_{\mathrm{sc}}/\rho\mathrm{Diff}_{\mathrm{sc}}$. The main obstacle for obtaining Fredholm estimates for $\hat{T}_s(\sigma)$ is the lack of ellipticity both in the usual differential sense near the horizon and in the scattering sense near $r = +\infty$. The principal symbol of $\hat{T}_s(\sigma)$ in the usual differential sense is

$$p : \xi \in T^*(M, +\infty) \times \mathbb{S}^2 \mapsto \rho^2 g_{M,a}^{-1}(\tilde{\xi}, \tilde{\xi})$$

where $\tilde{\xi} = \pi^*\xi$ with $\pi : R_{\mathrm{t}} \times (M, +\infty)_r \times \mathbb{S}^2 \to (M, +\infty)_r \times \mathbb{S}^2$ being the projection. Therefore, p vanishes on a conical subset of $T^*(M, +\infty) \times \mathbb{S}^2 \setminus \{0\}$ if and only if there exists a null covector $\tilde{\xi} \in T^*\mathcal{M}$ with $\tilde{\xi}(\partial_{\mathrm{t}}) = 0$. Such a $\tilde{\xi} \in T_x^*\mathcal{M}$ exists if and only if ∂_{t} is not timelike at x which is the case for $x \in \mathcal{E}$ as we have seen before (we remind that $\partial_{\mathrm{t}} = \partial_t = T$). We can check that the scattering principal symbol at infinity of $\hat{T}_s(\sigma)$ is also not invertible in the commutative algebra $\mathrm{Diff}_{\mathrm{sc}}/\rho\mathrm{Diff}_{\mathrm{sc}}$.

To overcome the difficulty posed by this lack of ellipticity, we need to complement estimates on elliptic phase-space regions with:

- Propagation of singularities estimates.
- Radial points estimates at the horizon following [59].
- Estimates near $r = +\infty$ based on [60, 61, 41].

The precise spaces where we can prove the Fredholm estimates are dictated by the regularity and decay thresholds in the radial point estimates. These thresholds are sensitive to the subprincipal term of $\hat{T}_s(\sigma)$ leading ultimately to different mapping properties for the resolvent $R(\sigma)$.

Remark 3.3. An important difficulty in the Fredholm estimates is due to the degeneration[8] of $\hat{T}_s(\sigma)$ at $\sigma = 0$. This difficulty is treated in [61].

[8]In the sense that the scattering characteristic set collapses at $\sigma = 0$ and this collapse has to be resolved to get uniform estimates.

The index zero property is obtained by a continuity argument from the high frequency estimates (which provides actual invertibility estimates instead of Fredholm ones). It then remains to prove that $\hat{T}_s(\sigma)$ has no kernel for $\sigma \in \mathbb{C}$ with $\Im(\sigma) \geq 0$. The key result in this direction is due to Whiting in [63] proving the absence of mode solutions for $\Im(\sigma) > 0$, later extended to the real axis in [1].

We introduce the weighted Sobolev spaces:

Proposition 3.4. *Let $s \in \mathbb{N}$ and $l \in \mathbb{R}$. The space $H^{s,l}$ is obtained by the completion of the space of smooth compactly supported functions in the norm:*

$$\|u\|^2_{H_b^{s,l}} := \sum_{\alpha+\beta\leq s} \int_M^{+\infty} \left\| r^l (r\partial_r)^\alpha u \right\|^2_{H_b^\beta(\mathbb{S}^2)} r^2 \mathrm{d}r$$

where $H^\beta(\mathbb{S}^2)$ is the usual Sobolev norm of order β on $\mathbb{S}^2$.

The outcome of the Fredholm estimates and absence of mode is summarized in the following propositions:[9]

Proposition 3.5. *Let K be a compact subset of the closed upper half-plane such that $0 \notin K$. Let $l < -3/2$. There exists $N \in \mathbb{N}$ such that for all $\tilde{r} \geq N$ and for all $\sigma \in K$: $R(\sigma) : H_b^{\tilde{r},l} \to H_b^{\tilde{r},l+1}$ is uniformly bounded.*

As previously mentioned, the bound in the previous proposition blows up as $\sigma \to 0$ and the uniform analysis following [61] enables to show that the blow up is of order $|\sigma|^{-1}$ and provides a uniform (up to $\sigma = 0$) bound for the resolvent and its derivatives in modified functional spaces:

Proposition 3.6. *Let $K \subset \{\Im(\sigma) \geq 0\}$ be a sufficiently small neighborhood of 0 in $\{\Im(\sigma) \geq 0\}$. Let $l \in (-(3/2) - s - |s|, -1/2)$, $\varepsilon \in [0, 1]$. There exists $N \in \mathbb{N}$ such that for all $\tilde{r} \geq N$ and for all $\sigma \in K \setminus \{0\}$:*

$$\|R(\sigma)\|_{\mathcal{L}(H_b^{\tilde{r},l-\varepsilon}, H_b^{\tilde{r},l})} \leq C\,|\sigma|^{-\varepsilon}$$

Crucially, the interval of allowed weights in Proposition 3.6 depends on s (this dependence comes from the subprincipal term in the operator $\hat{T}_s$).

To simplify the discussion, since we are not focusing on regularity, we introduce the Fréchet spaces:

$$\mathcal{A}^l := H^{\infty,l} = \cap_{s\in\mathbb{N}} H^{s,l}$$

whose seminorms are the Sobolev norms $(\|\cdot\|_{H^{s,l}})_{s\in\mathbb{N}}$.

3.3 High frequency estimate

Proposition (3.5) does not specify how the resolvent estimate behaves as $|\sigma| \to +\infty$. This is the purpose of the high frequency estimate. Since $\hat{f}(\sigma)$ has Schwartz bounds with respect to σ, it is enough to obtain polynomially growing bounds for $R(\sigma)$ and its derivatives near $|\sigma| = +\infty$.

[9] In this note, we do not aim to provide optimal or even precise statements regarding regularity.

We introduce the semiclassical parameter $h = 1/|\sigma|$, the rescaled spectral parameter $z = h\sigma$ and the rescaled operator:

$$\hat{T}_{s,h}(z) := h^2\hat{T}_s(h^{-1}z).$$

Since we are interested in the the behavior of $\hat{T}_s(\sigma)$ for $0 \leq \Im(\sigma) \leq C$, we assume $z = z_0 + O(h)$ with $z_0 \in \{-1, 1\}$. The semiclassical differential operator $\hat{T}_{s,h}(z)$ is non elliptic. The semiclassical characteristic set has a more complex structure than the classical characteristic set. To illustrate this point, we consider the case of trapped null geodesics.

Let $X := (M, +\infty) \times \mathbb{S}^2$ and let $\pi : \mathbb{R}_{\mathfrak{t}} \times X \to X$ be the second projection. The semiclassical principal symbol of $\hat{T}_{s,h}(z)$ at $\xi \in T^*_x X$ is

$$p_h(\xi) := \rho^2 G(-z_0 \mathrm{d}\mathfrak{t} + \pi^*\xi)$$

where the right-hand side is evaluated at any point of the form $(\mathfrak{t}, x)$ (since $\partial_{\mathfrak{t}} = T$ is Killing, it does not depend on the particular choice of $\mathfrak{t}$). Let $\mu : T^*(\mathbb{R}_{\mathfrak{t}} \times X) \to T^*X$ be the projection on[10] T^*X parallel to $\mathrm{d}t$ (in particular we have $\mu(-z_0\mathrm{d}\mathfrak{t} + \pi^*\xi) = \xi$). Let $\tilde{\xi} \in T^*_{(\mathfrak{t},x)}(\mathbb{R}_{\mathfrak{t}} \times X)$ such that $G(\tilde{\xi}, \tilde{\xi}) = 0$ and $\tilde{\xi}(\partial_{\mathfrak{t}}) = -z_0$. We have:

$$p_h(\mu(\tilde{\xi})) = \rho^2 G(\tilde{\xi}, \tilde{\xi}) = 0$$
$$H_{p_h}(\mu(\tilde{\xi})) = \rho^2 \mathrm{d}_{\tilde{\xi}}\mu(H_G(\tilde{\xi}))$$

where H_G is the Hamiltonian flow of G (generator of the geodesic flow on $T^*(\mathbb{R}_{\mathfrak{t}} \times X)$) and H_{p_h} is the Hamiltonian vector field associated to p_h. Denoting by Φ^s_h the Hamiltonian flow of p_h and denoting by Φ^s the geodesic flow on $T^*(\mathbb{R}_{\mathfrak{t}} \times X)$, we deduce:

$$\Phi^s_h(\mu(\tilde{\xi})) = \mu\Phi^{f(s)}(\tilde{\xi})$$

where f is a reparametrization function[11] (necessary because of the factor $\rho^2 > 0$). As a consequence, we see that if $\tilde{\xi}$ with $\tilde{\xi}(\partial_{\mathfrak{t}}) = -z_0$ is a point in phase space belonging to the lift of a trapped null geodesic, the corresponding point $\mu(\tilde{\xi})$ belongs to the characteristic set of p_h and the corresponding semiclassical Hamiltonian trajectory has a projection onto X which remains in a compact subset of $(r_+, +\infty) \times \mathbb{S}^2$. The set of such points in phase space is called the trapped set $\mathcal{K}$ and is stable by the flow and compact. The estimate of $\hat{T}_{s,h}(z)$ requires an estimate near $\mathcal{K}$ in addition to the semiclassical version of propagation of singularities, radial point estimates at the horizon (see [59]), semiclassical radial point estimates near $r = +\infty$ (see [60]). The trapped set $\mathcal{K}$ is normally hyperbolic which means in this context that there exists Γ_+ and Γ_- smooth codimension one orientable submanifolds stable by the Hamiltonian flow of p_h such that:

- Γ_+ and Γ_- intersect transversally and the intersection

  $$\tilde{\mathcal{K}} := \Gamma_+ \cap \Gamma_-.$$

  satisfies

  $$\mathcal{K} = \tilde{\mathcal{K}} \cap p^{-1}(\{0\})$$

[10]Using implicitly the identification between T^*X and $\ker(\partial_{\mathfrak{t}})$ provided by $\mathrm{d}\pi$

[11]with $f(0) = 0$. The function f depends on $\tilde{\xi}$.

- There exist smooth one-dimensional sub-bundles $\mathcal{V}_\pm \subset T(T^*X)$ over K stable by the flow of p_h such that:

$$T_{\tilde{K}}\Gamma_\pm = \mathcal{V}_\pm \oplus T\tilde{K}$$

- There exists $\nu > 0$ and $C > 0$ such that for all $s > 0$:

$$\sup_{\xi \in \mathcal{K}} \left\| (\mathrm{d}_\xi e^{\mp s H_{p_h}})_{|\mathcal{V}_\pm} \right\| \leq C e^{-\nu s} \tag{8}$$

  where the norm in (8) comes from an arbitrary metric on $\mathcal{V}_\pm$.

This normal hyperbolicity is the principal ingredient leading to the estimate near the trapped set which can be deduced from [64, 14].

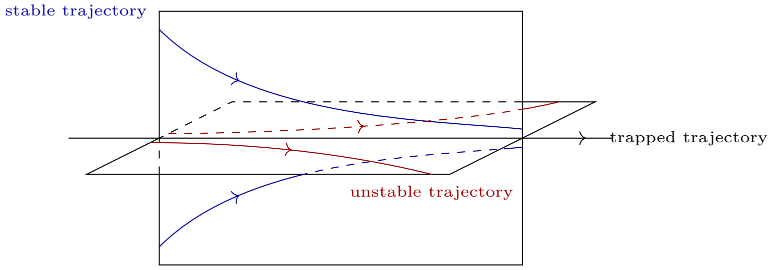

Figure 2: Representation of the normally hyperbolic trapped set.

3.4 Regularity of the resolvent on the real axis

To prove the regularity of the resolvent, we use extensively the resolvent identity:

$$R(\sigma) - R(\sigma') = R(\sigma)(\hat{T}_s(\sigma') - \hat{T}_s(\sigma))R(\sigma')$$

Let $r, l \in \mathbb{R}$. The operator $\frac{\hat{T}_s(\sigma') - \hat{T}_s(\sigma)}{\sigma - \sigma'}$ is uniformly bounded from $\mathcal{A}^l$ to $\mathcal{A}^{l+1}$ for $\sigma \in K$ where $K \subset \{\Im(\sigma) \geq 0\}$ is compact and $0 \notin K$. We deduce that:

$$\frac{R(\sigma) - R(\sigma')}{\sigma - \sigma'}$$

is locally bounded from $\mathbb{R} \setminus \{0\}$ to $\mathcal{L}(\mathcal{A}^l, \mathcal{A}^{l+1})$ and therefore $R(\sigma)$ is locally Lipschitz. Iterations of this procedure provide an arbitrarily high amount of regularity for $R(\sigma)$ from $\mathbb{R}\setminus\{0\}$ to $\mathcal{L}(\mathcal{A}^l, \mathcal{A}^{l+1})$.

The regularity at zero is more delicate. Indeed, if we want the resolvent to be uniformly bounded as $|\sigma| \to 0$, we have to take $\varepsilon = 0$ in Proposition 3.6 but then the resolvent does not provide any extra spatial decay and cannot compensate for the loss due to $(\hat{T}_s(0) - \hat{T}_s(\sigma))/\sigma$. Considering the fact that we must have $l \in (-(3/2) - s - |s|, -1/2)$ in Proposition 3.6, this explains why the resolvent has a limited regularity at $\sigma = 0$. To formalize, this argument, we introduce the sequence defined by:

- $w_0 = \hat{f}(\sigma)$.

- While $w_k \in C^\infty((-1,1)_\sigma, \mathcal{A}^l)$ with $l > -(3/2) - s - |s|$, we define

$$w_{k+1} = \frac{\hat{T}_s(0) - \hat{T}_s(\sigma)}{\sigma} R(0) w_k.$$

Let N be the index of the last element in the sequence (i.e. the first integer such that $\forall l > -(3/2) - s - |s|, w_N \notin C^\infty((-1,1)_\sigma, \mathcal{A}^l)$). By induction, we get:

$$\sigma^{k+1} R(\sigma) w_{k+1} = \sigma^k \left(R(\sigma) - R(0)\right) w_k$$

$$R(\sigma)\hat{f}(\sigma) = \sigma^N R(\sigma) w_N + \sum_{k=0}^{N-1} \sigma^k R(0) w_k \tag{9}$$

where the second term of the right-hand side of (9) is smooth with respect to σ. Since we are only interested in the most singular part of $R(\sigma)\hat{f}(\sigma)$, we can neglect it as well as[12] the term $\sigma^N R(\sigma)(w_N(\sigma) - w_N(0))$.

The analysis of $R(0)$ provides a development for each w_k as $r \to +\infty$ and we deduce that $w_N(0) \sim r^{s+|s|} c$ for some function on the sphere c. It can then be proved that since $w_N(0) - r^{s+|s|} c$ has a better decay with respect to r, $R(\sigma)\hat{f}(\sigma) = \sigma^N R(\sigma) r^{s+|s|} c$ modulo more regular term. A last step involving singular analysis is needed to further simplify this last expression and obtain an explicit description of the singularity. We refer the reader to [47] for details on this last part.

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