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Yvan Martel

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Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
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Centre de mathématiques Laurent Schwartz
CMLS, École polytechnique, CNRS, Université Paris-Saclay
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Asymptotic stability of small solitons for one-dimensional nonlinear Schrödinger equations

Yvan Martel

Laboratoire de mathématiques de Versailles
UVSQ, Université Paris-Saclay, CNRS,
45 avenue des États-Unis, 78035 Versailles Cedex, France

Abstract

We review results from two recent articles [34, 35] on the asymptotic completeness of small standing solitary waves for a class of one-dimensional nonlinear Schrödinger equations. The models considered are perturbations of the integrable cubic 1D Schrödinger equation. The notion of internal modes plays an important role and part of the discussion concerns their existence. In case an internal mode exists, the proof of asymptotic stability is more delicate and relies on a nonlinear variant of the Fermi golden rule.

Contents

1 Near cubic 1D Schrödinger equations

In these notes, we focus on the one-dimensional Schrödinger equation with a double power, cubic and quintic, nonlinearity

$$\begin{cases} \mathrm{i}\partial_t\psi + \partial_x^2\psi + |\psi|^2\psi + \sigma|\psi|^4\psi = 0 & (t,x)\in\mathbb{R}\times\mathbb{R} \\ \psi(0) = \psi_0 & x\in\mathbb{R} \end{cases} \tag{1}$$

where $\sigma = \pm 1$ determines the sign of the quintic perturbation and where the initial data ψ_0 is taken in the Sobolev space $H^1(\mathbb{R})$. The Cauchy problem for (1) is locally well-posed in $H^1(\mathbb{R})$ (see for instance [3]). Moreover, for any solution ψ in $H^1(\mathbb{R})$, the mass, momentum and energy

$$\int |\psi|^2, \quad \Im\int \psi\partial_y\bar{\psi}, \quad \int\Big(\frac{1}{2}|\partial_x\psi|^2 - \frac{1}{4}|\psi|^4 - \frac{\sigma}{6}|\psi|^6\Big)$$

are conserved, as long as $\psi(t)$ exists. If $\sigma = -1$, or if $\sigma = +1$ and the initial data is sufficiently small in H^1, then the corresponding solution is global and bounded in the energy space H^1. We recall the invariances by Galilean transform, translation and phase: if ψ is a solution of (1) then, for any $\beta, x_0, \gamma \in \mathbb{R}$, the function

$$\zeta(t,x) = \mathrm{e}^{\mathrm{i}(\beta x - \beta^2 t + \gamma)}\psi(t, x - 2\beta t - x_0)$$

is also a solution of (1).

Let us now introduce the solitary waves. Let $\Omega_- = (0, \frac{3}{16})$ and $\Omega_+ = (0, +\infty)$. For any $\omega \in \Omega_\sigma$, there exists a unique even positive solution $\phi_\omega \in H^1(\mathbb{R})$ of the equation

$$\phi_\omega'' - \omega\phi_\omega + \phi_\omega^3 + \sigma\phi_\omega^5 = 0 \quad x\in\mathbb{R}$$

given by $\phi_\omega(x) = \sqrt{\omega}Q_\omega(\sqrt{\omega}x)$, where the function Q_ω solves the equation

$$Q_\omega'' - Q_\omega + Q_\omega^3 + \sigma\omega Q_\omega^5 = 0 \quad x\in\mathbb{R}$$

and is explicitly defined by

$$Q_\omega(y) = \sqrt{\frac{4}{1 + a_\omega\cosh 2y}} \quad \text{with} \quad a_\omega = \sqrt{1 + \frac{16}{3}\sigma\omega}\,. \tag{2}$$

The formula (2) can be found, for example, in [44] and [46]. With such notation, the function $\psi(t,x) = \mathrm{e}^{\mathrm{i}\omega t}\phi_\omega(x)$ is a standing wave solution of (1). Moreover, the invariances of the equation generate a larger family of traveling waves, of the form

$$\psi(t,x) = \mathrm{e}^{\mathrm{i}(\beta x - \beta^2 t + \omega t + \gamma)}\phi_\omega(x - 2\beta t - x_0)$$

for any parameters $\omega \in \Omega_\sigma$ and $\beta, x_0, \gamma \in \mathbb{R}$.

Observe that when $\omega \to 0$, one has $Q_\omega \to Q$, where

$$Q(y) = \sqrt{\frac{4}{1+\cosh 2y}} = \sqrt{2}\,\mathrm{sech}(y) \quad \text{solves} \quad -Q'' + Q - Q^3 = 0.$$

More generally, for a solution ψ of (1), by changing variables

$$\psi(t,x) = \sqrt{\omega_0}\zeta(s,y), \quad s = \omega_0 t,\ x = \sqrt{\omega_0}y,$$

one obtains a solution ζ of the equation

$$\mathrm{i}\partial_s \zeta + \partial_y^2 \zeta + |\zeta|^2\zeta + \sigma\omega_0|\zeta|^4\zeta = 0.$$

This means that for small solutions, which corresponds to taking ω_0 small, equation (1) is a perturbation of the integrable focusing cubic Schrödinger equation

$$\mathrm{i}\partial_t\psi + \partial_x^2\psi + |\psi|^2\psi = 0 \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{3}$$

For $\sigma = +1$, the other asymptotic problem $\omega \to +\infty$, which corresponds to $\omega^{\frac{1}{4}}Q_\omega \to Q_\infty$, where

$$Q_\infty(y) = \frac{3^{1/4}}{\sqrt{\cosh 2y}} \quad \text{solves} \quad -Q_\infty'' + Q_\infty - Q_\infty^5 = 0,$$

is very interesting but it is an open problem not to be discussed here. For $\sigma = -1$, the limit $\omega \to 3/16$ is also interesting.

In these notes, we shall focus on the stability properties of small solitary waves. Of course, equation (1) is only a particular example of general semilinear perturbations of the integrable equation (3) of the form

$$\mathrm{i}\partial_t\psi + \partial_x^2\psi + |\psi|^2\psi + g(|\psi|^2)\psi = 0, \tag{4}$$

where the function $g : [0,+\infty) \to \mathbb{R}$ is sufficiently regular and satisfies $g(s) = o(s)$.

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2 The classical orbital stability result

The stability property of the solitary wave $\psi(t,x) = \mathrm{e}^{\mathrm{i}\omega t}\phi_\omega(x)$ as a solution of (1) is a classical and well-understood question. We recall from [44] the orbital stability result for general perturbations of the initial data in the energy space $H^1(\mathbb{R})$.

Theorem 1 (Orbital stability, [44, Theorem 3])**.** *Let $\sigma = \pm 1$. For any $\omega_0 \in \Omega_\sigma$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\psi_0 \in H^1(\mathbb{R})$, $\|\psi_0 - \phi_{\omega_0}\|_{H^1(\mathbb{R})} < \delta$ and ψ is the solution of* (1) *with $\psi(0) = \psi_0$, then*

$$\sup_{t\in\mathbb{R}} \inf_{(\gamma,y)\in\mathbb{R}^2} \|\psi(t,\cdot+y) - \mathrm{e}^{\mathrm{i}\gamma}\phi_{\omega_0}\|_{H^1(\mathbb{R})} < \varepsilon.$$

Remark. Note that the stability holds for any $\omega_0 \in \Omega_\sigma$. In particular, for $\sigma = +1$, solitary waves of any scale are stable. The freedom left by the infimum in the two parameters γ and y in Theorem 1 is necessary. For example, for $\beta \neq 0$ small, the initial data

$$\psi_0(x) = \mathrm{e}^{\mathrm{i}\beta x}\phi_{\omega_0}(x)$$

is close in $H^1(\mathbb{R})$ to ϕ_{ω_0} and the corresponding solution (we use the Galilean transform to determine it)

$$\psi(t,x) = \mathrm{e}^{\mathrm{i}(\beta x - \beta^2 t + \omega_0 t)}\phi_{\omega_0}(x - 2\beta t)$$

is close to $\mathrm{e}^{\mathrm{i}\omega t}\phi_{\omega_0}(x)$ for all time t, but only up to time-dependent phase and translation. If the initial data is even (which is the framework of the rest of this paper), only the parameter γ is relevant in the stability statement.

Remark. The proof of the stability result in [44] follows from the general variational arguments of [4, 19, 56]. Moreover, the same statement holds for nonlinear Schrödinger equations with a large class of nonlinearities, under a simple stability property, checked in [44] for (1). Interestingly, the orbital stability result is proved using only the conservation of mass and energy.

3 The asymptotic stability result

In the framework of the stability result, we discuss the more precise *asymptotic stability* property of the family of small standing waves of (1), under *even* perturbations of the initial data in the energy space.

Theorem 2 (Asymptotic stability of small solitary waves, [34, 35])**.** *Let $\sigma = \pm 1$. For $\omega_0 > 0$ sufficiently small, there exists $\delta > 0$ such that if $\psi_0 \in H^1(\mathbb{R})$ is even and satisfies $\|\psi_0 - \phi_{\omega_0}\|_{H^1(\mathbb{R})} < \delta$, then there exist $\omega_+ > 0$ and a $\mathcal{C}^1$ function $\gamma : [0,+\infty) \to \mathbb{R}$ with $\lim_{+\infty} \gamma' = \omega_+$, such that the solution ψ of* (1) *satisfies*

$$\lim_{t\to+\infty} \mathrm{e}^{-\mathrm{i}\gamma(t)}\psi(t) = \phi_{\omega_+} \quad \textit{uniformly on compact sets of } \mathbb{R}.$$

Remark. The range of applicability of Theorem 2 is much more restricted than the one of Theorem 1. It applies to *sufficiently small* solitary waves and under a symmetry assumption on the initial data. We expect the symmetry assumption to be technical, since no additional spectral difficulty appears in the non symmetric case. In contrast, the smallness assumption is necessary in the approach, the closeness to the integrable case being strongly used in the computations. It will certainly require new key ingredients to deal with all the solitary waves, at least for $\sigma = +1$. For $\sigma = -1$, Theorem 2 holds without the symmetry assumption and for $\omega_0 \in (0, 1/8]$ (see [34]). Thus, in this case, the question is open only for the interval $\omega_0 \in (1/8, 3/16)$. For reasons that will be explained below, the proof of the result for $\sigma = +1$ is much more involved than for the case $\sigma = -1$.

Remark. The orbital stability property in Theorem 1 implies that for all time, the solution stays close to the family of solitary waves and more precisely, close to the initial solitary wave ϕ_{ω_0}, up to a phase (for even data). In particular, in the conclusion of Theorem 1, one can replace ϕ_{ω_0} by ϕ_ω for any ω close to ω_0. In contrast, the asymptotic stability property, as stated in Theorem 2, says that as $t \to +\infty$, the solution converges, locally in space and up to a time-dependent phase, to a final asymptotic solitary wave characterized by a unique frequency ω_+. It follows from the orbital stability that in the context of Theorem 2, the quantity $|\omega_+ - \omega_0|$ is arbitrarily small for $\|\psi_0 - \phi_{\omega_0}\|_{H^1(\mathbb{R})}$ small. By the time reversibility of equation (1), the same result holds for $t \to -\infty$, with a possibly different parameter ω_-.

Remark. For the integrable Schrödinger equation (3), the asymptotic stability of solitary waves as stated in Theorem 2 is *not* true. Indeed, formula (2.15) page 333 of [45] with the choice of parameters $b_1 = b_2 = 1$, $\xi_1 = \xi_2 = 0$, $\eta_1 = 1/2$ and $\eta_2 = \eta/2$ provides an explicit periodic solution of (3) with a 2-*parallel-soliton* structure, which is arbitrarily close in $H^1(\mathbb{R})$ to the soliton $\sqrt{2}\,\mathrm{sech}(x)$ when η, parameter related to the size of the second soliton, is small. Recall that such explicit expressions of parallel multi-solitons were obtained by applying the Inverse Scattering Transform theory to the integrable equation (3); see [57].

However, the notion of asymptotic stability depends on the topology used and techniques of complete integrability were successfully applied in [16] to prove the asymptotic stability of solitons of (3) in suitable L^2 weighted spaces. Indeed, considering initial data close to a solitary wave in a weighted space is a way to remove the possibility of arbitrarily small solitons (which have a large weighted norm). Another approach to asymptotic stability for solitons of (3), not based on the integrable structure, is given in the recent preprint [30].

One can conclude from this discussion that the addition to the equation of the nonlinear perturbative term $\sigma|u|^4u$ not only destroys the integrability structure but also drastically modifies the dynamical properties of flow of (1) in a vicinity of the solitary waves in the energy space.

Remark. We now justify that for an initial perturbation in the energy space, the convergence for the supremum norm *on any compact set of* $\mathbb{R}$, as obtained in Theorem 2, is optimal. Indeed, [38] shows the existence of solutions of general nonlinear Schrödinger equations behaving asymptotically as the sum of *decoupled* solitary waves. By decoupled, we mean that the speeds of the solitary waves are two by two different (excluding parallel multi-solitons as discussed in the previous remark). We state the result more precisely: for any $0 < \omega < \omega_0$ and any $\beta > 0$, there exists an even solution ψ of (1) with the asymptotic behavior

$$\lim_{t\to+\infty} \|\psi(t) - (q_0 + q_+ + q_-)(t)\|_{H^1(\mathbb{R})} = 0,$$

where q_0 and $q_\pm$ are the solitary waves explicitly given by

$$q_0(t,x) = \mathrm{e}^{\mathrm{i}\omega_0 t}\phi_{\omega_0}(x), \quad q_\pm(t,x) = \mathrm{e}^{\mathrm{i}(\pm\beta x - \beta^2 t + \omega t)}\phi_\omega(x \mp 2\beta t).$$

Such a solution may be called a 3-soliton or more accurately, since equation (1) is not completely integrable, an *asymptotic* 3*-solitary wave.* (Such a solution also exists with any number of solitary waves.) Choosing $0 < \omega \ll \omega_0$, the solitary waves q_+ and q_- are arbitrarily small in H^1 norm compared to q_0. Therefore, the existence of the solution ψ shows that the solitary wave q_0 is *not* asymptotically stable for the supremum norm on the whole $\mathbb{R}$, for any small perturbation in the energy space.

In the literature (see references in Section 9), stronger notions of asymptotic stability are often considered, and explicit decay rates are obtained. Such results hold for small perturbations of the initial data in suitable *weighted* spaces. Small solitons, like $q_\pm$ defined above, do belong to such weighted spaces but have large norms, and so they are not acceptable perturbations. Working in weighted spaces thus provides more precise asymptotic results and may allow to deal with the integrable case (as discussed above), while working in the energy space allows the presence of any number of arbitrarily small solitary waves and to distinguish the specific properties of the integrable versus non integrable cases.

Remark. In [48, 49], the result of Theorem 2 was generalized to equation (4) for a large class of nonlinear perturbations g. Moreover, the case of the equation $\mathrm{i}\partial_t\psi + \partial_x^2\psi + |\psi|^{p-1}\psi = 0$ is treated in [14] for p close to 3. The construction of the internal mode in this case is due to [8]. See also [15] for some another range of p. Interestingly, the approach in [14, 15] involves dispersive estimates of different nature than [34, 35].

In the next sections, we give an idea of the proof of Theorem 2.

4 Linearization around the solitary wave family

In these notes, $\omega_0 > 0$ is sufficiently small and we consider a global solution $\psi(t,x)$ of (1) which is close to ϕ_{ω_0} for all $t \geq 0$. Then, it is standard to define time-dependent $\mathcal{C}^1$ functions $\gamma : [0,+\infty) \to \gamma(s) \in \mathbb{R}$ and $\omega : [0,+\infty) \to \omega(s) \in (0,+\infty)$ such that the function $u = u_1 + \mathrm{i}u_2$ defined by

$$\psi(t,x) = \exp(\mathrm{i}\gamma(s))\sqrt{\omega(s)}\,\big(Q_{\omega(s)}(y) + u_1(s,y) + \mathrm{i}u_2(s,y)\big)$$

where s and y are the rescaled time and space variables, respectively defined by

$$dt = \frac{ds}{\omega(s)}, \quad x = \frac{y}{\sqrt{\omega(s)}},$$

satisfies the below orthogonality relations

$$\int u_2(s)\Lambda_\omega Q_\omega(s) = \int u_1(s) Q_\omega(s) = 0 \tag{5}$$

where

$$\Lambda = \frac{1}{2} + \frac{1}{2}y\partial_y, \quad \Lambda_\omega = \Lambda + \omega\partial_\omega.$$

In such variables, the result of Theorem 2 is equivalent to the properties $u(s) \to 0$ uniformly on bounded intervals of $\mathbb{R}$ and $\omega(s) \to \omega_+$ as $s \to +\infty$.

Setting

$$f_\omega(\psi) = |\psi|^2\psi + \sigma\omega|\psi|^4\psi,$$

and

$$\begin{aligned} L_+ &= -\partial_y^2 + 1 - f'_\omega(Q_\omega) = -\partial_y^2 + 1 - 3Q_\omega^2 - 5\sigma\omega Q_\omega^4\,, \\ L_- &= -\partial_y^2 + 1 - f_\omega(Q_\omega)/Q_\omega = -\partial_y^2 + 1 - Q_\omega^2 - \sigma\omega Q_\omega^4\,, \end{aligned}$$

the functions $\gamma(s)$, $\omega(s)$ and $u(s,y)$ satisfy

$$\begin{cases} \dot{u}_1 = L_- u_2 + \mu_2 + p_2 - q_2 \\ \dot{u}_2 = -L_+ u_1 - \mu_1 - p_1 + q_1 \end{cases} \tag{6}$$

where μ_k and p_k are related to the time variation of the parameters

$$\begin{aligned} &\mu_1 = (\dot{\gamma} - 1)Q_\omega, \quad \mu_2 = -\frac{\dot{\omega}}{\omega}\Lambda_\omega Q_\omega, \\ &p_1 = (\dot{\gamma} - 1)u_1 + \frac{\dot{\omega}}{\omega}\Lambda u_2, \quad p_2 = (\dot{\gamma} - 1)u_2 - \frac{\dot{\omega}}{\omega}\Lambda u_1, \end{aligned}$$

while q_k are related to the nonlinearity

$$q_1 = \Re\left\{f_\omega(Q_\omega+u) - f_\omega(Q_\omega) - f'_\omega(Q_\omega)u_1\right\},$$
$$q_2 = \Im\left\{f_\omega(Q_\omega+u) - \mathrm{i}(f_\omega(Q_\omega)/Q_\omega)u_2\right\}.$$

Note that (5)-(6) is a differential system for (γ,ω,u). Indeed, differentiating (5) and using (6), one find expressions for $(\dot\gamma-1)$ and $\dot\omega$ (we do not give them here), which are sufficient to prove the bounds

$$|\dot\gamma - 1| + |\dot\omega| \lesssim \|u\|_{\mathrm{loc}}^2. \tag{7}$$

(In these notes, we will use the notation $\|v\|_{\mathrm{loc}}$ for an L^2 weighted norm of v of the form $\|e^{-\omega_0/2}v\|_{L^2}$, without giving more details.) We point out that the above quadratic estimate in u is due to the specific choice of orthogonality conditions (5). Looking back to the system (6), we see that the terms in μ_k are quadratic in u, while the terms in p_k are cubic and the terms in q_k are at least quadratic. Thus, even taking into account the modulation parameters γ and ω, the linearization of (1) around a soliton involves the simple linear system

$$\begin{cases}\dot U_1 = L_- U_2\\ \dot U_2 = -L_+ U_1\end{cases} \tag{8}$$

complemented with the orthogonality conditions (5). We study this system in the next section.

5 The spectral problem

Recall that by explicit computations, one has $L_-Q_\omega = 0$ and $L_+(\Lambda_\omega Q_\omega) = -Q_\omega$. In particular, the linear system has some simple explicit solutions, of the form

$$U(s,y) = a\Lambda_\omega Q_\omega(y) + \mathrm{i}(as+b)Q_\omega(y)$$

for real constants $a,b\in\mathbb{R}$. However, since $\int Q_\omega \Lambda_\omega Q_\omega \neq 0$ (this property is related to the stability of the solitary wave Q_ω), the orthogonality conditions (5) imply $a=b=0$. In other words, the orthogonality relations (5) rules out the even solutions related to the invariances of the equation.

More generally, the spectral problem

$$\begin{cases}L_+V_1 = \lambda V_2\\ L_-V_2 = \lambda V_1\end{cases} \tag{9}$$

is relevant for the dynamics of (8). Indeed, if there exists a solution (λ, V_1, V_2) of the system (9), then (U_1, U_2) defined by

$$U_1(s,y) = \sin(\lambda s)V_1(y) \quad \text{and} \quad U_2(s,y) = \cos(\lambda s)V_2(y) \tag{10}$$

solves the linear evolution system (8). At the linear level, the possible existence of such a time periodic solution (U_1, U_2) represents a serious difficulty to prove the asymptotic stability property. Understanding the spectral problem (9) is thus a key step. This is also where the analysis diverges between the cases $\sigma = -1$ and $\sigma = +1$ since according to the formal arguments in [46], the problem (9) admits an eigenvalue for $\sigma = +1$ and no eigenvalue for $\sigma = -1$ (up to the invariances). For the threshold integrable case, there exists a resonance, but we will not discuss further this delicate issue. To study rigorously the eigenvalue problem (9) and then the evolution problem (8), it is convenient to use an auxiliary problem, introduced for the first time in [34] in the context of the nonlinear Schrödinger equation and called the *transformed problem*. Note that similar ideas were previously developed for wave-type and KdV-type equations in [26, 33].

We define the operators

$$M_+ = -\partial_y^2 + 1 + \sigma\frac{\omega}{3}Q_\omega^4, \quad M_- = -\partial_y^2 + 1 - \sigma\omega Q_\omega^4, \tag{11}$$

and

$$S = \partial_y - \frac{Q_\omega'}{Q_\omega}, \qquad S^* = -\partial_y - \frac{Q_\omega'}{Q_\omega}.$$

The introduction of M_+ and M_- is motivated by the below identity

$$S^2 L_+ L_- = M_+ M_- S^2. \tag{12}$$

The proof of (12) follows from a direct computation, see [5, §3.4], [34, Lemma 7].

Remark. The above identity is inspired by the elementary conjugaison relation

$$SL_- = M_+ S$$

which is simply deduced from $L_- = S^*S$ and $M_+ = SS^*$. The interest of such identities lies on the properties of the transformed operators M_+ and M_-, which are more favorable than the ones of L_+ and L_- from the spectral point of view. Indeed, from (11), we observe that the potentials involved in M_+ and M_- are small for ω small. Moreover, since these potentials of M_+ and M_- have opposite signs, depending on the value of σ, one of these potentiels is repulsive. Concretely, by the identity (12), one factorises the operator S^2 on the right, which removes the null directions of L_+L_- related to the invariances. Note that the introduction of such an identity for linearized Schrödinger problems is reminiscent of the mechanism of *reduction of eigenvalues* (see [12, 17, 26]).

For ω_0 small, it is proved in [34] that the operator M_+M_- has no eigenvalue for $\sigma = -1$, while an eigenvalue for M_+M_- is constructed in [35] for $\sigma = +1$. Below, we present precise results which provide a justification to the claims in [46] for ω_0 small. Interestingly, for the threshold case $\sigma = 0$, which corresponds to the integrable pure cubic case, one has $M_+ = M_- = -\partial_y^2 + 1$, and then $+1$ is a resonance for the operator M_+M_-, associated to the constant function. Such a simple expression for M_+ and M_- for $\sigma = 0$ has been a strong motivation for working close to the integrable case, i.e. by perturbation theory, for the class of equations (4) for small solitary waves.

We start with the result for $\sigma = -1$.

Proposition 1 (Non existence of internal mode for $\sigma = -1$, [34]). *Let $\sigma = -1$. For all $\omega > 0$ sufficient small, if (λ, W_1, W_2) satisfies*

$$\begin{cases} M_+W_1 = \lambda W_2 \\ M_-W_2 = \lambda W_1 \end{cases} \tag{13}$$

then $W_1 = W_2 = 0$.

As a consequence, if (λ, V_1, V_2) satisfies (9), then $V_1 = V_2 = 0$, or $V_1 \in \operatorname{span}(Q_\omega)$, $V_2 \in \operatorname{span}(Q'_\omega)$ and $\lambda = 0$.

In fact, from [34], the above lemma holds for the range $0 < \omega \le \frac{1}{8}$. Moreover, we note that the result holds without symmetry assumption.

Now, we turn to the case $\sigma = +1$. A key observation is that if $\lambda \neq 0$ and (W_1, W_2) satisfy (13) then by the identity (12),

$$L_-L_+(S^*)^2W_1 = (S^*)^2M_-M_+W_1 = \lambda^2(S^*)^2W_1.$$

Thus, setting $V_1 = (S^*)^2W_1$ and $V_2 = \lambda^{-1}L_+V_1$, the pair (V_1, V_2) solves (9) with the same λ.

Proposition 2 (Construction of the internal mode for $\sigma = +1$, [35]). *Let $\sigma = +1$. For all $\omega > 0$ sufficiently small, there exist*

$$\lambda = 1 - \frac{64}{81}\omega^2 + O(\omega^3),$$

and non zero even functions W_1, W_2 satisfying (13). Moreover, setting $V_1 = (S^*)^2W_1$ and $V_2 = \lambda^{-1}L_+V_1$, (λ, V_1, V_2) solves (9). Finally, on $\mathbb{R}$,

$$\left|V_1 - (1 - Q_0^2)\mathrm{e}^{-\alpha|y|}\right| + \left|V_2 - \mathrm{e}^{-\alpha|y|}\right| \lesssim \omega\mathrm{e}^{-\alpha|y|}.$$

where $\alpha = \sqrt{1-\lambda}$.

The construction of the internal mode in [35] is based on the fact that for ω small, the transformed system (13) can be rewritten as a *weakly coupled* eigenvalue problem, entering the theory developed in [40] (extending arguments of [47, 52] for the scalar case). In practice, after some reformulations, the existence of λ follows from the Implicit Function Theorem.

Remark. From the construction, one obtains expansions $V_1 = 1-Q_0^2+\omega R_1+O(\omega^2)$ and $V_2 = 1+\omega R_2+O(\omega^2)$, where R_1, R_2 are explicit functions independent of ω. This means that (λ, V_1, V_2) bifurcates from the resonance $(1, 1-Q_0^2, 1)$ of the integrable case, which corresponds in the present setting to $\omega = 0$.

6 The Fermi golden rule

This section concerns only the case $\sigma = +1$. Indeed, in the case where $\sigma = -1$, i.e. in the absence of internal mode, one can directly apply the dispersive estimates of Section 7 to (u_1, u_2). For $\sigma = +1$, to prove the asymptotic stability property, one has to extract and study the component related to the internal mode by decomposition, introducing $v = v_1 + \mathrm{i}v_2$,

$$u_1(s,y) = v_1(s,y) + b_1(s)V_1(s,y), \quad u_2(s,y) = v_2(s,y) + b_2(s)V_2(s,y)$$

(note that V_1 and V_2 are functions of y but do also depend on s through the parameter ω) where b_1 and b_2 are chosen so that

$$\int v_1 V_2 = \int v_2 V_1 = 0.$$

Then, (v_1, v_2) satisfies the linearised system

$$\begin{cases} \dot{v}_1 = L_- v_2 + \mu_2 + p_2^\perp - q_2^\perp - r_2^\perp \\ \dot{v}_2 = -L_+ v_1 - \mu_1 - p_1^\top + q_1^\top + r_1^\top \end{cases} \tag{14}$$

where the error terms are mainly projections of the error terms of the system for (u_1, u_2). Moreover, the time-dependent function $b = b_1 + \mathrm{i}b_2$ satisfies

$$\begin{cases} \dot{b}_1 = \lambda b_2 + B_2 \\ \dot{b}_2 = -\lambda b_1 - B_1 \end{cases} \tag{15}$$

where B_1 and B_2 are error terms. Once the component (b_1, b_2) of the solution is identified, the strategy to prove that it converges to 0 is specific and different from the treatment of the infinite-dimensional residual component (v_1, v_2) by dispersive estimates in Section 7. Indeed, while the dispersive estimates for (v_1, v_2) are linear by nature and based on the properties of the operators L_+, L_-, but not on the expressions of the nonlinear error terms in (14), the linear part of the system in b is not sufficient to understand the behavior of (b_1, b_2). One needs to use a nonlinear variant of the Fermi golden rule, i.e. a specific property of the nonlinear terms involved in the systems (14) and (15). The pioneering approach in [1, 53, 54] consisted in solving formally the system (14), taking into account the terms of the form $b_k b_j$ in the second member and in putting them back into the equation of b. Then, in the equation of b, there are quadratic terms in b, which are oscillatory, and cubic terms in b, either directly present in the equation, or coming from terms of the form $b_j v_k$ and the previous formal resolution of (14). Formally, up to oscillatory and higher order terms (both unimportant to determine the long time behavior at the main order), one finds the ODE

$$\dot{b} = \mathrm{i}\lambda b - \Gamma^2 |b|^2 b$$

where the property $\Gamma \neq 0$, called the *Fermi golden rule*, implies nonlinear damping of b. Following exactly this methodology requires rather strong estimates on v to be able to justify the above equation of b, which do not seem accessible by virial-type identities. Therefore, we rely on a slightly different strategy, which is formally equivalent but less demanding to be made rigorous. We identify the quadratic terms in b in the equation of v, writing

$$\begin{cases} \dot{v}_1 = L_- v_2 - b_1 b_2 F + O(|bv| + |v|^2) \\ \dot{v}_2 = -L_+ v_1 + b_1^2 G + b_2^2 H + O(|bv| + |v|^2) \end{cases}$$

where F, G and H are explicit functions

$$F = 2V_1 V_2 (Q_\omega + 2\omega Q_\omega^3), \quad G = V_1^2 (3Q_\omega + 10\omega Q_\omega^3), \quad H = V_2^2 (Q_\omega + 2\omega Q_\omega^3).$$

This is simply deduced from the explicit expressions of q_1 and q_2. From the system in b, we will only need the linear part. In particular, setting $d_1 = b_1^2 - b_2^2$, $d_2 = 2b_1 b_2$, we find at the main order

$$\begin{cases} \dot{d}_1 = 2\lambda d_2 \\ \dot{d}_2 = -2\lambda d_1 \end{cases}$$

Then, we consider non trivial even bounded functions g_1, g_2 such that

$$\begin{cases} L_+ g_1 = 2\lambda g_2 \\ L_- g_2 = 2\lambda g_1 \end{cases}$$

(the existence of such functions is standard, but the proof is easier using again the identity (12)) and we define the new quantity

$$\mathbf{J} = d_1 \int v_2 g_1 \chi_A - d_2 \int v_1 g_2 \chi_A$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth even cut-off function such that

$$\chi = 1 \text{ on } [0,1],\ \chi = 0 \text{ on } [2,+\infty),\ \chi' \leq 0 \text{ on } [0,+\infty)$$

and $\chi_A(y) = \chi(y/A)$ for A large. (Observe that without the cut-off χ_A, the integrals $\int v_j g_k$ would not be well-defined.) Then, using (14) and (15), up to oscillatory terms and higher order terms, one finds

$$\dot{\mathbf{J}} = \Gamma |b|^4$$

where $\Gamma := \int (G - H) g_1 + \int F g_2$.

As in the other approach, we are reduced to proving that $\Gamma \neq 0$ for $\omega > 0$ small. In a situation where $\Gamma = 0$ for $\omega = 0$ (the integrable case), we use the explicit expansions of Q_ω and (V_1, V_2) for $\omega > 0$ close to 0, to find $\Gamma = \Gamma_0 \omega + O(\omega^2)$ for some explicit $\Gamma_0 > 0$ independent of ω. Taking into account all the error terms, the final rigorous estimate states that for $\omega > 0$ small, A large, for any $s > 0$,

$$\int_0^s |b|^4 \lesssim 1 + \frac{1}{A} \int_0^s \|v\|_{\mathrm{loc}}^2 \tag{16}$$

where $\|v\|_{\mathrm{loc}}$ is an L^2 weighted norm of v. Note that this part is inspired by the treatment of the internal mode for Klein-Gordon type equation in [23] and [24]. The main new points in [35] are to extend the strategy of [23] to Schrödinger-type equations and to prove the fact that $\Gamma \neq 0$ for $\omega > 0$ close to 0.

The next step is to prove dispersive estimates for (u_1, u_2) in the case $\sigma = -1$ where there is no internal mode, and for (v_1, v_2) in the case $\sigma = +1$ where there exists an internal mode.

7 Dispersive estimates

We first illustrate on a simple Schrödinger-type linear system how to apply (localized) virial identities to prove dispersive estimates. Assume that (w_1, w_2) satisfies the linear system

$$\begin{cases} \dot{w}_1 = -\partial_y^2 w_2 + w_2 + P_2 w_2 \\ \dot{w}_2 = \partial_y^2 w_1 - w_1 - P_1 w_1 \end{cases}$$

where P_1 and P_2 are functions of y, say, in the Schwartz class. A simple formal computation gives

$$\frac{d}{ds} \int (2y\partial_y w_2 + w_2) w_1 = 2 \int (\partial_y w_1)^2 - \int y P_1' w_1^2 + 2 \int (\partial_y w_2)^2 - \int y P_2' w_2^2 \,.$$

Thus, if the quantity $\int(2y\partial_y w_2 + w_2)w_1$ is defined and bounded, and $yP_k' \leq 0$ on $\mathbb{R}$ (such a potential will be called repulsive), then $\int |\partial_y w|^2$ is time-integrable. In practice, $\int(2y\partial_y w_2 + w_2)w_1$ is not defined for H^1 solutions, and one needs to define a *localized virial* functional of the form

$$\mathcal{I} = \int (\Theta_A w_2)w_1, \quad \Theta_A = 2\Phi_A\partial_y + \Phi_A'$$

where Φ_A is a bounded, increasing, approximation of y on $\mathbb{R}$ at a scale $A \gg 1$. Then,

$$\begin{aligned}\frac{d}{ds}\mathcal{I} = 2\int(\partial_y w_1)^2\Phi_A' - \frac{1}{2}\int w_1^2\Phi_A''' - \int \Phi_A P_1' w_1^2 \\ + 2\int(\partial_y w_2)^2\Phi_A' - \frac{1}{2}\int w_2^2\Phi_A''' - \int \Phi_A P_2' w_2^2.\end{aligned}$$

Without giving any detail, let us say that there is a relatively simple way to estimate the terms $\int w_1^2\Phi_A'''$ and to approximate $\Phi_A P_2'$ by yP_2'. Thus, if w is globally bounded in H^1, provided that $yP_k' \leq 0$, it is possible to prove that the time-dependent quantity $\int(|\partial_y w|^2 + |w|^2)e^{-\frac{|y|}{A}}$ is time-integrable, which is a local dispersive estimate on w.

For the systems (6) or (14), the potentials of L_+ and L_- are not repulsive, and this is natural since there are special solutions of the linear system (8) for which an unconditional dispersive estimate would be impossible. One possibility would be to put the orthogonality conditions (5) into play, but this does not seem easy for our model (see [41] for the mass critical case). This is where we need to rely on the identity (12) and on the transformed problem (13). Following [34, 35], we set (replace v by u in the case $\sigma = -1$)

$$w_1 = X_\theta^2 M_- S^2 v_2, \quad w_2 = -X_\theta^2 S^2 L_+ v_1,$$

where $X_\theta = (1-\theta\partial_y^2)^{-1}$ is a smoothing operator, close to the identity ($0 < \theta \ll 1$). Such a regularization is necessary to have $w_1, w_2 \in H^1$, but we will not insist on this point in these notes. The pair of functions (w_1, w_2) then satisfies a nonlinear system which is perturbative (quadratic terms and error terms are omitted here) of the form

$$\begin{cases}\dot{w}_1 = M_- w_2 \\ \dot{w}_2 = -M_+ w_1\end{cases} \tag{17}$$

In the case $\sigma = -1$, this is sufficient to conclude by a virial identity in w. Indeed, M_+ does not have a repulsive potential, but M_- has a repulsive potential, which is larger in absolute value than the one of M_+. A slight modification of the above virial functional then proves the dispersive estimate for small ω.

In the case $\sigma = +1$, the situation is more involved, because of the existence of the internal mode. Not only we need to use the decomposition (v, b) and to control b by the Fermi golden rule as described in the previous section, but the dispersive estimates are also more complicated to get on w. Technically, now the potential of M_+ is repulsive, but it is smaller in absolute value than the one of M_-, which is not repulsive. Actually, once reduced to understanding the dispersion for the equation of w, we need to take into account the periodic solution (10). It rules out the possibility of proving a simple dispersive estimate as before. Our strategy is to establish another operator identity

$$UM_+M_- = KU \quad \text{where} \quad U = \partial_y - \frac{W_2'}{W_2}$$

and where K is an explicit fourth order differential operator of the form

$$K = \partial_y^4 - 2\partial_y^2 + K_2\partial_y^2 + K_1\partial_y + K_0 + 1$$

for *small* Schwartz functions K_j. Note that $UW_2 = 0$, thus it is again a factorization procedure used to eliminate the remaining eigenfunction W_2. The above identity leads us to introduce a *second transformation* $(0 < \vartheta \ll 1)$

$$z_1 = X_\vartheta U w_2, \quad z_2 = -X_\vartheta U M_+ w_1.$$

Here, $z_1 \in H^2$ and $z_2 \in L^2$. Again, we do not comment on regularity issues and on the necessity to define z using X_ϑ. Formally, the couple (Uw_2, UM_+w_1) is the new unknown, which satisfies the transformed system

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -Kz_1 \end{cases}$$

at the linear order (quadratic terms and error terms are omitted). The operator K has two remarkable properties. It is a perturbation of $(-\partial_y^2 + 1)^2$ for ω_0 small and its potential K_0 is *repulsive* (in some sense) which makes it possible to prove the dispersive estimate via a virial argument on K, i.e. on the variable z. The fact that K does not factorize as two simple second order operators is not an issue here. The exact repulsivity property of K_0 to be used is given by [52], relating the absence of eigenvalue for a second order differential operator to the sign of the *integral* of its potential, provided it is sufficiently small. Here, this sign of $\int K_0$ is checked by using again the expansion of (λ, W_1, W_2) around the (transformed) resonance $(1, 1, 1)$ of the integrable case.

8 Outline of the proof of asymptotic stability

Two virial arguments. The introduction of transformed problems and of the necessary regularisation arguments breaks the structure of the nonlinear terms, which is usually required to treat them by a virial argument. Thus, to make the argument complete and rigorous, one has to localize the virial argument on the transformed problem. This provides estimates on (z_1, z_2) only on compacts sets in space, with error terms outside this compact set. These errors need to be controlled using other estimates. The strategy designed in [26] for the nonlinear Klein-Gordon equation, and extended to the nonlinear Schrödinger equation in [34, 35], is to use a first localized virial argument to estimate the functions (v_1, v_2) at a large spacial scale A, in terms of local norms and in terms of the internal mode component b. At the level of (v_1, v_2), the structure of the nonlinear terms is preserved and only the spectral argument is missing, which justifies the large error term in local norm. The estimate obtained at this point is

$$\int_0^s \left(\|e^{-|y|/A}\partial_y v\|^2 + \frac{1}{A^2}\|e^{-|y|/A}v\|^2 \right) \lesssim 1 + \int_0^s \left(\|v\|_{\mathrm{loc}}^2 + |b|^4 \right). \tag{18}$$

Another localized virial argument on the second transformed problem z is then used at a scale B, with $1 \ll B \ll A$. It proves

$$\int_0^s \left(\|\partial_y^2 z_1\|_{\mathrm{loc}}^2 + \|\partial_y z_1\|_{\mathrm{loc}}^2 + \|z_1\|_{\mathrm{loc}}^2 + \|z_2\|_{\mathrm{loc}}^2 \right) \lesssim 1 + \frac{1}{\sqrt{A}} \int_0^s \|v\|_{\mathrm{loc}}^2. \tag{19}$$

Here, spectral properties (as discussed in the previous section) are essential.

The last step is to exchange information between the functions (v_1, v_2) and (z_1, z_2) by suitable estimates. The most delicate ones being the so-called *coercivity estimates* (reminiscent of coercivity properties proved in [56]), of the form

$$\|v\|_{\mathrm{loc}} \lesssim \|\partial_y^2 z_1\|_{\mathrm{loc}} + \|\partial_y z_1\|_{\mathrm{loc}} + \|z_1\|_{\mathrm{loc}} + \|z_2\|_{\mathrm{loc}}. \tag{20}$$

The set of orthogonality relations for (v_1, v_2) is required in this step. We skip technical details on the choice of the weight functions in the definition of the different definitions of norms $\|\cdot\|_{\mathrm{loc}}$ used in this step.

End of the proof of Theorem 2. Using first (20) and then (19), we obtain for $s > 0$,

$$\int_0^s \|v\|_{\mathrm{loc}}^2 \lesssim \int_0^s \left(\|\partial_y^2 z_1\|_{\mathrm{loc}}^2 + \|\partial_y z_1\|_{\mathrm{loc}}^2 + \|z_1\|_{\mathrm{loc}}^2 + \|z_2\|_{\mathrm{loc}}^2 \right) \lesssim 1 + \frac{1}{\sqrt{A}} \int_0^s \|v\|_{\mathrm{loc}}^2.$$

Therefore, taking A sufficiently large (depending on ω_0), then passing to the limit as $s \to +\infty$, and taking ε sufficiently small, we have proved the key estimate

$$\int_0^{+\infty} \|v\|_{\mathrm{loc}}^2 \lesssim 1.$$

By (18) and (16), passing to the limit $s \to \infty$ it follows that

$$\int_0^{+\infty} \big(|b|^4 + \|\partial_y v\|_{\mathrm{loc}}^2 + \|v\|_{\mathrm{loc}}^2\big) \lesssim \int_0^{+\infty} \big(|b|^4 + \|e^{-|y|/A}\partial_y v\|^2 + \|e^{-|y|/A} v\|^2\big) \lesssim 1 \quad (21)$$

(at this point, A is fixed). In particular, there exists a sequence $s_n \to +\infty$ such that

$$\lim_{n\to+\infty} |b(s_n)|^4 + \|\partial_y v(s_n)\|_{\mathrm{loc}}^2 + \|v(s_n)\|_{\mathrm{loc}}^2 = 0.$$

We have obtained the asymptotic completeness on a subsequence of times and using (21) again, it is not difficult to obtain a convergence property for the whole sequence of time.

Indeed, setting $\mathcal{M} = |b|^4 + \|v\|_{\mathrm{loc}}^2$, by simple computations using (14) and (15), one obtains the estimate $|\dot{\mathcal{M}}| \lesssim |b|^4 + \|\partial_y v\|_{\mathrm{loc}}^2 + \|v\|_{\mathrm{loc}}^2$. Let $s > 0$. Integrating this on (s, s_n) for n such that $s_n > s$, we obtain

$$\mathcal{M}(s) \le \mathcal{M}(s_n) + \int_s^{s_n} |\dot{\mathcal{M}}| \lesssim \mathcal{M}(s_n) + \int_s^{s_n} \big(|b|^4 + \|\partial_y v\|_{\mathrm{loc}}^2 + \|v\|_{\mathrm{loc}}^2\big),$$

and so $\mathcal{M}(s) \lesssim \int_s^{+\infty}(|b|^4 + \|\partial_y v\|_{\mathrm{loc}}^2 + \|v\|_{\mathrm{loc}}^2)$ by passing to the limit $n \to +\infty$. Thus, using (21), $\lim_{s\to+\infty} \mathcal{M}(s) = 0$.

Finally, by the equation of $\dot\omega$ and (21), it is possible to prove that $\omega(s)$ has a finite limit ω_+ as $s \to +\infty$. One obtains $\lim_{+\infty} \dot\gamma = 1$ by (7), which implies $\lim_{t\to+\infty} d\gamma/dt = \omega_+$ by the change of variable.

9 Some related articles

Classical references. The study of the asymptotic stability of solitary waves started with a few pioneering articles published in the Nineties: [1, 53, 54] in the absence of internal mode and [2, 46, 51, 55] in the presence of internal mode, with the emergence of the *nonlinear Fermi golden rule*. The survey [22] describes several relevant models perturbative of (3).

Closely related articles. As mentioned in the previous sections, the proof of Theorem 2 in [34, 35] relies on virial techniques developed for one-dimensional wave-type equations, such as the ϕ^4 model in [24], the nonlinear Klein-Gordon equation in [26] and general scalar fields models [23, 27]. Before being used for wave equations, localized virial techniques were introduced to study blowup and asymptotic stability of solitons for nonlinear dispersive equations, like the generalized Korteweg-de Vries equation [33, 36, 37] and the critical nonlinear Schrödinger equation [41]. The specific strategy of using a transformed problem and two virial arguments was introduced in [26] and then extended to the nonlinear Schrödinger equation (1) in [34]. The argument in [35] to treat the presence of an internal mode is adapted from [23, 24].

Other related works. The literature on asymptotic stability is abondant. For wave-type equations, we refer to [13, 18, 29, 31, 32, 39], which contain some of the most advanced results in different directions. Restricting now to Schrödinger-type models, we quote a few surveys [10, 11, 25, 50] and some of the most recent articles in various settings [6, 9, 20, 21, 28, 42]. Some other articles [7, 12, 20, 43] concern nonlinear Schrödinger equations with a potential.

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