Séminaire Laurent Schwartz

EDP et applications

Année 2023-2024

Exposé n^{o} XVI (2 avril 2024)

Eliot Pacherie

SOME ASYMPTOTIC PROFILES FOR THE VISCOUS BURGERS EQUATION WITH INFINITE MASS

https://doi.org/10.5802/slsedp.167

Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
http://www.ihes.fr/

Centre de mathématiques Laurent Schwartz
CMLS, École polytechnique, CNRS, Université
Paris-Saclay
F-91128 PALAISEAU CEDEX
http://www.math.polytechnique.fr/

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2266-0607

Séminaire Laurent-Schwartz — EDP et applications
Centre de mathématiques Laurent Schwartz, 2023-2024
Exposé n° XVI, 1-8

Some asymptotic profiles for the viscous Burgers equation with infinite mass

Eliot Pacherie *

CNRS and Cergy University

Abstract

This is a summary of the papers [6], [7] done in collaboration with Tej-Eddine Ghoul and Nader Masmoudi and [2] done in collaboration with Nicola de Nitti. We are interested in the long time behavior of solutions of the viscous Burgers equation for a class of initial data with infinite mass. We show that, up to a rescaling, they converge to a limit profile that has a discontinuity. We investigate this profile and explain how to understand this discontinuity point.

1 A discontinuous asymptotic profile

We consider the viscous Burgers equation

$$\partial_t f - \partial_x^2 f + f\partial_x f = 0$$

on the real line with an initial data $f_0 \in C^1_{\text{loc}}(\mathbb{R}, \mathbb{R})$ that satisfies

$$f_0(x) \sim \frac{\kappa_\pm}{|x|^\alpha}, f_0(x) \sim \frac{-\alpha\kappa_\pm}{|x|^{1+\alpha}} \tag{1.1}$$

when $x \to \pm\infty$ for some $\kappa_\pm > 0$ and $\alpha \in]0,1[$. This function is not integrable as it decays too slowly at infinity, but the solution of the viscous Burgers equation for this initial data is globally well posed for positive time. We are interested in the long time behavior of this solution, and see [3], [4], [5], [9], [10], [11] for other results for different types of initial data and for similar equations. Here, the function u converges (in $L^\infty(\mathbb{R})$ for instance) to 0 when $t \to +\infty$, so we want to do a rescaling to see the asymptotic profile of the solution.

It has been shown in Theorem 1.5 of [6] that there exists $z_c \in \mathbb{R}$ depending on $\alpha, \kappa_\pm$ such that

$$t^{\alpha/(1+\alpha)} f(t^{1/(1+\alpha)}z, t) \to \mathrm{p}(z) \tag{1.2}$$

for any $z = z_c$ when $t \to +\infty$, where p is a bounded function that does not cancel.

*email: ep2699@nyu.edu

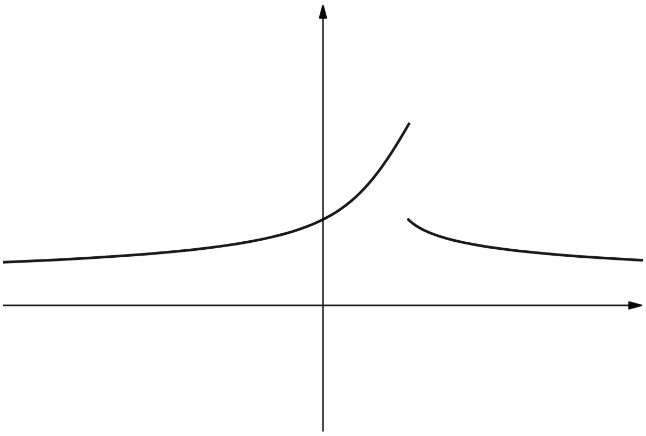


Figure 1: Graph of the function $z \to \mathrm{p}(z)$. The point of discontinuity is z_c.

The convergence is uniform in z as long as we remove a neighborhood of z_c. The function p is smooth except at z_c, where it is discontinuous. This result was done for $\kappa_+ = \kappa_-$ in [6] but the generalisation is not too difficult to prove. Remark also that this converges depends only on the behavior at $\pm\infty$ of the initial data, hence changing it by a compactly supported function (or even an integrable function) does not affect the long time behavior. In that sense, the profile p is a global attractor in the class of functions behaving like $\kappa_\pm/|x|^\alpha$ at $\pm\infty$ in a C^1 way. Let us discuss this result in the next subsections, then we will give the key elements of its proof in subsection 1.4.

1.1 Comparison with the heat equation

If we consider instead the heat equation $\partial_t g - \partial_x^2 g = 0$ on the real line with an initial data

$$g_0(x) \sim \frac{\kappa}{|x|^\alpha}, g_0'(x) \sim \frac{-\alpha\kappa}{|x|^{1+\alpha}}$$

for some $\kappa > 0$ and $\alpha \in]0,1[$ when $x \to \pm\infty$, then we can show (see Proposition 1.1 of [7]) that

$$t^{\alpha/2} g\left(t^{1/2} z, t\right) \to \frac{\kappa}{\sqrt{4\pi}}\left(\frac{1}{|x|^\alpha} * e^{-x^2/4}\right)(z)$$

when $t \to +\infty$, uniformly in $z \in \mathbb{R}$.

There are two main differences compared to (1.2): the scalings in time are not the same, and for the heat equation, the limit profile is continuous (the convolution of two functions has the smoothness of the smoothest function of the two). Remark also that this means that $\|g\|_{L^\infty(\mathbb{R})} \sim t^{-\alpha/2}$ when $t \to +\infty$, while for the viscous Burgers equation, $\|f\|_{L^\infty(\mathbb{R})} \sim t^{-\alpha/(1+\alpha)}$, and that $\alpha/(1+\alpha) > \alpha/2$ since $\alpha < 1$. This means that the Burgers term improves the decay of the solution in this class of initial data. This implies that in the viscous Burgers equation, because of this scaling, the viscosity becomes negligible for large times.

This can be seen if we look at the equation on $h(z, \varepsilon(t)) = t^{\alpha/(1+\alpha)} f\left(t^{1/(1+\alpha)} z, t\right)$. With

$$\varepsilon(t) := t^{(\alpha-1)/(\alpha+1)} \to 0$$

when $t \to +\infty$, then

$$\frac{1-\alpha}{1+\alpha}\varepsilon\partial_\varepsilon h + \frac{\alpha}{1+\alpha} h + \partial_z h\left(\frac{z}{1+\alpha} - h\right) + \varepsilon\partial_z^2 h = 0. \tag{1.3}$$

The term $\varepsilon\partial_\varepsilon h$ corresponds to the derivative in time, and $\varepsilon\partial_z^2 h$ the viscosity, that is, in this scale, multiplied by $\varepsilon \ll 1$. This means that the asymptotic profile p is in some sense a profile for the Burgers equation, without viscosity. We are going to focus more on this in subsection 1.3.

1.2 The boundary layer interpretation of the discontinuity

We will show in subsection 1.3 that although z_c depends on α, κ_+ and κ_-, the function p for $z < z_c$ depends only on α and κ_-, and for $z > z_c$ it depends only on α and κ_+. This can be interpreted in the following way: for $z < z_c$, we see the effect of the tail of the initial data at $-\infty$, and for $z > z_c$ we see the tail at $+\infty$. The point z_c is where both effect clash.

This can be interpreted as a boundary layer effect, that is a layer of transition between two dominating effect. This is more classical in PDEs with actual boundaries (where the boundary condition dominates near the boundary, but no longer as soon as we are a little away from it). Another way to interpret this “clash” at z_c is the following. If we look at the equation (1.3) for $h = \mathrm{p} + g$ and we linearized in g, we see that this linear equation transports the function g towards $z = z_c$.

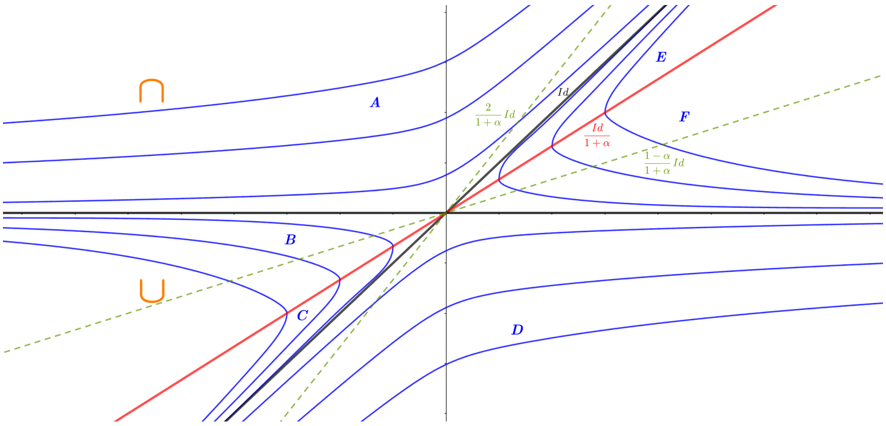
This implies, at least formally, that p is a stable solution (and even a global attractor) at this scale because any perturbation of it is transported to be “hidden” in the discontinuity. We will see later that, in fact, if we zoom near $z = z_c$, we will not have a limit profile that is a global attractor for all initial data satisfying (1.1).

1.3 The construction of the profile p

Taking $\varepsilon = 0$ (that is $t = +\infty$) in equation (1.3) yields the equation satisfied by p: for $z = z_c$,

$$\left(\frac{z}{1+\alpha} - \mathrm{p}\right)\partial_z \mathrm{p} + \frac{\alpha}{1+\alpha}\mathrm{p} = 0. \tag{1.4}$$

Remark that this is a degenerate first order ODE (as the term factor of $\partial_z \mathrm{p}$) can cancel if $\mathrm{p}(z) = z/(1+\alpha)$. In the following figure, the blue lines are the solutions of this equation, the red line is $\mathrm{Id}/(1+\alpha)$ where the degeneracy of the ODE happens. Remark that 0 and Id (the two black lines) are also solutions of the equation.



We can show that all solutions that are in the sector

$$A = \{(z, y) \in \mathbb{R}^2, y > \max(0, z)\}$$

behave like $\kappa_1/|x|^\alpha$ for some $\kappa_1 > 0$ when $x \to -\infty$, and all solutions in the sector

$$F = \left\{(z, y) \in \mathbb{R}^2, \frac{z}{1+\alpha} > y > 0, z > 0\right\}$$

behave like $\kappa_2/|x|^\alpha$ for some $\kappa_2 > 0$ when $x \to +\infty$. The profile p is then obtained in the following way: in A there exists a unique solution behaving like $\kappa_-/|x|^\alpha$ at $-\infty$, say Y_-, and we define $\mathrm{p}(z) = Y_-(z)$ for $z < z_c$, and similarly, in the sector F, there exists a unique solution Y_+ behaving like $\kappa_+/|x|^\alpha$ at $+\infty$, and $\mathrm{p}(z) = Y_+(z)$ for $z > z_c$.

We can check that $z_c > 0$, and in fact, z_c itself has a geometric interpretation: it is the only value such that

$$\frac{Y_+(z_c) + Y_-(z_c)}{2} = \frac{z_c}{1+\alpha}.$$

That is once the two blue lines of sector A and F are chosen by the values of κ_+ and κ_-, the jump between them happened at the only point where the red line can be in the middle of the jump. The two dotted green lines are such that the red line is the middle between them and the two black lines. This is a Rankine-Huigonot condition, we will come back to it in subsection 2.1.

It is of interested to note that Y_- is an unbounded exact solution of the rescaled Burgers equation, and Y_+ is also an exact solution of the rescaled Burgers equation, which is bounded but not defined on the full real line. These two objects are thus very degenerate, especially because we are looking at a problem where the solutions are bounded and on the full real line. The profile p is still composed of pieces of $Y_\pm$, in a way to hide the unboundedness of Y_- and the problem of the definition of Y_+. In other words, in this problem with bounded solutions, we see pieces of unbounded profiles.

1.4 Sketch of the proof

The proof of (1.2) is done in [6]. It relies on the Hopf-Cole formula [1], [8] and a rescaling. We define the function

$$H_t(y,z) = \frac{-(z-y)^2}{4} - \frac{t^{-(1-\alpha)/(1+\alpha)}}{2}\int_0^{yt^{1/(1+\alpha)}} f_0,$$

and we check that

$$t^{\alpha/(1+\alpha)} f(zt^{1/(1+\alpha)}, t) = \frac{\int_{\mathbb{R}} t^{\alpha/(1+\alpha)} f_0(yt^{1/(1+\alpha)}) e^{t^{(1-\alpha)/(1+\alpha)} H_t(y,z)} dt}{\int_{\mathbb{R}} e^{t^{(1-\alpha)/(1+\alpha)} H_t(y,z)} dt}.$$

When $t \to +\infty$, for $y = 0$ we have that $t^{\alpha/(1+\alpha)} f_0\left(yt^{1/(1+\alpha)}\right) \to \kappa_\pm/|y|^\alpha$, where the $\pm$ in $\kappa_\pm$ is the sign of y. Since $(1-\alpha)/(1+\alpha) > 0$, for t large, we can show that $H_t(y,z)$ is of size independent of time, and thus (at fixed $z \in \mathbb{R}$), $t^{(1-\alpha)/(1+\alpha)} H_t(y,z)$ is mainly concentrated near the maximums of $H_t(.,z)$. Therefore, the value of $\int_{\mathbb{R}} e^{t^{(1-\alpha)/(1+\alpha)} H_t(y,z)} dt$ will be almost equal to the same integral but only in a neighborhood of the maximums of $H_t(.,z)$.

We can deduce from these two arguments that, if for t large enough the function $H_t(.,z)$ has only one global maximum, denoted $y_*(z,t)$, then

$$\left| t^{\alpha/(1+\alpha)} f(t^{\alpha/(1+\alpha)} z) - \frac{\kappa}{|y_*(z,t)|^\alpha} \right| \leqslant K_\nu t^{-(1-\alpha)/2(1+\alpha)} \tag{1.5}$$

where $K_\nu > 0$ depends on

$$\nu := \max\{|H_t(y_1,z) - H_t(y_2,z)|, y_1, y_2 \text{ are local maxima of } H_t(.,z)\},$$

with, in particular, $K_\nu \to +\infty$ when $\nu \to 0$.

Then, by working on the function $y \to H_t(y,z)$, we show that for t large enough and any $z \in \mathbb{R}$, this function has only at most two candidates, denoted $y_\pm(z,t)$, that can reach the global maximum of $H_t(.,z)$. When $t \to +\infty$, they converge respectively to solutions of the implicit problems

$$z = y_\pm(z) + \frac{\kappa_\pm}{|y_\pm(z)|^\alpha}.$$

We then show that there exists $z_c > 0$ such that, for $z < z_c$ the maximum is reached only by $y_-(z,t)$, and for $z > z_c$, and only by $y_+(z,t)$. We can check that for t large enough, $|y_-(z,t) - y_+(z,t)|$ is never 0, and in (1.5) we have

$$y_*(z,t) = \begin{cases} y_+(z,t) \text{ if } z > z_c \\ y_-(z,t) \text{ if } z < z_c \end{cases},$$

leading to the discontinuity of $\mathrm{p}(z) = \lim_{t\to+\infty} \kappa_\pm/|y_\pm(z)|^\alpha$ for $z = z_c$ (where the $\pm$ is the sign of $z - z_c$). We can check that this definition of p is compatible with equation (1.4).

2 Investigating the discontinuity

The fact that (1.2) happens for $z = z_c$ implies that there is still some information to be understood near $z = z_c$ when $t \to +\infty$ for solutions of the viscous Burgers equation. Our goal here is to show and explain some results about that.

Remark that imposing simply $f_0(x) \quad \kappa_\pm/|x|^\alpha$ when $x \to \pm\infty$ is not very precise, in the sense that $f_0(x) - \kappa_\pm/|x|^\alpha$ can be for instance non integrable, or identically 0 at infinity. The profile p simply depends on the first order of the tails at $\pm\infty$ of f_0, but what happens neat $z = z_c$ will depend on the rest of f_0. We want to give some examples of that.

2.1 Construction of an approximate profile and application

2.1.1 The profile h_ε

Our goal here is to construct an approximation of p for large time that is continuous, and converges to p in some sense when $t \to +\infty$. Looking at (1.3), let us fix some $\varepsilon > 0$ small and try to solve the ODE problem

$$\begin{cases} \dfrac{\alpha}{1+\alpha}\mathrm{h}_\varepsilon + \partial_z \mathrm{h}_\varepsilon \left(\dfrac{z}{1+\alpha} - \mathrm{h}_\varepsilon\right) + \varepsilon \partial_z^2 \mathrm{h}_\varepsilon = 0 \\ \mathrm{h}_\varepsilon(z) \quad \dfrac{\kappa_\pm}{|z|^\alpha} \text{ when } z \to \pm\infty. \end{cases} \tag{2.1}$$

This is a second order ODE with two conditions at $\pm\infty$, so not yet a Cauchy problem. Furthermore, in the limit $\varepsilon \to 0$ this is a doubly degenerate equation: the term with two derivatives disappear, and the equation at $\varepsilon = 0$ is itself degenerate as the term in front of $\partial_z \mathrm{h}_\varepsilon$ can cancel.

We can solve this problem by a shooting method:

Lemma 2.1 (Proposition 1.3 of [7]) *For any $\varepsilon > 0$ small enough, there exists $z_c(\varepsilon) > 0$ and $a(\varepsilon) \quad \mathbb{R}$ with*

$$z_c(\varepsilon) \to z_c, a(\varepsilon) \to \frac{\mathrm{p}(z_c^-) - \mathrm{p}(z_c^+)}{2}$$

when $\varepsilon \to 0$ such that the solution of the Cauchy problem

$$\begin{cases} \dfrac{\alpha}{1+\alpha}\mathrm{h}_\varepsilon + \partial_z \mathrm{h}_\varepsilon \left(\dfrac{z}{1+\alpha} - \mathrm{h}_\varepsilon\right) + \varepsilon \partial_z^2 \mathrm{h}_\varepsilon = 0 \\ \mathrm{h}_\varepsilon(z_c(\varepsilon)) = \dfrac{z_c(\varepsilon)}{1+\alpha}, \quad \mathrm{h}_\varepsilon(z_c(\varepsilon)) = \dfrac{-a(\varepsilon)^2}{2\varepsilon} \end{cases}$$

satisfy (2.1).

We can also show that $\mathrm{h}_\varepsilon - \mathrm{p} \;{}_{L^\infty(\mathbb{R}\setminus[z_c-y,z_c+y])} \to 0$ when $\varepsilon \to 0$ for any fixed $y > 0$.

With $\varepsilon(t) = t^{(\alpha-1)/(\alpha+1)} \to 0$ when $t \to +\infty$, the function $\mathrm{h}_{\varepsilon(t)}$ is thus a smooth approximation of p. This means that for large times, we expect the function f solving the viscous Burgers equation to be of the form

$$f(x,t) \quad t^{-\alpha/(1+\alpha)}\mathrm{h}_{\varepsilon(t)}(t^{-1/(1+\alpha)}x).$$

2.1.2 An example of new profile near the discontinuity

With this profile, we can construct an initial data for which, in large time, we can describe what is happening near $z = z_c$.

Proposition 2.2 *Consider the viscous Burgers equation $\partial_t f - \partial_x^2 f + f\partial_x f = 0$ for an initial time $T > 0$ large, and with the initial condition*

$$f_T(x) = T^{-\alpha/(1+\alpha)}\mathrm{h}_{\varepsilon(T)}(T^{-1/(1+\alpha)}x).$$

Then, with $\mathrm{a} := (\mathrm{p}(z_c^+) - \mathrm{p}(z_c^-))/2$ *we have that for any $x \quad \mathbb{R}$,*

$$t^{\alpha/(1+\alpha)} f\left(t^{1/(1+\alpha)}\left(z_c + t^{-(1-\alpha)/(1+\alpha)}x\right), t\right) \to \mathrm{a}\tanh(2\mathrm{a}x) + \frac{\mathrm{p}(z_c^+) + \mathrm{p}(z_c^-)}{2}.$$

This is a direct corollary of Theorem 1.4 of [7]. The initial data f_T satisfy $f_T(x) \sim \kappa_\pm/|x|^\alpha$ when $x \to \pm\infty$ and as such (1.2) apply. Here, we take $z = z_c + t^{-(1-\alpha)/(1+\alpha)}x$ for some $x \in \mathbb{R}$, so $z \to z_c$ when $t \to +\infty$. When $x \to \pm\infty$ the profile we obtain here converges to $\mathrm{p}(z_c^\pm)$, meaning that we have "resolved" the discontinuity. That is, near $z = z_c$, up to a new rescaling in time, we see a smooth profile, that connect the two limits of p on the right and left of the discontinuity. This is also a feature seen in several boundary layer problems.

2.1.3 A family of profiles near the discontinuity

In [7], we compute such profiles for initial data at time T that are not necessarily equal to f_T but that are close to it using energy arguments.

In that case, the result of Proposition 2.2 is modified. Take for instance an initial data at time T of the form $f_T + g_T$, where g_T is compactly supported and small in $H^1(\mathbb{R})$ (the smallness depends on T). Let us denote $f_t(x) = t^{-\alpha/(1+\alpha)}\mathrm{h}_{\varepsilon(t)}(t^{-1/(1+\alpha)}x)$ and write the solution of the viscous Burgers equation as $f_t + g$.

For the viscous Burgers with initial data of finite mass (i.e. $\left|\int_{\mathbb{R}} f_0\right| < +\infty$), the mass of the solution is a conserved quantity. It also appears in the profile for large time of the solution, see [11]. Here, $\int_{\mathbb{R}} f_T = +\infty$, but remarkably, if $\left|\int_{\mathbb{R}} g_T\right| < +\infty$ then $\int_{\mathbb{R}} g$ is a conserved quantity.

This conserved mass will still be visible in the limit profile. We can show that, for such an initial data of the form $f_T + g_T$ described above, with $M = \int_{\mathbb{R}} g_T$ small, there exists a function $x \to \nu_M(x)$ depending only on M such that for all $x \in \mathbb{R}$,

$$t^{\alpha/(1+\alpha)} f\left(t^{1/(1+\alpha)}\left(z_c + t^{-(1-\alpha)/(1+\alpha)}x\right), t\right) \longrightarrow \mathrm{a}\tanh(2\mathrm{a}x) + \frac{\mathrm{p}(z_c^+) + \mathrm{p}(z_c^-)}{2} + \nu_M(x)$$

when $t \to +\infty$. The function ν_M does not cancel (except if $M = 0$) and $\int_{\mathbb{R}} \nu_M = M$. We see that, although by (1.2) the profile p is a global attractor for $z = z_c$, this is not the case for $z = z_c$, as ν_M still depends on the initial data. This is in fact more generic than the situation described above, see subsection 2.2.

Although not proven, it is likely that there exists $X_M \in \mathbb{R}$ such that

$$\mathrm{a}\tanh(2\mathrm{a}x) + \nu_M(x) = \mathrm{a}\tanh(2\mathrm{a}(x - X_M)).$$

2.1.4 Generalisation to other equations

The proof of these results does not rely on the Hopf-Cole formula, but on ODE technics and energy estimates. As such, it can be applied to other equations than the viscous Burgers equation, for instance

$$\partial_t f - \partial_x^2 f + \partial_x\left(\frac{f^2}{2} + f^3\right) = 0.$$

In that case, after the rescaling in time, the term $\partial_x(f^3)$ will be negligible compared to the Burgers term, and the energy estimates will still hold. But for this equation, we can not use the Hopf-Cole formula, so the convergence (1.2) for general initial data (and not just small perturbation of a specific profile) is an open problem in this equation.

2.2 Embedded discontinuous profiles

Here, we want to show that the results of subsection 2.1 are not generic. They rely on the fact that $f_0 - \kappa_\pm/|x|^\alpha$ is integrable at $\pm\infty$. If this is not the case, then it is not possible to resolve the discontinuity as described above. Here is an example of that.

Proposition 2.3 ([2]) *Consider the solution of the viscous Burgers equation $\partial_t f - \partial_x^2 f + f\partial_x f = 0$ with an initial data f_0 satisfying*

$$f_0(x) = \frac{\kappa_1}{|x|^\alpha} + \frac{\kappa_2}{|x|^\beta} + o\left(1/|x|^\beta\right), f_0(x) = -\frac{\alpha\kappa_1}{|x|^{1+\alpha}} - \frac{\beta\kappa_2}{|x|^{1+\beta}} + o\left(1/|x|^{1+\beta}\right)$$

when $x \to \pm\infty$ for $\kappa_1, \kappa_2 > 0$, $\alpha \in]0,1[$ and $\beta \in \left]\alpha, \frac{1+\alpha}{2}\right[$. Then, for any $x = 0$,

$$t^{(\beta-\alpha)/(1+\alpha)}\left(t^{\alpha/(1+\alpha)} f\left(t^{1/(1+\alpha)}\left(z_c + t^{-(\beta-\alpha)/(1+\alpha)}x\right), t\right) - \mathrm{p}\left(z_c + t^{(\beta-\alpha)/(1+\alpha)}x\right)\right) \longrightarrow \mathrm{q}(x)$$

when $t \to +\infty$, *where* q *is constant on* $]-\infty, 0[$ *and on* $]0, +\infty[$, *and is discontinuous at* $x = 0$ *for almost all* $\beta \in]\alpha, \frac{1+\alpha}{2}[$.

In this result, we zoom around the discontinuity by taking $z = z_c + t^{-(\beta-\alpha)/(1+\alpha)}x$ for some $x \in \mathbb{R}$, yet when $t \to +\infty$ the limit profile still has a discontinuity at $x = 0$. By taking an initial data of the form

$$f_0(x) = \sum_{n \in \mathbb{N}} \frac{2^{-n}}{|x|^{\alpha_n}} + o(1/|x|)$$

for a well chosen increasing sequence $(\alpha_n)_{n \in \mathbb{N}}$, we have an example where the asymptotic profile has a discontinuity, and although we can keep zooming on it, with new scales, the discontinuity does not disappear after any arbitrary number of rescaling (see Theorem 1.2 of [2]).

3 Some related open problems

The problem of the long time behavior of the viscous Burgers equation with infinite mass initial condition is a good playing ground to see and study degenerate asymptotic profiles. We give here a few follow up questions that can be of interest:

- What happens in the critical case $\alpha = 1$?
- Does (1.2) still hold if we consider the Burgers equation without viscosity ? What can happen near the discontinuity then ?
- Does the approach of section 2.1 work for $\partial_t f - \partial_x^2 f + f^2 \partial_x f = 0$ or other similar equations ?
- In Proposition 2.3, the limit profile q is piecewise constant and the convergence is not uniform in x. Can we improve on that, and can we find specific initial data, in the spirit of section 2.1, where we can “resolve” the second discontinuity ?
- The convergence (1.2) holds for $\kappa_+, \kappa_- > 0$, but we can in fact show (in [7]) that it holds for any $\kappa_+, \kappa_- = 0$ except if $\kappa_- < 0, \kappa_+ > 0$. What is happening in this last case ?

References

[1] Julian D. Cole. On a quasi-linear parabolic equation occurring in aerodynamics. *Quart. Appl. Math.*, 9:225–236, 1951.

[2] N. De Nitti and E. Pacherie. Nested discontinuous asymptotic profiles for the viscous Burgers equation with infinite mass. arXiv:2406.10874

[3] Daniel B. Dix. Large-time behaviour of solutions of Burgers’ equation. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 132(4):843–878, 2002.

[4] Miguel Escobedo, Juan Luis Vazquez, and Enrique Zuazua. Asymptotic behaviour and source-type solutions for a diffusion-convection equation. *Archive for Rational Mechanics and Analysis*, 124:43–65, 01 1993.

[5] Thierry Gallay and Arnd Scheel. Viscous shocks and long-time behavior of scalar conservation laws. *Communications on Pure and Applied Analysis*, 2023. arXiv:2306.13341

[6] Tej-Eddine Ghoul, Nader Masmoudi, and Eliot Pacherie. Nonlinear enhanced dissipation in viscous Burgers type equations II. *Communications in Partial Differential Equations*, 48(10-12):1286–1322, 2023.

[7] Tej-Eddine Ghoul, Nader Masmoudi, and Eliot Pacherie. Nonlinear enhanced dissipation in viscous Burgers type equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire (2024)*, to appear. arXiv:2211.16839

[8] Eberhard Hopf. The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.*, 3:201–230, 1950.

[9] Shoshana Kamin and Lambertus A. Peletier. Large time behaviour of solutions of the heat equation with absorption. *Annali Della Scuola Normale Superiore Di Pisa-classe Di Scienze*, 12:393–408, 1985.

[10] Pavol Quittner and Philippe Souplet. Superlinear parabolic problems. Blow-up, global existence and steady states. Birkhäuser Advanced Texts. Basler Lehrbücher. Birkhäuser, Cham.

[11] Enrique Zuazua. Asymptotic behavior of scalar convection-diffusion equations. arXiv:2003.11834