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ZERO-DISPERSION LIMIT FOR THE BENJAMIN-ONO EQUATION

ELLIOT BLACKSTONE, LOUISE GASSOT, PATRICK GÉRARD, AND PETER D. MILLER

ABSTRACT. We consider the Benjamin-Ono equation on the line with a small dispersion parameter going to zero. After the shock time for the underlying inviscid Burgers equation, a dispersive shock wave appears in the solution when the parameter is small enough. We show that the solution is asymptotic to the multi-phase solution of Dobrokhotov and Krichever (generalizing periodic traveling waves) for the Benjamin-Ono equation, modulated by slow-varying parameters that depend only on the branches of the Burgers equations obtained by the method of characteristics. The proof relies on a solution formula of the Benjamin-Ono equation established by Gérard [28] and that we simplify for rational initial data in [7]. A paper on the zero-dispersion asymptotics will appear soon in [8].

CONTENTS

1. INTRODUCTION

We are interested in the Benjamin-Ono equation (BO)

$$\partial_t u + \partial_x(u^2) = \varepsilon \partial_x |\partial_x| u \tag{1.1}$$

on the line and on the torus. The operator $|\partial_x|$ is the Fourier multiplier $\widehat{|\partial_x| f}(\xi) = |\xi|\widehat{f}(\xi)$, where $\xi \in \mathbb{R}$ on the line, and $\xi \in \mathbb{Z}$ on the torus. This equation, named after [4, 65], is an asymptotic model, derived in a small-amplitude and long-wave limit, for internal water waves propagating in one direction. It applies to gravity-driven motions of the pycnocline separating a lower-density upper fluid layer from a higher-density lower fluid layer in the situation that the lower layer is assumed to be infinitely deep. The solution u is a measure of the vertical displacement of the interface at position x and time t. The parameter ε measures the relative strength of the dispersion compared to the nonlinear effects. See Saut [67] for a description of the equation and an extensive bibliography, and also [47, Chapter 3.5.3] for references on the zero-dispersion limit problem for the Benjamin-Ono equation.

1.1. **Dispersive shock wave.**

Taking $\varepsilon = 0$ in (1.1), we get the inviscid Burgers or Hopf equation

$$\partial_t u + \partial_x(u^2) = 0. \tag{1.2}$$

The solution is well-defined as long as $t \in (T_-, T_+)$ with

$$T_- := -\frac{1}{2\max_{x\in\mathbb{R}} u_0'(x)} < 0 < T_+ := -\frac{1}{2\min_{x\in\mathbb{R}} u_0'(x)}. \tag{1.3}$$

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For general initial data $u_0 \in \mathcal{C}^2$, a shock may appear if the breaking time T_+ is finite. The breaking points are parameterized by the inflection points ξ of the initial data. Generically, one can assume that $u_0'(\xi) \neq 0$ and $u_0''(\xi) = 0$, and in this case, a shock appears at

$$(x_\xi, t_\xi) = \left(\xi - \frac{u_0(\xi)}{u_0'(\xi)}, -\frac{1}{2u_0'(\xi)} \right). \tag{1.4}$$

In the case of a small parameter $\varepsilon > 0$, the solution u is expected to stay close to the solution to (1.2) before the breaking time, but the solution also continues to exist after the breaking time because of global well-posedness for the Benjamin-Ono equation.

Regarding the Cauchy problem for the Benjamin-Ono equation, the first result on global well-posedness dates back to [66]. One can mention among other references that this result has then been improved via a gauge transformation in [69, 61]. Thanks to integrable techniques, global well-posedness is now known to hold in the Sobolev spaces H^s when $s > -\frac{1}{2}$ both on $\mathbb{R}$ [45] and on $\mathbb{T}$ [34] (see also [37] for well-posedness in a refined space), where the threshold is sharp in the sense that ill-posedness holds when $s < -\frac{1}{2}$ on $\mathbb{R}$ and when $s = -\frac{1}{2}$ on $\mathbb{T}$.

The shock happening for the inviscid Burgers equation is converted into a *dispersive shock wave* for solutions to the Benjamin-Ono equation with small parameter $\varepsilon > 0$: the solution becomes strongly oscillatory in a localized region that turns out to be precisely the region where the inviscid Burgers solution is multivalued. The dispersive shock wave is expected to be described by a modulated periodic wave-train, at least on a formal level using Whitham approximation theory.

1.2. **Main result: Benjamin-Ono equation on the line.** In the paper [8], we give an asymptotic expansion of the dispersive shock wave for (1.1). The main objects that shape the dispersive shock wave are the multi-phase solutions obtained by Dobrokhtov and Krichever [20] by using the Lax pair structure of (1.1). The description of the modulation parameters is based on a multivalued function constructed from the method of characteristics for the inviscid Burgers equation (1.2).

Multi-phase solutions. We first introduce the J-phase solutions. In the case $J = 1$, these solutions are the periodic traveling wave solutions that have been classified without the use of integrability in [1]. They are of the form

$$u(t, x; \varepsilon) = MU_r(M\varepsilon^{-1}(x - c_r t) + \alpha) + a \tag{1.5}$$

where $M > 0$, $\alpha \in \mathbb{T}$, $a \in \mathbb{R}$, and for some parameter $0 < r < 1$, the velocity is given by $c_r = (1 + r^2)/(1 - r^2)$ and the profile is given by

$$U_r(\theta) := \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta)}. \tag{1.6}$$

The maximum amplitude is $a+M(1+r)/(1-r)$ and the peak-to-trough variation is $4Mr/(1-r^2)$.

In the general case, the J-phase solutions were defined by Dobrokhotov and Krichever [DK91]. More recently, in the special case of spacial periodic solutions, they have been connected to finite gap solutions, for which only the first J Birkhoff coordinates may be nonzero, see [62], and [33, Section 7] for a formula using a $J \times J$ matrix defined from the first J Birkhoff coordinates of the initial data. Multi-phase solutions are given by the following formula.

Proposition 1.1 (J-phase solution, [20]). *The J-phase solutions are characterized by real constant parameters $R_0 < R_1 < R_2 < \cdots < R_{2J}$, and nonzero complex parameters $\gamma_1, \ldots, \gamma_J$*

satisfying

$$|\gamma_j|^2 = -\frac{(R_{2j}-R_0)\prod_{\substack{k=1\\k\neq j}}^{J}(R_{2j-1}-R_{2k-1})\cdot\prod_{\substack{i=1\\i\neq j}}^{J}(R_{2j}-R_{2i})}{(R_{2j-1}-R_0)\prod_{k=1}^{J}(R_{2j}-R_{2k-1})\cdot\prod_{i=1}^{J}(R_{2j-1}-R_{2i})}, \quad j=1,\ldots,J. \tag{1.7}$$

They are given by

$$u(t,x;\varepsilon) = R_0 + \sum_{j=1}^{J}(R_{2j-1}-R_{2j}) - 2\varepsilon \mathrm{Im}\left(\frac{\partial}{\partial x}\log(\det(\mathbf{M}(t,x;\varepsilon)))\right) \tag{1.8}$$

in which $\mathbf{M}(t,x;\varepsilon)$ *is a* $J\times J$ *matrix with elements*

$$M_{jk} := \gamma_j\, \mathrm{e}^{\mathrm{i}\theta_j^{\mathrm{L}}(t,x)/\varepsilon}\,\delta_{jk} + \frac{1}{R_{2j-1}-R_{2k}}, \quad 1\le j,k\le J \tag{1.9}$$

and $\theta_j^{\mathrm{L}}(t,x)$ *is a linear phase given by*

$$\theta_j^{\mathrm{L}}(t,x) := (R_{2j-1}-R_{2j})x - (R_{2j-1}^2 - R_{2j}^2)t, \quad j=1,\ldots,J. \tag{1.10}$$

Characteristic lines for the inviscid Burgers equation. We now introduce the required tools to define the modulation parameters. Before the breaking time T_+, equation (1.2) can be solved by using the method of characteristics: the solution $u^B(t,x)$ satisfies the implicit equation

$$u^B(t,x) = u_0(x - 2tu^B(t,x)), \tag{1.11}$$

which is equivalent by introducing $y := x - 2tu^B(t,x)$ to

$$y + 2tu_0(y) = x. \tag{1.12}$$

When $t > T_+$, equation (1.12) still makes sense but may have several solutions.

More precisely, we consider an initial data $u_0 \in L^2(\mathbb{R})\cap\mathcal{C}^1(\mathbb{R})$ with $|u_0(x)|+|u_0'(x)|\to 0$ as $|x|\to+\infty$, and $t\in\mathbb{R}$. Then thanks to the Sard theorem, the set $K_t(u_0)$ of critical values of $y\mapsto y+2tu_0(y)$ is compact and has zero Lebesgue measure [31]. On each connected component of $K_t(u_0)^c$, there is an integer $J\ge 0$ such that for x in this connected component, equation (1.12) has $2J+1$ solutions

$$y_0(t,x) > \cdots > y_{2J}(t,x). \tag{1.13}$$

Therefore, at the point (t,x), the multivalued function u^B has $2J+1$ branches

$$u_0^B(t,x) < \cdots < u_{2J}^B(t,x), \tag{1.14}$$

$$u_k^B(t,x) := u_0(y_k(t,x)), \quad 0\le k\le 2J. \tag{1.15}$$

Characteristic lines as critical points. We now define a function $h:\mathbb{R}\to\mathbb{R}$ by

$$h(y) := \frac{1}{4t}(y-x)^2 + \int_0^y u_0(y')\,\mathrm{d}y' \tag{1.16}$$

so that

$$h'(y) = \frac{y-x}{2t} + u_0(y). \tag{1.17}$$

The determination of the logarithm implies that if $y\in\mathbb{R}$, then $h(y)\in\mathbb{R}$. The real critical points of $h(y)$ are the intercepts of characteristic lines through (t,x) given by $(y_k(t,x))_{0\le k\le 2J}$. One can note that the quantity $h(y)$ is the objective function that one minimizes over y in the Lax-Oleinik formula [23] to obtain the correct (i.e., entropy) weak solution of (1.2). However, an important point is that the Benjamin-Ono equation is a dispersive (instead of viscous) regularization of (1.2), thus all of the real critical points of $h(y)$ will play an equal role rather than just the minimizer.

Modulation parameters. First, we define nonlinear phases by

$$\theta_j(t,x) := h(y_{2j-1}(t,x)) - h(y_{2j}(t,x)), \quad j = 1,\dots,J(t,x). \tag{1.18}$$

Then, we define nonlinear phase corrections. For this, we define

$$g(y) := \frac{y + 2tu_0(y) - x}{\displaystyle\prod_{k=0}^{2J(t,x)} (y + 2tu_k^{\mathrm{B}}(t,x) - x)}. \tag{1.19}$$

The function g is real-valued for $y \in \mathbb{R}$, and analytic when u_0 is analytic. Moreover, we have that $g(y) > 0$ for $y \in \mathbb{R}$, and $g(y) = y^{-2J(t,x)}(1 + \mathcal{O}(y^{-1}))$ as $y \to \infty$. From these properties of $g(y)$, it is easy to check that[1]

$$\Phi(y) := -J\frac{\pi}{2} + \frac{1}{2\pi}\int_0^{+\infty} \ln\left(\frac{g(y-s)}{g(y+s)}\right)\frac{\mathrm{d}s}{s}, \quad y \in \mathbb{R} \tag{1.20}$$

is a well-defined real-valued function of $y \in \mathbb{R}$. We define phase corrections by

$$\varphi_j(t,x) := \frac{\pi}{2} + \Phi(y_{2j-1}(t,x)) - \Phi(y_{2j}(t,x)), \quad j = 1,\dots,J(t,x). \tag{1.21}$$

Finally, we define

$$R_j(t,x) := u_j^B(t,x), \quad j = 0,\dots,2J(t,x) \tag{1.22}$$

and introduce the complex functions

$$\gamma_j(t,x) := |\gamma_j(t,x)|\,\mathrm{e}^{\mathrm{i}\varphi_j(t,x)}, \quad j = 1,\dots,J(t,x), \tag{1.23}$$

where $|\gamma_j(t,x)|^2$ is defined by the condition (1.7) from Proposition 1.1.

Zero-dispersion limit profile. Using the modulation parameters, we define a function $u^{\mathrm{ZD}}(t,x;\varepsilon)$ for $(t,x) \in \mathbb{R}^2 \setminus K_t(u_0)$ and $\varepsilon > 0$. A plot of $u^{\mathrm{ZD}}(t,x;\varepsilon)$ can be found in [8, Figure 2].

Definition 1.2 (Zero-dispersion profile)**.** If $J(t,x) = 0$ then we set:

$$u^{\mathrm{ZD}}(t,x;\varepsilon) := u_0^{\mathrm{B}}(t,x). \tag{1.24}$$

If $J(t,x) > 0$, we define $u^{\mathrm{ZD}}(t,x;\varepsilon)$ as a J-phase profile similarly as in Proposition 1.1 with θ_j, R_j and γ_j given in (1.18), (1.22) and (1.23) respectively:

$$u^{\mathrm{ZD}}(t,x;\varepsilon) := R_0(t,x) + \sum_{j=1}^{J(t,x)} (R_{2j-1}(t,x) - R_{2j}(t,x)) - 2\varepsilon\mathrm{Im}\left(\frac{\partial}{\partial x}\log(\det(\mathbf{M}(t,x;\varepsilon)))\right), \tag{1.25}$$

$$M_{jk}(t,x;\varepsilon) := \gamma_j(t,x)\,\mathrm{e}^{\mathrm{i}\theta_j(t,x)/\varepsilon}\,\delta_{jk} + \frac{1}{R_{2j-1}(t,x) - R_{2k}(t,x)}, \quad 1 \le j,k \le J(t,x). \tag{1.26}$$

Note that when $J(t,x) = 1$, the above formula simplifies as

$$u^{\mathrm{ZD}}(t,x;\varepsilon) = u_0^{\mathrm{B}}(t,x) + (u_2^{\mathrm{B}}(t,x) - u_1^{\mathrm{B}}(t,x))U_{r(t,x)}\left(\frac{\theta_1(t,x)}{\varepsilon} + \varphi_1(t,x)\right), \tag{1.27}$$

[1] Note that the constant term $-J\pi/2$ in (1.20) does not play any role in the definition of the phase corrections, but arises by representing Φ as a particular complex argument in [8].

where $U_r(\theta)$ is given by (1.6) and

$$0 < r(t,x) := \sqrt{\frac{u_1^{\mathrm{B}}(t,x) - u_0^{\mathrm{B}}(t,x)}{u_2^{\mathrm{B}}(t,x) - u_0^{\mathrm{B}}(t,x)}} < 1. \tag{1.28}$$

Our main result is as follows.

Theorem 1.3 (Zero-dispersion asymptotics [8]). *Assume that $u_0 \in L^2(\mathbb{R})$ is a rational initial condition, and let (t,x) with $t \geq 0$. Under a generic condition on u_0 and (t,x), as $\varepsilon \to 0$,*

$$u(t,x;\varepsilon) = u^{\mathrm{ZD}}(t,x;\varepsilon) + \mathcal{O}(\varepsilon) \tag{1.29}$$

where the convergence is pointwise for such (t,x) and also uniform on compact subsets of any connected component of $K_t(u_0)^c$.

Note that as long as $J(t,x) \leq 1$, then we can show that the approximate profile u^{ZD} is bounded by $|u^{\mathrm{ZD}}(t,x;\varepsilon)| \leq 9\|u_0\|_{L^\infty_x}$. A consequence is then the following.

Corollary 1.4 (Convergence in L^2 [8]). *Assume that $u_0 \in L^2(\mathbb{R})$ is a rational initial condition. If $t \geq 0$ is such that $J(t,x) \leq 1$ for every generic x where $J(t,x)$ is well-defined, then as $\varepsilon \to 0$,*

$$u(t,\diamond;\varepsilon) = u^{\mathrm{ZD}}(t,\diamond;\varepsilon) + o_{L^2}(1). \tag{1.30}$$

The profile u^{ZD} still makes sense when u_0 is not a rational initial data, however, the proof of the zero-dispersion asymptotics is open in this case. The convergence towards such an approximate profile for the zero-dispersion limit on $\mathbb{T}$ is also an open problem.

1.3. **Weak limit for the Benjamin-Ono equation.** The first rigorous results on the zero-dispersion limit problem for the Benjamin-Ono equation (1.1) mostly focus on the weak limit of a particular sequence of solutions u^ε associated to a fixed initial data u_0, as $\varepsilon \to 0$.

In [58], Miller and Xu consider sufficiently smooth and decaying-positive initial data u_0 (see [58, Definition 3.1]). The scattering data associated to u_0 consists of a finite number of Lax eigenvalues [72], and a reflection coefficient [73], from which one can try to implement the inverse scattering transform first proposed in [25]. In the zero-dispersion limit, the formal asymptotic behavior of the scattering data was derived in [51, 52], see [10] for a more detailed explanation.

Using approximate scattering data, one can define a sequence of approximate initial data $u_0(\diamond;\varepsilon)$ under the form of a multi-soliton, the number of solitons being proportional to ε^{-1}. Miller and Xu [58] show that the solutions associated to the approximate initial data $u_0(\diamond;\varepsilon)$ have a weak limit in $L^2(\mathbb{R})$, which is uniform on compact time intervals

$$u(t,\diamond;\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} \overline{u}(t,\diamond)[u_0]. \tag{1.31}$$

Moreover, the weak limit has a simple expression as the signed sum of branches of the inviscid Burgers equation constructed in (1.14):

$$\overline{u}(t,x)[u_0] = \sum_{k=0}^{2J(t,x)} (-1)^k u_k^B(t,x). \tag{1.32}$$

One can check that this expression is consistent with Theorem 1.3 in the sense that

$$u^{\mathrm{ZD}}(t,\diamond;\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} \overline{u}(t,\diamond)[u_0] \tag{1.33}$$

with $\overline{u}(t,\diamond)[u_0]$ satisfying (1.32). The approximation of the scattering data was rigorously justified for rational initial data such that for generic $\lambda \in \mathbb{R}$, the equation $u_0(x) = -\lambda$ has either zero or two solutions x in [57], using exact formulas derived in [56]. The formula for the weak limit admits an extension to the whole Benjamin-Ono hierarchy [59]. In the periodic case, the weak limit of solutions has the exact same form (1.31), (1.32) when the initial data is bell-shaped [27]. The study of the scattering data [26], coupled with a continuity argument thanks to a solution formula [28] detailed below, enables to remove the approximation of the initial data and consider instead the exact initial data $u_0(\diamond;\varepsilon) = u_0$.

The above results were generalized for the Benjamin-Ono equation on the line in [31] and [15], using the solution formula from [28] extended to any L^2 initial data in [15]:

- for every $u_0 \in L^2(\mathbb{R}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R})$ with $|u_0(x)|/|x| \to 0$ as $|x| \to \infty$, and for every $t \in \mathbb{R}$, the weak limit $\overline{u}(t,\diamond)[u_0]$ in (1.31) exists;
- for every $u_0 \in L^2(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$ such that $|u_0| + |u_0'| \to 0$ as $|x| \to \infty$, then $\overline{u}(t,\diamond)[u_0]$ satisfies (1.32).

Finally, using the approximate initial data $u_0(\diamond;\varepsilon)$ and the explicit formula for multi-solitons [51, 68] in cases when the reflection coefficient vanishes, it is possible to transfer the zero-dispersion limit problem into the study of large matrices of size $N(\varepsilon) \times N(\varepsilon)$, where $N(\varepsilon)$ is proportional to ε^{-1}. Some numerical simulations allowed to formulate conjectures explaining how to grasp the oscillations of the dispersive shock locally around a fixed point in [10].

One can remark that formula (1.32) was already introduced by Brenier in the context of the transport-collapse method [12, 13]. More precisely, let $S(t)[u_0]$ be the entropy solution associated to the inviscid Burgers equation, then by iterating the process of taking the signed sum of branches as in (1.32) over small time steps t/n, the the following nonlinear Trotter formula holds [14]

$$S(t)[u_0] = \lim_{n\to+\infty} \left(\overline{u}\left(\frac{t}{n}\right)[\cdot]\right)^n [u_0]. \tag{1.34}$$

Note that the entropy solution $S(t)[u_0]$ corresponds to the zero-viscosity limit $\varepsilon \to 0$ for the viscous Burgers equation

$$\partial_t u + \partial_x(u^2) = \varepsilon \partial_{xx} u. \tag{1.35}$$

These two limits do not coincide in general: $S(t,\diamond)[u_0] \neq \overline{u}(t,\diamond)[u_0]$, see for instance [58, Section 6.1] for more details.

1.4. **The Korteweg-de Vries equation.** The analogous zero-dispersion limit problem for the Korteweg-de Vries (KdV) equation

$$\partial_t u + \partial_x(u^2) = \frac{\varepsilon^2}{3} \partial_{xxx} u \tag{1.36}$$

was first analyzed rigorously by Lax and Levermore [48, 49]. We refer to [47, Chapter 2.6] for an overview of the results on the zero-dispersion limit problem for this equation.

The formal limit $\varepsilon = 0$ for equation (1.36) is also the inviscid Burgers equation (1.2). However, one can see from [49] in the case of negative initial data (vanishing reflection coefficient) and [70] in the case of positive initial data (no Lax eigenvalues) that the weak limit of solutions has an expression as the second derivative of the solution of a variational problem, which is different from $\overline{u}(t,\diamond)[u_0]$. The form of oscillations of the dispersive shock wave was then derived in [71], and this result was strengthened using the steepest descent method in [19]. The oscillations are described by the modulated one-phase solution of the KdV equation consistent with Whitham modulation theory.

One can note that the exact multi-phase solutions for Benjamin-Ono recalled in Proposition 1.1 are rational trigonometric functions, and hence are far simpler than the corresponding Korteweg-de Vries solutions which are built instead from Riemann theta functions of hyperelliptic curves of genus J [24]. It also turns out that in the multi-phase modulation theory for Benjamin-Ono, the parameters R_j are replaced with functions $R_j(t,x)$ defined in (1.22) that are all required to satisfy individually equation (1.2), which is far simpler than the Whitham modulation system of the Korteweg-de Vries equation which takes the form of a *coupled* system of $2J+1$ equations in Riemann invariant form [24]. The formalism of the Whitham approximation for the Benjamin-Ono equation is developed for a step initial data in [54], then for a wider class of initial data in [53, 44]. We refer to [55] for a survey of scattering transform techniques and their use in the study of various integrable equations, and their connection to Whitham modulation theory.

Near the boundaries of the Whitham zones, several asymptotic regimes have been evidenced in [38], see also [47, Figure 2.15] for an illustration. At the earliest breaking point (x_ξ, t_ξ), a gradient catastrophe happens for the underlying inviscid Burgers equation. Dubrovin [21] conjectured that the solution of the KdV equation should exhibit a universal form obtained as a specific solution of the second equation in the Painlevé-I hierarchy as $\varepsilon \to 0$. This conjecture has been validated for the KdV equation by Claeys and Grava for positive initial data [16]. Regarding the Benjamin-Ono equation, Masoero, Raimondo and Antunes [50] extended Dubrovin's perturbation analysis to a larger class of equations and formally derived a new universality class in terms of nonlocal variant of a Painlevé equation. At the harmonic/dispersive edge, a numerical study [39] rigorously confirmed in [17] gives an asymptotic expansion in terms of the Hastings-McLeod solution of the Painlevé II equation. At the opposite (or soliton) edge, numerics [40] and theorical results [18] describe the solution as a train of KdV solitons. A comparison of these results with the Benjamin-Ono equation was investigated in [60].

1.5. **Non-integrable cases.** The zero-dispersion limit is much less understood in the non-integrable cases, for which rigorous approaches are based on the study of Whitham's modulation equations. For a comprehensive survey on the topic, we refer to [5] and references therein. With the notation of [5], the idea is to justify a two-scale asymptotic expansion combining slow and fast scales: with $T = \varepsilon t$ and $X = \varepsilon x$, the solution is sought in the form

$$u(t,x;\varepsilon) = U_0\left(T, X, \frac{\phi_{(\varepsilon)}(T,X)}{\varepsilon}\right) + \varepsilon U_1\left(T, X, \frac{\phi_{(\varepsilon)}(T,X)}{\varepsilon}\right) + \mathcal{O}(\varepsilon^2). \tag{1.37}$$

The function $\phi_{(\varepsilon)}$ also has an expansion

$$\phi_{(\varepsilon)} = \phi_0(T,X) + \varepsilon\phi_1(T,X) + \mathcal{O}(\varepsilon^2). \tag{1.38}$$

Moreover, the functions $U_j(T,X,\theta)$ are one-periodic in the variable θ.

Writing $k = \partial_X\phi_0$ and $\omega = \partial_T\phi_0$, the twice differentiability of ϕ_0 implies the first modulation equation $\partial_T k - \partial_X\omega = 0$. Then, plugging the ansatz in the equation, the zeroth-order implies that U_0 must be linked to one of the periodic traveling wave solutions $\underline{U}$ of period $1/k$ by the relation $U_0(T,X,\xi k(T,X)) = \underline{U}(\xi)$. Finally, plugging for instance the ansatz in the conservation laws (mean value and impulse), one gets the last two modulation equations, that are evolution equations for the averaged values over one period $\langle\underline{U}\rangle$ and $\frac{1}{2}\langle\underline{U}^2\rangle$.

The game is then to match the modulated periodic wave trains with their limits at $\pm\infty$, which are assumed to be constant states: this is called the Gurevich–Pitaevskii problem [41]. There are two types of junction. For the first type, the amplitude of the wave train goes to zero. In equation (1.1), this situation happens at the harmonic edge corresponding to the left-side of the dispersive shock wave, around the point $X_-(t)$ (see also [10, Figure 2]). For the second type, the wavelength goes to infinity and the peaks converge to a train of solitary waves, with the final wave decaying to the right to match the background constant. In equation (1.1), this situation happens at the soliton edge corresponding to the right-side of the dispersive shock wave, near the point $X_+(t)$.

One can mention that regarding Benjamin-Ono related equations, the study of generalized nonlocal BO-type equations via non-integrable techniques was the object of study of [22].

2. Solution formulas for the Benjamin-Ono equation

The main ingredient in the study of the Benjamin-Ono equation (1.1) is a new explicit formula for the solution discovered in [28]. More details on the justification of this explicit formula and its application to the weak limit of solutions as $\varepsilon \to 0$ can be found in the proceeding [29].

2.1. **The Hardy space.** We denote by $L^2_+(\mathbb{R})$ the set of complex-valued L^2 functions with Fourier transform supported only on the nonnegative frequencies $\xi \geq 0$:

$$L^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \mid \operatorname{supp}(\widehat{f}) \subset [0,+\infty)\}. \tag{2.1}$$

Functions in $L^2_+(\mathbb{R})$ have a holomorphic extension on the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ thanks to the Paley-Wiener theorem, so one can write

$$L^2_+(\mathbb{R}) = \{f \in \operatorname{Hol}(\mathbb{C}_+) \mid \sup_{y>0} \int_{\mathbb{R}} |f(x+\mathrm{i}y)|^2 \,\mathrm{d}x < +\infty\}. \tag{2.2}$$

Similarly, on the torus $\mathbb{T} := \mathbb{R} \pmod{2\pi\mathbb{Z}}$, the Hardy space $L^2_+(\mathbb{T})$ is the space of complex-valued functions in $L^2(\mathbb{T})$ such that the only nonzero Fourier coefficients are nonnegative. Setting $z = e^{\mathrm{i}x}$ for $x \in \mathbb{T}$, the space $L^2_+(\mathbb{T})$ identifies to holomorphic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$:

$$L^2_+(\mathbb{T}) = \{f \in \operatorname{Hol}(\mathbb{D}) \mid \sup_{r<1} \int_0^{2\pi} |f(r\,\mathrm{e}^{\mathrm{i}x})|^2 \,\mathrm{d}x < +\infty\}. \tag{2.3}$$

We denote by Π the orthogonal projector from $L^2(\mathbb{R})$ onto $L^2_+(\mathbb{R})$, called the Szegő projector (and similarly on the torus). An important operator associated to the Benjamin-Ono equation and its integrability properties is the Lax operator [63, 11]. Given $u \in L^2$, the operator $L_u(\varepsilon)$ is an unbounded self-adjoint operator on L^2_+ with domain $\operatorname{Dom}(L_u(\varepsilon)) = H^1 \cap L^2_+$, defined as

$$L_u(\varepsilon) = -\mathrm{i}\varepsilon\partial_x - T_u, \tag{2.4}$$

where T_u is the Toeplitz operator with symbol u defined as

$$T_u h = \Pi(uh). \tag{2.5}$$

2.2. **Solution formula.** We denote by X^* the adjoint operator of the multiplication by x on $L^2_+(\mathbb{R})$. The domain of X^* is the set of functions such that the restriction $\widehat{f}_{|(0,+\infty)}$ of $\widehat{f}$ to $(0,+\infty)$ is in the Sobolev space H^1:

$$\operatorname{Dom}(X^*) = \{f \in L^2_+(\mathbb{R}) \mid \widehat{f}_{|(0,+\infty)} \in H^1((0,+\infty))\}. \tag{2.6}$$

Moreover, as soon as $f \in L^2_+(\mathbb{R})$ and $\widehat{f}_{|(0,1)} \in H^1((0,1))$, we define

$$I_+(f) = \widehat{f}(0^+). \tag{2.7}$$

For any function $f \in L^2_+(\mathbb{R}) \cap \operatorname{Dom}(X^*)$, we have

$$X^* f(y) = y f(y) + \frac{1}{2\mathrm{i}\pi} I_+(f). \tag{2.8}$$

According to [28], the following solution formula holds.

Theorem 2.1 (Solution formula on the line). *Given an initial data $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the real-valued solution u to (1.1) decomposes as $u = \Pi u + (\Pi u)^*$, where for $x \in \mathbb{C}_+$,*

$$\Pi u(t,x) = \frac{1}{2\mathrm{i}\pi} I_+[(X^* + 2\mathrm{i}t\varepsilon\partial_y + 2tT_{u_0} - x\,\mathrm{Id})^{-1}(\Pi u_0)]. \tag{2.9}$$

On the torus, the solution formula reads as follows. Let $S = T_{\mathrm{e}^{\mathrm{i}x}}$ be the shift operator, and $S^* = T_{\mathrm{e}^{-\mathrm{i}x}}$ its adjoint.

Theorem 2.2 (Solution formula on the torus). *Given and initial data $u_0 \in L^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$, the real-valued solution u to (1.1) on $\mathbb{T}$ decomposes as $u = \Pi u + (\Pi u)^* - \langle u_0 \mid \mathbf{1}\rangle$, where for $z \in \mathbb{D}$,*

$$\Pi u(t,z) = \langle (\mathrm{Id} - z\,\mathrm{e}^{\mathrm{i}t}\,\mathrm{e}^{2\mathrm{i}tL_{u_0}(\varepsilon)}\,S^*)^{-1}(\Pi u_0) \mid \mathbf{1}\rangle. \tag{2.10}$$

The existence of solution formulas is far from being systematic, even in the context of integrable equations. For instance, we are not aware of a similar result for the KdV equation. Let us however mention some other examples. The solution formula on the line extends to the matrix generalization of (1.1) called the spin Benjamin-Ono equation [6, 30]. A similar explicit formula was already known for the Szegő equation on the torus [32], then proved on the line in [36]. Recently, solution formulas have been proved for the focusing and defocusing Calogero-Moser derivative NLS equation, for which integrability properties present many similarities with the Benjamin-Ono equation [35], both on the torus [2] and on the line [45]. Using the solution formula in the zero-dispersion limit, the existence of a weak limit was established for this equation on the line in [3].

The solution formulas (2.9) and (2.10) are continuous with respect to the parameters t, ε and the initial condition u_0. This makes them especially convenient in order to study the zero-dispersion limit problem. We expect that formula (2.9) also allows the study of long-time behavior of solutions to the Benjamin-Ono equation on the line (1.1) [9]. Besides global well-posedness, the only known long-time properties for the Benjamin-Ono equation on the line are dispersive estimates [43] and the existence of generalized action-angle coordinates on the N-soliton manifolds [68]. On the torus, almost-periodicity of all the L^2 solutions is a corollary of the existence of Birkhoff coordinates [33].

3. Solution formula for rational initial data

In the proof of Theorem 1.3, the first step is to transform the solution formula (2.9) into a new solution formula when the initial data u_0 is a rational function [7]. If u_0 has $2N$ poles, the solution formula is a quotient of two $(N+1)\times(N+1)$ determinants, each coefficient in the determinant formula being an oscillatory or exponential integral depending on u_0, for which it is easier to derive small-ε asymptotics.

Let us consider a rational initial data u_0 in $L^2(\mathbb{R})$. Then up to a genericity assumption, u_0 is of the form

$$u_0(x) = \sum_{n=1}^{N} \frac{c_n}{x-p_n} + \frac{c_n^*}{x-p_n^*}, \quad c_n, p_n \in \mathbb{C}\setminus\{0\}, \quad \mathrm{Im}(p_n) > 0, \tag{3.1}$$

so that we have

$$\Pi u_0(y) = \sum_{n=1}^{N} \frac{c_n^*}{y-p_n^*}. \tag{3.2}$$

3.1. A new solution formula.

Given (1.16), we choose for $y<0$ with $|y|$ sufficiently large:

$$h(y) := \frac{1}{4t}(y-x)^2 + \sum_{n=1}^{N} \left[c_n \log(y-p_n) + c_n^* \log(y-p_n^*)\right], \tag{3.3}$$

where the complex logarithms denote the principal branches. We analytically continue $y \mapsto h(y)$ to a maximal domain that is generally more complicated than implied by using the principal branch of the logarithm and taking $y=z$ to be complex. For this purpose, we start by allowing for general branch cuts $\{\Gamma_n, \bar{\Gamma}_n\}_{n=1}^N$ assumed only to have the following properties, see Figure 1 (reproduced from [7]), left-hand panel.

Definition 3.1 (Branch cuts of h)**.** The branch cuts $\Gamma_1, \dots, \Gamma_N$ are pairwise disjoint piecewise-smooth curves each emanating from exactly one of the poles $\{p_n\}_{n=1}^N$ and tending to $z=\infty$ in the direction asymptotic to the ray $\arg(z) = 3\pi/4$. All of these branch cuts are assumed to lie in a half-plane $\mathrm{Im}(z) > -\delta$ for some $\delta>0$ sufficiently small (in particular, we assume $\delta < \min_n\{\mathrm{Im}(p_n)\}$.

The branch cuts $\bar{\Gamma}_1, \dots, \bar{\Gamma}_N$ are straight horizontal rays each emanating from exactly one of the conjugate poles $\{p_n^*\}_{n=1}^N$ and extending to $z=\infty$ in the left half-plane.

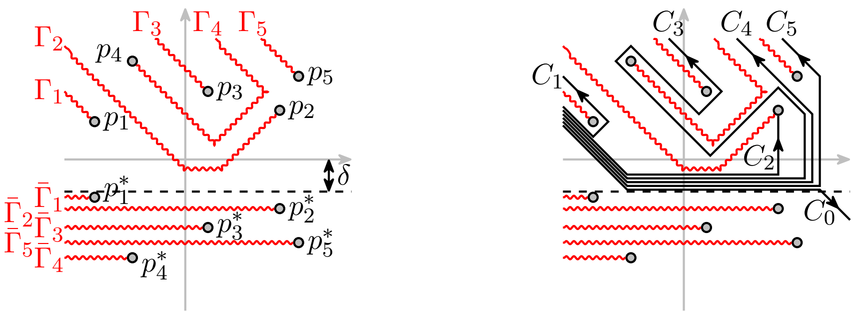

Figure 1. Left: admissible branch cuts of $h(z)$ in the z-plane for a rational initial condition with $N = 5$. Right: corresponding contours C_1, C_2, C_3, C_4, C_5 and C_0 for a situation where 2 is the only exceptional index. Source : [7].

Given the branch cuts, we index the poles $\{p_n\}_{n=1}^N$ such that in the vicinity of $z = \infty$ in the upper half-plane, the branch cuts $\Gamma_1, \dots, \Gamma_N$ are ordered left-to-right. We hence obtain a well-defined function $z \mapsto h(z)$ by analytic continuation from large negative real values of z where $h(z)$ is given by (1.16) to the domain

$$z \in \mathbb{C} \setminus (\Gamma_1 \cup \dots \cup \Gamma_N \cup \bar\Gamma_1 \cup \dots \cup \bar\Gamma_N). \tag{3.4}$$

We now define some relevant contours in the upper half plane, see Figure 1, right-hand panel.

Definition 3.2 (Contours)**.** Let C_m, $m = 1, \dots, N$ denote the contour defined by one of the following alternatives:

- If $\mathrm{i}c_m/\varepsilon$ is a strictly negative integer, then C_m originates at $z = \infty$ in the direction $\arg(z) = 3\pi/4$ to the left of all branch cuts of h, lies in the domain of analyticity of h, and terminates at $z = p_m$. We call such an index m *exceptional.*
- Otherwise, C_m originates and terminates at $z = \infty$ in the direction $\arg(z) = 3\pi/4$ and encircles with counterclockwise orientation precisely the branch cuts of $h(z)$ emanating from each of the points $z = p_n$, $1 \le n \le m$. Such an index m will be called *non-exceptional.*

Finally, we let C_0 denote a path originating at $z = \infty$ in the direction $\mathrm{e}^{3\mathrm{i}\pi/4}$ to the left of all the diagonal branch cuts of $h(z)$ in the upper half-plane and terminating at $z = \infty$ in the direction $\mathrm{e}^{-\mathrm{i}\pi/4}$ to the right of all branch cuts of $h(z)$ in the lower half-plane.

Up to a genericity condition on the initial data, we will assume that the contours C_m do not have any exceptional index. For some applications it is sufficient to choose straight-ray cuts in Definition 3.1, for which Γ_n is simply $p_n + \mathrm{e}^{3\pi\mathrm{i}/4}\,\mathbb{R}_+$. Furthermore, the branch cuts $\bar\Gamma_j$ in the lower half-plane can be chosen to be quite arbitrary as long as they do not intersect any of the contours of integration C_n.

Finally, we introduce a related matrix $\widetilde{\mathbf{B}}(t,x)$, the first column of which is the same as that of $\mathbf{B}(t,x)$ while Let $\mathbf{A}, \mathbf{B}$ be two $(N+1)\times(N+1)$ matrices defined for $1 \le j \le N+1$ and $2 \le k \le N+1$ as

$$A_{j1} := \int_{C_{j-1}} u_0(z)\,\mathrm{e}^{-\mathrm{i}h(z)/\varepsilon}\,\mathrm{d}z, \quad A_{jk} := \int_{C_{j-1}} \frac{\mathrm{e}^{-\mathrm{i}h(z)/\varepsilon}\,\mathrm{d}z}{z - p_{k-1}}, \tag{3.5}$$

$$B_{j1} := \int_{C_{j-1}} \mathrm{e}^{-\mathrm{i}h(z)/\varepsilon}\,\mathrm{d}z, \quad B_{jk} := \int_{C_{j-1}} \frac{\mathrm{e}^{-\mathrm{i}h(z)/\varepsilon}\,\mathrm{d}z}{z - p_{k-1}} = A_{j,k}. \tag{3.6}$$

$$\widetilde{B}_{jk} = \mathrm{e}^{\mathrm{i}(x-p_{k-1})^2/(4t\varepsilon)}\,B_{jk} = \mathrm{e}^{\mathrm{i}(x-p_{k-1})^2/(4t\varepsilon)} \int_{C_{j-1}} \frac{\mathrm{e}^{-\mathrm{i}h(z)/\varepsilon}}{z - p_{k-1}}\,\mathrm{d}z, \quad k = 2, \dots, N+1. \tag{3.7}$$

The elements of $\mathbf{A}$, $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ are functions of $(t, x; \varepsilon)$.

Theorem 3.3 (Solution of Benjamin-Ono for rational initial data)**.** *Let $\varepsilon > 0$. The solution of the Cauchy initial-value problem for the Benjamin-Ono equation* (1.1) *with rational initial condition $u(x,0) = u_0(x)$ of the form* (3.1) *is*

$$u(t,x;\varepsilon) = \Pi u(t,x;\varepsilon) + \Pi u(t,x;\varepsilon)^* = 2\mathrm{Re}(\Pi u(t,x;\varepsilon)), \quad t > 0, \tag{3.8}$$

$$\Pi u(t,x;\varepsilon) = \frac{\det(\mathbf{A}(t,x;\varepsilon))}{\det(\mathbf{B}(t,x;\varepsilon))}. \tag{3.9}$$

An equivalent formula is

$$\Pi u(t,x;\varepsilon) = \mathrm{i}\varepsilon \frac{\partial}{\partial x} \log(\det(\widetilde{\mathbf{B}}(t,x;\varepsilon))). \tag{3.10}$$

Also, we have $\det(\mathbf{B}(t,x;\varepsilon)) \neq 0$ and $\det(\widetilde{\mathbf{B}}(t,x;\varepsilon)) \neq 0$ for all $(t,x) \in \mathbb{R}^2$ with $t > 0$ and all $\varepsilon > 0$.

Formula (3.9) extends by continuity to the case $\mathrm{Im}(x) = 0$, contrary to the solution formula from Theorem 2.1 that assumes that $\mathrm{Im}(x) > 0$.

This solution formula resembles the N-soliton formula, which is also a ratio of determinants, and which is also a rational function of x when $t = 0$ (see for instance (1.19) in [68]). However, the N-soliton solutions remain rational for all time $t > 0$, while in general this is not true of $u(t,x;\varepsilon)$. We wish to emphasize that the exact formula for $u(t,x;\varepsilon)$ represents also solutions that in the setting of the Fokas-Ablowitz inverse-scattering transform correspond to nonzero reflection coefficients. See [56] where the scattering data are computed explicitly for general rational initial conditions of the form (3.1); generally these solutions consist of both solitonic and reflective/dispersive components.

Regarding the Calogero-Moser derivative NLS equation, a similar formula as (3.9) for rational initial data can be derived from the general solution formula, however, no formulation comparable to (3.10) is known. This could be explained by the different nature between the two equations. In fact, the solutions of the Benjamin-Ono equation stay bounded with respect to every Sobolev norm [64, 11]. On the contrary, for the Calogero-Moser derivative NLS equation, solutions with initial data of mass greater than the mass of the ground state can exhibit turbulent behavior [42] or even blow-up [46].

3.2. **Idea of proof for the new solution formula.** The proof of this formula relies on the general solution formula from Theorem 2.1. More precisely, in formula (2.9), we first solve the following partial differential equation on $\mathbb{C}_+$:

$$(X^* + 2\mathrm{i}t\varepsilon\partial_y + 2tT_{u_0} - x\,\mathrm{Id})f = \Pi u_0, \tag{3.11}$$

where $f \in L^2_+(\mathbb{R})$ and $f \in \mathrm{Dom}(X^*)$.

This equation admits several simplifications. First, thanks to (2.8), there exists $\lambda \in \mathbb{C}$ such that

$$X^* f(y) = yf(y) - \lambda. \tag{3.12}$$

Then, when $\mathrm{Im}(p) > 0$, one can see from the decomposition into a holomorphic and a anti-holomorphic part

$$\frac{f(z)}{z-p} = \frac{f(z) - f(p)}{z-p} + \frac{f(p)}{z-p} \tag{3.13}$$

that $\Pi\left(\frac{f(z)}{z-p}\right) = \frac{f(z)-f(p)}{z-p}$. Therefore,

$$\Pi(u_0 f)(y) = u_0 f - \sum_{n=1}^{N} \frac{c_n f(p_n)}{y - p_n}. \tag{3.14}$$

Taking the complex numbers $\lambda, f(p_1), \dots, f(p_N)$ as parameters, equation (3.11) becomes an ODE on the complex upper half-plane. More precisely, setting $V_n := c_n(2tf(p_n) - 1)$ for $1 \le n \le N$, the function f has the integral representation

$$f(y) = -\frac{\mathrm{i}}{2t\varepsilon}\,\mathrm{e}^{\mathrm{i}h(z)/\varepsilon}\int_{\ell(y)}\left(u_0(z) + \lambda + \sum_{n=1}^{N}\frac{V_n}{z - p_n}\right)\mathrm{e}^{-\mathrm{i}h(z)/\varepsilon}\,\mathrm{d}z, \tag{3.15}$$

where $\ell(y)$ is a path going from infinity in the upper left corner of $\mathbb{C}_+$ (to the left of all branch cuts) to y, avoiding all the branch cuts. Then f belongs to $L^2_+(\mathbb{R})$ if f is analytic at the poles $p_1, \dots, p_N$, and also if the norm

$$\sup_{y>0}\int_{\mathbb{R}}|f(x + \mathrm{i}y)|^2\,\mathrm{d}x \tag{3.16}$$

is finite. It turns out that these conditions are equivalent to the following equalities [8]:

$$\int_{C_j}\left(u_0(z) + \lambda + \sum_{n=1}^{N}\frac{V_n}{z - p_n}\right)\mathrm{e}^{-\mathrm{i}h(z)/\varepsilon}\,\mathrm{d}z = 0, \quad 0 \le n \le N. \tag{3.17}$$

This is a linear system of $(N+1)$ equations in the $(N+1)$ unknowns $\lambda, V_1, \dots, V_N$ of the form

$$\mathbf{B}\mathbf{x} = -\mathbf{b}, \tag{3.18}$$

where $\mathbf{x} = (\lambda, V_1, \dots, V_N)^\top$ and $\mathbf{b} = (A_{1,1}, \dots, A_{1,N+1})^\top$. By Cramer's rule, one can solve

$$\lambda = -\frac{\det(\mathbf{A}(t,x;\varepsilon))}{\det(\mathbf{B}(t,x;\varepsilon))}. \tag{3.19}$$

It only remains to note that thanks to the solution formula (2.9) and by definition of λ, we have $\Pi u(t,x;\varepsilon) = \frac{1}{2\mathrm{i}\pi}I_+(f) = -\lambda$.

4. Zero-dispersion asymptotics and steepest descent method

The small-ε asymptotics of the solution $u(t,\diamond;\varepsilon)$ can be deduced from the solution formula (3.9) by applying the steepest descent method [8].

Theorem 4.1 (Steepest descent method)**.** *Let C be a contour of integration. Assume the following:*

- *C passes through exactly one stationary point y_j of the phase h (i.e. a zero of h'), at which the tangent to the contour C makes an angle of θ_j with the direction $\arg(z - y_j) = 0$;*
- *the critical point y_j is a non-degenerate saddle point $h''(y_j) \neq 0$;*
- *the real part of the phase $z \mapsto \mathrm{Re}(\mathrm{i}h(z))$ is maximal at y_j only;*
- *the imaginary part of the phase $z \mapsto \mathrm{Im}(\mathrm{i}h(z))$ is constant along C.*

Then for any function g on $\mathbb{C} \setminus (\Gamma_1 \cup \dots \cup \Gamma_N \cup \bar{\Gamma}_1 \cup \dots \cup \bar{\Gamma}_N)$. analytic at y_j,

$$\int_C g(z)\,\mathrm{e}^{\mathrm{i}h(z)/\varepsilon}\,\mathrm{d}z = \frac{\sqrt{2\pi\varepsilon}}{\sqrt{|h''(y_j)|}}\,\mathrm{e}^{\mathrm{i}\theta_j + \mathrm{i}h(y_j)/\varepsilon}(g(y_j) + \mathcal{O}(\varepsilon)). \tag{4.1}$$

The steepest descent angle θ_j satisfies

$$\theta_j = -\frac{1}{2}\arg(h''(y_j)) \pm \frac{\pi}{2}, \tag{4.2}$$

the sign $\pm$ is chosen depending on the orientation of the contour C, see Figure 4 for an illustration.

When the contour C passes several different critical points, one can cut this contour into several parts to see that the right-hand side becomes the sum of contributions of all the critical points.

We note that there are many possible choices of the contours $C_0, \dots, C_N$ as long as they satisfy Definition 3.2, due to the fact that the integrals in (3.5), (3.6) have holomorphic integrands outside of the branch cuts. But that alone is not enough because calculation of the determinants

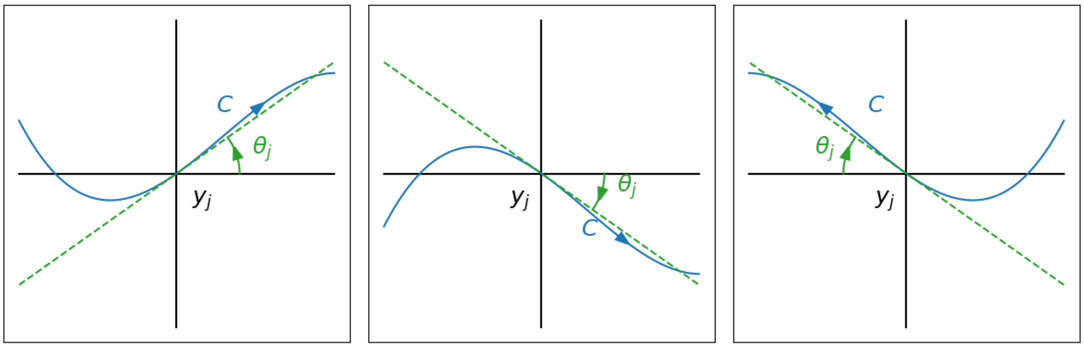

FIGURE 2. Steepest descent angle in various situations

can lead to cancellations from the leading order terms unless the contours are carefully chosen. The goal of the nonlinear steepest descent method is to choose the contours so that one can apply Theorem 4.1.

For the rational initial condition (3.1), there are $2N+1$ (possibly complex) solutions of the characteristic equation (1.12), which are the critical points of the phase functional h defined in (1.16). More precisely, on connected components of $K_t(u_0)^c$, equation (1.12) has $2J+1$ real points for some $J \geq 0$. We denote the other zeroes by $y_m(t,x)$ for $2J+1 \leq m \leq 2N$ where

$$\operatorname{Im}(y_{2j}(t,x)) > 0, \quad y_{2j-1}(t,x) = y_{2j}(t,x)^*, \quad J+1 \leq j \leq N. \tag{4.3}$$

In [8], we show that it is possible to choose the contours $C_0, \dots, C_N$ as steepest descent contours passing through different critical points in formula (2.9). More precisely, we show that there is a one-to-one correspondence between these contours and the set consisting of the union of:

- the $N-J$ critical points y of $h(\cdot)$ with $\operatorname{Im}(y) > 0$;
- the J pairs of consecutive real critical points (y_{2j}, y_{2j-1}) with $j = 1, \dots, J$;
- the critical point y_0.

The existence of such contours relies on a combinatorial argument. The case $N=1$ needs less complex tools, and is presented in detail as a first example in [8].

Choosing these adapted contours, we apply the steepest decent method in each coefficient of the matrices $\mathbf{A}$ and $\mathbf{B}$ in the solution formula (3.9). We then simplify the subsequent asymptotic expansion to derive Theorem 1.3.

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