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STABLE AND UNSTABLE STOKES WAVES

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Stable and unstable Stokes waves

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Abstract. In the last years much progress has been achieved concerning the problem of determining the stability/instability of Stokes waves, i.e. periodic traveling solutions of the pure gravity water waves equations in an ocean of depth $\mathtt{h} > 0$, subject to longitudinal perturbations. In this talk we review some of these results focusing the attention on the existence of unstable spectral bands away from zero for Stokes waves of small amplitude $\varepsilon > 0$. In [11] we prove that the unstable spectrum is the union of "isolas" of approximately elliptical shape, parameterized by integers $\mathtt{p} \geq 2$, with semiaxis of size $|\beta_1^{(\mathtt{p})}(\mathtt{h})|\varepsilon^{\mathtt{p}} + O(\varepsilon^{\mathtt{p}+1})$ where $\beta_1^{(\mathtt{p})}(\mathtt{h})$ is a *nonzero* analytic function of the depth $\mathtt{h}$ that depends on the Taylor coefficients of the Stokes waves up to order $\mathtt{p}$. [1]

1 Introduction

Stokes waves, discovered in 1847 by Stokes in the work [29], are among the most renowned solutions of the gravity water waves equations. They are space periodic traveling waves that, despite the dispersive effects of the linear wave dynamics, survive for all times moving steadily in the x-direction thanks to nonlinear effects. The first mathematically rigorous existence proof of Stokes waves was given by Nekrasov [26], Levi Civita [22] and Struik [30] almost one century ago. In the last years also the existence of quasi-periodic traveling Stokes waves –which are the nonlinear superposition of Stokes waves moving with rationally independent speeds– has been proved in [4, 19, 5] by means of KAM methods.

A question of fundamental physical importance concerns the stability/instability of the Stokes waves.

The mathematical problem can be formulated as follows. Consider a 2π-periodic Stokes wave with amplitude $0 < \varepsilon \ll 1$ in an ocean with depth $\mathtt{h} > 0$. The linearized water waves equations at the Stokes waves are, in the inertial reference frame moving with the speed of the wave, a linear time-independent system of the form

$$\partial_t h(t,x) = \mathcal{L}_\varepsilon(\mathtt{h})\,[h(t,x)] \tag{1.1}$$

where $\mathcal{L}_\varepsilon(\mathtt{h})$ is a Hamiltonian pseudo-differential linear operator with 2π-periodic coefficients, cfr. (2.4). The dynamics of (1.1) for perturbations $h(t,\cdot)$ in $L^2(\mathbb{R})$ is fully determined by answering the following

- QUESTION: *What is the $L^2(\mathbb{R})$ spectrum of $\mathcal{L}_\varepsilon(\mathtt{h})$?*

Despite its simplicity, such a problem remained largely unsolved. Supported by numerical simulations, it has been conjectured in [18] the *existence* of a *figure* "8" close to the origin in the spectrum and of *infinitely many isolas* of approximately elliptic shape, centered along the imaginary axis, and shrinking exponentially fast away from the origin. The works [12, 27] (in finite and infinite depth respectively) proved the existence of a local spectral band forming a cross amidst zero in the complex plane. Next the existence of the figure 8 has been fully proved in the works [6, 8, 9]. Very recently the conjecture about the existence of infinitely many unstable isolas away from zero has been proved in [11]. A statement of the latter result is the following :

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Theorem 1.1. *[11] For any integer* $\mathtt{p} \geq 2$*, there exist* $\varepsilon_1^{(\mathtt{p})} > 0$ *and a closed set of isolated depths* $\mathcal{S}^{(\mathtt{p})} \subset (0, +\infty)$ *such that for any* $\mathtt{h} \in (0, +\infty) \setminus \mathcal{S}^{(\mathtt{p})}$ *and* $0 < \varepsilon \leq \varepsilon_1^{(\mathtt{p})}$*, the spectrum* $\sigma_{L^2(\mathbb{R})}\big(\mathcal{L}_\varepsilon(\mathtt{h})\big)$ *of the operator* $\mathcal{L}_\varepsilon(\mathtt{h})$ *contains* $\mathtt{p} - 1$ *disjoint "isolas" in the complex upper half-plane away from the origin and* $\mathtt{p} - 1$ *disjoint symmetrical isolas in the lower half-plane. For any* $\ell = 2, \dots, \mathtt{p}$*, the* ℓ*-th isola in the upper half complex plane is approximated by an ellipse with semiaxes of size proportional to* ε^ℓ*, centered on the imaginary axis at a point* $\mathrm{i}\,\omega_*^{(\ell)}(\mathtt{h}) + O(\varepsilon^2)$*, see Figure 1, where the positive numbers* $\omega_*^{(\ell)}(\mathtt{h})$ *form a monotone sequence* $0 < \omega_*^{(2)}(\mathtt{h}) < \omega_*^{(3)}(\mathtt{h}) < \dots < \omega_*^{(\ell)}(\mathtt{h}) < \dots$ *with* $\lim_{\ell \to +\infty} \omega_*^{(\ell)}(\mathtt{h}) = +\infty$.

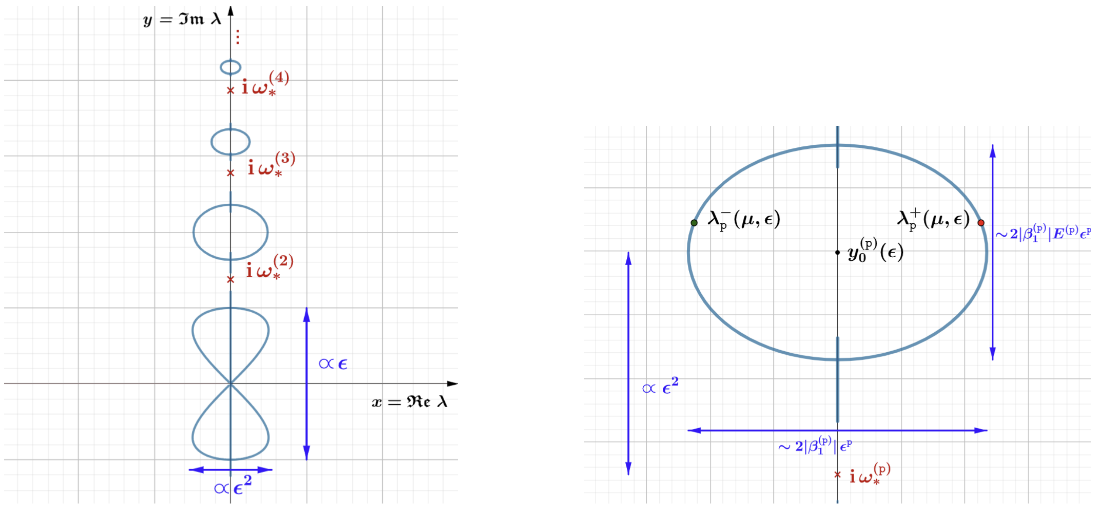

Figure 1: Spectral bands with non zero real part of the $L^2(\mathbb{R})$-spectrum of $\mathcal{L}_\varepsilon$. On the right, zoom of the $\mathtt{p}$-th isola of modulational instability. Its center $y_0^{(\mathtt{p})}(\varepsilon)$ is $O(\varepsilon^2)$ distant from $\mathrm{i}\,\omega_*^{(\mathtt{p})}$ and its size is $\propto \varepsilon^{\mathtt{p}}$. As shown in Theorem 2.7, two eigenvalues $\lambda_{\mathtt{p}}^{\pm}(\mu, \varepsilon)$ of the Floquet operator $\mathcal{L}_{\mu,\varepsilon}$ span the $\mathtt{p}$-th isola for Floquet exponents $\mu \in (\mu_\wedge^{(\mathtt{p})}(\varepsilon), \mu_\vee^{(\mathtt{p})}(\varepsilon))$ and recollide on the imaginary axis at the ends of the interval.

A more complete statement is given in Theorem 2.7.

Together with the previous works [6, 8, 9] about the spectrum $\sigma_{L^2(\mathbb{R})}\big(\mathcal{L}_\varepsilon(\mathtt{h})\big)$ close to 0 this result provides a complete description of the unstable spectrum of $\mathcal{L}_\varepsilon(\mathtt{h})$ inside *any* horizontal strip in the complex plane.

Let us now present the physical and mathematical state of the art on the modulational instability problem of Stokes waves.

In the sixties Benjamin and Feir [3, 2], Whitham [32], Lighthill [23] and Zakharov [33] discovered, through experiments and formal arguments, that small amplitude Stokes waves in sufficiently deep water are unstable when subject to long-wave perturbations. More precisely, their works predicted that 2π-periodic Stokes waves evolving in a tank of depth

$$\mathtt{h} > \mathtt{h}_{\mathrm{WB}} := 1.363...$$

and length $2\pi/\mu$, with Floquet exponent $\mu \ll 1$, are unstable. This phenomenon is nowadays called "Benjamin-Feir" instability. It is supported by an enormous amount of physical observations and numerical simulations and it has been detected in several dispersive PDE models derived from water waves.

The first mathematically rigorous proof of local branches of Benjamin-Feir spectrum was obtained by Bridges-Mielke [12] in finite depth (via spatial dynamics and center manifold theory), see also [21], and recently by Nguyen-Strauss [27] in deep water (via Lyapunov-Schmidt decomposition). We also mention the nonlinear instability result in Chen-Su [13]. The results in Berti-Maspero-Ventura [6, 8, 9] proved, via a novel symplectic version of Kato perturbation theory for eigenvalues of Hamiltonian operators, that these local branches can be globally continued to form a complete figure "8" as shown in Figure 1. The paper [6] describes the spectrum in deep water $\mathtt{h} = +\infty$, the paper [7] for any depths $\mathtt{h} > \mathtt{h}_{\rm WB}$, whereas [9] is devoted to determine the stability/instability nature of the Stokes waves close to the critical transition depth $\mathtt{h}_{\rm WB}$. This latter question had stimulated a controversial debate that remained open for many years.

Preliminary numerical investigations about the existence of unstable spectrum away from the origin were performed by McLean [24, 25] in the eighties. More recently Deconinck and Oliveras [18] were able to obtain the first complete plots of these high-frequency instabilities, which appear as small isolas centered on the imaginary axis. In contrast to Benjamin-Feir instability, they occur at all values of the depth. Deconinck and Oliveras were able to plot the first few of these isolas, and conjectured the existence of infinitely many of them, parametrized by integers $\mathtt{p} \geq 2$, going to infinity along the imaginary axis and shrinking as $O(\varepsilon^{\mathtt{p}})$. Such isolas become, even for moderate values of $\mathtt{p}$, numerically invisible.

A formal perturbation method to describe them has been developed in [16] for the first two isolas closest to the origin (i.e. for $\mathtt{p} = 2, 3$ in the language of Theorem 1.1). The first high-frequency instability isola (i.e. $\mathtt{p} = 2$) has been rigorously proved in [21] in finite depth and in [10] for the deep-water case. We also mention the very recent paper [17] which, relying on the spectral approach in [6], describes the first instability isola for Stokes waves under transversal perturbations. Other results of transverse instability are in [20] for the gravity-capillary case and [28] for solitary waves. All these mathematically rigorous results about high frequency modulational instability regard only the first isola ($\mathtt{p} = 2$) of unstable eigenvalues.

One of the main challenges for analytical investigations of high-frequency instabilities is that the $\mathtt{p}$-th isola has exponentially small size $\propto \varepsilon^{\mathtt{p}}$, spanned for an interval of Floquet exponents of width $\propto \varepsilon^{\mathtt{p}}$. Furthermore, the $\mathtt{p}$-th isola depends on the Taylor expansion of the Stokes waves of order $\mathtt{p}$, resulting in analytical manipulations of enormous increasing difficulty as $\mathtt{p} \to +\infty$. These facts explain why the conjecture about the existence of infinitely many high-frequency modulational instability isolas –proved in Theorems 1.1 and 2.7– is particularly challenging. We now describe in detail this result.

2 Infinitely many isolas of modulational instability

The water waves equations. We consider the Euler equations for a 2-dimensional incompressible and irrotational fluid under the action of gravity which fills the time dependent region

$$\mathcal{D}_\eta := \big\{(x,y) \in \mathbb{T} \times \mathbb{R} \ : \ -\mathtt{h} < y < \eta(t,x)\big\}\,, \quad \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}\,,$$

with depth $\mathtt{h} > 0$ and space periodic boundary conditions. The irrotational velocity field is the gradient of the scalar potential $\Phi(t,x,y)$ determined as the unique harmonic solution of

$$\Delta\Phi = 0 \ \text{ in } \ \mathcal{D}_\eta\,, \quad \Phi(t,x,\eta(t,x)) = \psi(t,x)\,, \quad \Phi_y(t,x,-\mathtt{h}) = 0\,.$$

The time evolution of the fluid is determined by two boundary conditions at the free surface. The first is that the fluid particles remain, along the evolution, on the free surface (kinematic boundary condition), and the second one is that the pressure of the fluid is equal, at the free surface, to the constant atmospheric pressure (dynamic boundary condition). Then, as shown by Zakharov [34] and Craig-Sulem [14], the time evolution of the fluid is determined by the following equations for the unknowns $(\eta(t,x), \psi(t,x))$,

$$\eta_t = G(\eta)\psi\,, \quad \psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2\,, \tag{2.1}$$

where $g > 0$ is the gravity constant and $G(\eta) := G(\eta, \mathtt{h})$ denotes the Dirichlet-Neumann operator

$$[G(\eta)\psi](x) := \Phi_y(x,\eta(x)) - \Phi_x(x,\eta(x))\eta_x(x)\,.$$

The equations (2.1) are the Hamiltonian system

$$\partial_t \begin{bmatrix} \eta \\ \psi \end{bmatrix} = \mathcal{J} \begin{bmatrix} \nabla_\eta \mathscr{H} \\ \nabla_\psi \mathscr{H} \end{bmatrix}, \qquad \mathcal{J} := \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix},$$

where ∇ denote the L^2-gradient, and the Hamiltonian

$$\mathscr{H}(\eta, \psi) := \frac{1}{2} \int_{\mathbb{T}} \left(\psi\, G(\eta)\psi + g\eta^2 \right) \mathrm{d}x$$

is the sum of the kinetic and potential energy of the fluid.

In addition of being Hamiltonian, the water waves system (2.1) is time reversible with respect to the involution

$$\rho \begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} := \begin{bmatrix} \eta(-x) \\ -\psi(-x) \end{bmatrix}, \quad \text{i.e. } \mathscr{H} \circ \rho = \mathscr{H}\,, \tag{2.2}$$

and it is invariant under space translations.

In the sequel, with no loss of generality, we set the gravity constant $g = 1$.

Stokes waves. The water waves equations (2.1) admit periodic traveling solutions –called Stokes waves–

$$\eta(t, x) = \breve{\eta}(x - ct)\,, \quad \psi(t, x) = \breve{\psi}(x - ct)\,,$$

for some speed c (which depends on the amplitude of the wave) and 2π-periodic profiles $\breve{\eta}(x), \breve{\psi}(x)$. Bifurcation of small amplitude Stokes waves has been rigorously proved in [22, 26] in infinite depth, and in [30] in finite depth, and nowadays can be deduced by a direct application of the Crandall-Rabinowitz bifurcation theorem from a simple eigenvalue as in [7]. The speed of Stokes waves of small amplitude is close to the one predicted by the linear theory,

$$\mathtt{c_h} := \sqrt{\tanh(\mathtt{h})}\,.$$

In order to prove the existence of *all* the instability isolas of the Stokes waves, we need some novel structural information about their Taylor expansion at *any* order. We introduce the following definition.

Definition 2.1. Let $\ell \in \mathbb{N}$. We denote by $\mathtt{Evn}_\ell$ the vector space of real trigonometric polynomials

$$f(x) = \begin{cases} f^{[0]} + f^{[2]} \cos(2x) + \cdots + f^{[\ell]} \cos(\ell\, x) & \text{if } \ell \text{ is even}\,, \\ f^{[1]} \cos(x) + f^{[3]} \cos(3x) + \cdots + f^{[\ell]} \cos(\ell\, x) & \text{if } \ell \text{ is odd}\,, \end{cases}$$

with real coefficients $f^{[i]}$, $i = 0, \ldots, \ell$ and by $\mathtt{Odd}_\ell$ the vector space of real trigonometric polynomials

$$g(x) = \begin{cases} g^{[2]} \sin(2x) + g^{[4]} \sin(4x) + \cdots + g^{[\ell]} \sin(\ell\, x) & \text{if } \ell \text{ is even}\,, \\ g^{[1]} \sin(x) + g^{[3]} \sin(3x) + \cdots + g^{[\ell]} \sin(\ell\, x) & \text{if } \ell \text{ is odd}\,, \end{cases}$$

with real coefficients $g^{[i]}$, $i = 1, \ldots, \ell$.

Note that a function $f \in \mathtt{Evn}_\ell$, resp. $\mathtt{Odd}_\ell$, is not just an even, resp. odd, trigonometric real polynomial of degree less or equal to ℓ, but it possesses only harmonics of the *same* parity of ℓ.

In order to describe the coefficients of the Stokes wave we also introduce the following definition.

Definition 2.2. An analytic function $g : (0, +\infty) \to \mathbb{R}$ belongs to $\mathcal{Q}(\mathtt{c}_\mathtt{h}^2)$ if there exist polynomials $p(y)$ and $q(y)$ with *integer* coefficients and without common factors, such that $q(y) \neq 0$ for any $0 < y \leq 1$, and $g(\mathtt{h}) = \frac{\mathtt{p}(\mathtt{c}_\mathtt{h}^2)}{q(\mathtt{c}_\mathtt{h}^2)}$ for any $\mathtt{h} > 0$, where $\mathtt{c}_\mathtt{h}^2 = \tanh(\mathtt{h})$. Note that $\lim_{\mathtt{h} \to +\infty} g(\mathtt{h}) = \frac{p(1)}{q(1)} \in \mathbb{Q}$.

The next result, proved in [11], provides a careful description of the Taylor expansion of the Stokes waves at *any* order.

Theorem 2.3. ***(Stokes waves)*** *For any $\mathtt{h}_* > 0$ there exist $\varepsilon_* := \varepsilon_*(\mathtt{h}_*) > 0$ and for any $\mathtt{h} \geq \mathtt{h}_*$ a unique family of real analytic solutions $(\eta_\varepsilon(x), \psi_\varepsilon(x), c_\varepsilon)$, defined for any $|\varepsilon| < \varepsilon_*$, of*

$$\begin{cases} c\,\eta_x + G(\eta, \mathtt{h})\psi = 0\,, \\ c\,\psi_x - \eta - \dfrac{\psi_x^2}{2} + \dfrac{1}{2(1+\eta_x^2)}\big(G(\eta,\mathtt{h})\psi + \eta_x\psi_x\big)^2 = 0\,, \end{cases}$$

such that $\eta_\varepsilon(x)$ and $\psi_\varepsilon(x)$ are 2π-periodic, $\eta_\varepsilon(x)$ is even and $\psi_\varepsilon(x)$ is odd, of the form

$$\eta_\varepsilon(x) = \varepsilon\cos(x) + \sum_{\ell\geq 2}\varepsilon^\ell \eta_\ell(x)\,, \quad \psi_\varepsilon(x) = \varepsilon \mathtt{c}_{\mathtt{h}}^{-1}\sin(x) + \sum_{\ell\geq 2}\varepsilon^\ell\psi_\ell(x)\,, \quad c_\varepsilon = \mathtt{c}_{\mathtt{h}} + \sum_{\ell\geq 2,\ \ell \text{ even}} \varepsilon^\ell c_\ell\,.$$

The following properties hold: for any $\ell \in \mathbb{N}$

1. *the function $\eta_\ell(x)$, resp. $\psi_\ell(x)$, is a trigonometric polynomial in $\mathtt{Evn}_\ell$, resp. in $\mathtt{Odd}_\ell$;*
2. *the coefficients $\eta_\ell^{[0]}, \dots, \eta_\ell^{[\ell]}$ and $\mathtt{c}_{\mathtt{h}}\psi_\ell^{[1]}, \dots \mathtt{c}_{\mathtt{h}}\psi_\ell^{[\ell]}$ (according to Definition 2.1) and $\mathtt{c}_{\mathtt{h}} c_\ell$ belong to $\mathcal{Q}(\mathtt{c}_{\mathtt{h}}^2)$ in the sense of Definition 2.2.* (2.3)

Linearization at the Stokes waves. In a reference frame moving with the speed c_ε, the linearized water waves equations at the Stokes waves turn out to be[2] the linear system (1.1) where $\mathcal{L}_\varepsilon$ is the Hamiltonian and reversible real operator

$$\mathcal{L}_\varepsilon := \mathcal{L}_\varepsilon(\mathtt{h}) := \begin{bmatrix} \partial_x \circ (\mathtt{c}_{\mathtt{h}} + p_\varepsilon(x)) & |D|\tanh((\mathtt{h}+\mathtt{f}_\varepsilon)|D|) \\ -(1+a_\varepsilon(x)) & (\mathtt{c}_{\mathtt{h}} + p_\varepsilon(x))\partial_x \end{bmatrix} = \mathcal{J}\begin{bmatrix} 1 + a_\varepsilon(x) & -(\mathtt{c}_{\mathtt{h}} + p_\varepsilon(x))\partial_x \\ \partial_x \circ (\mathtt{c}_{\mathtt{h}} + p_\varepsilon(x)) & |D|\tanh((\mathtt{h}+\mathtt{f}_\varepsilon)|D|) \end{bmatrix}, \tag{2.4}$$

where $\mathtt{f}_\varepsilon$ is a real number, analytic in ε, and $p_\varepsilon(x), a_\varepsilon(x)$ are 2π-periodic, even, real analytic functions. As a corollary of the properties (2.3) of the Stokes waves we deduce that the functions $p_\varepsilon(x)$, $a_\varepsilon(x)$, as well as the constant $\mathtt{f}_\varepsilon$ in (2.4), admit the Taylor expansions

$$p_\varepsilon(x) = \sum_{\ell\geq 1}\varepsilon^\ell p_\ell(x)\,, \quad a_\varepsilon(x) = \sum_{\ell\geq 1}\varepsilon^\ell a_\ell(x)\,, \quad \mathtt{f}_\varepsilon = \sum_{\ell\geq 2\,,\,\ell \text{ even}}\varepsilon^\ell \mathtt{f}_\ell\,,$$

where

1. the functions $p_\ell(x)$, $a_\ell(x)$ are trigonometric polynomials in $\mathtt{Evn}_\ell$;
2. the coefficients $\mathtt{c}_{\mathtt{h}} p_\ell^{[0]}, \dots, \mathtt{c}_{\mathtt{h}} p_\ell^{[\ell]}$, $a_\ell^{[0]}, \dots, a_\ell^{[\ell]}$ and $\mathtt{f}_\ell$ belong to $\mathcal{Q}(\mathtt{c}_{\mathtt{h}}^2)$ in the sense of Definition 2.2.

Spectral Bloch-Floquet bands. Since the operator $\mathcal{L}_\varepsilon$ in (2.4) has 2π-periodic coefficients, the starting point to obtain Theorem 1.1 is the Bloch-Floquet decomposition

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_\varepsilon) = \bigcup_{\mu\in[-\frac12,\frac12)} \sigma_{L^2(\mathbb{T})}(\mathcal{L}_{\mu,\varepsilon}) \qquad \text{where} \qquad \mathcal{L}_{\mu,\varepsilon} := \mathcal{L}_{\mu,\varepsilon}(\mathtt{h}) := e^{-\mathrm{i}\,\mu x}\,\mathcal{L}_\varepsilon\, e^{\mathrm{i}\,\mu x}$$

is the complex *Hamiltonian* and *reversible* pseudo-differential operator

$$\mathcal{L}_{\mu,\varepsilon} := \begin{bmatrix} (\partial_x + \mathrm{i}\,\mu)\circ(\mathtt{c}_{\mathtt{h}} + p_\varepsilon(x)) & |D+\mu|\tanh\big((\mathtt{h}+\mathtt{f}_\varepsilon)|D+\mu|\big) \\ -(1+a_\varepsilon(x)) & (\mathtt{c}_{\mathtt{h}} + p_\varepsilon(x))(\partial_x + \mathrm{i}\,\mu) \end{bmatrix} \tag{2.5}$$

[2]After conjugating with the "good unknown of Alinhac" and the "Levi-Civita" transformations, we refer to [1, 6, 7, 9].

$$= \underbrace{\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}}_{=\mathcal{J}} \underbrace{\begin{bmatrix} 1 + a_\varepsilon(x) & -(\mathtt{c_h} + p_\varepsilon(x))(\partial_x + \mathrm{i}\,\mu) \\ (\partial_x + \mathrm{i}\,\mu) \circ (\mathtt{c_h} + p_\varepsilon(x)) & |D+\mu| \tanh\big((\mathtt{h} + \mathtt{f}_\varepsilon)|D+\mu|\big) \end{bmatrix}}_{=:\, \mathcal{B}(\mu,\varepsilon) = \mathcal{B}(\mu,\varepsilon)^*},$$

that we regard as an operator with domain $H^1(\mathbb{T}) := H^1(\mathbb{T}, \mathbb{C}^2)$ and range $L^2(\mathbb{T}) := L^2(\mathbb{T}, \mathbb{C}^2)$, equipped with the complex scalar product

$$(f,g) := \frac{1}{2\pi} \int_0^{2\pi} (f_1 \overline{g_1} + f_2 \overline{g_2})\, \mathrm{d}x\,, \quad \forall f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in L^2(\mathbb{T}, \mathbb{C}^2)\,,$$

and with the sesquilinear, skew-Hermitian and non-degenerate *complex symplectic form*

$$\mathcal{W}_c : L^2(\mathbb{T}, \mathbb{C}^2) \times L^2(\mathbb{T}, \mathbb{C}^2) \to \mathbb{C}\,, \quad \mathcal{W}_c(f,g) := (\mathcal{J} f, g)\,. \tag{2.6}$$

The complex operator $\mathcal{L}_{\mu,\varepsilon}$ is reversible, namely

$$\mathcal{L}_{\mu,\varepsilon} \circ \overline{\rho} = \overline{\rho} \circ \mathcal{L}_{\mu,\varepsilon}\,, \qquad \text{equivalently} \qquad \mathcal{B}(\mu,\varepsilon) \circ \overline{\rho} = -\overline{\rho} \circ \mathcal{B}(\mu,\varepsilon)\,,$$

where $\overline{\rho}$ is the complex involution (cfr. (2.2))

$$\overline{\rho} \begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} := \begin{bmatrix} \overline{\eta}(-x) \\ -\overline{\psi}(-x) \end{bmatrix}\,. \tag{2.7}$$

Since the functions $\varepsilon \mapsto a_\varepsilon$, p_ε are analytic as maps from $\{|\varepsilon| < \varepsilon_0\} \to H^1(\mathbb{T})$ and $\varepsilon \mapsto \mathtt{f}_\varepsilon$ is analytic as well, the operator $\mathcal{L}_{\mu,\varepsilon}$ in (2.5) is analytic in (μ,ε). The operator $\mathcal{L}_{\mu,\varepsilon}$ is analytic also w.r. to $\mathtt{h} > 0$.

The spectrum of $\mathcal{L}_{\mu,\varepsilon}$ on $L^2(\mathbb{T})$ is discrete and its eigenvalues span, as μ varies, the continuous *spectral bands* of $\sigma_{L^2(\mathbb{R})}\big(\mathcal{L}_\varepsilon\big)$. The parameter μ is often referred to as the Floquet exponent. A direct consequence of the Bloch-Floquet decomposition is that any solution of (1.1) can be decomposed as a linear superposition of Bloch-waves

$$h(t,x) = e^{\lambda t} e^{\mathrm{i}\,\mu x} v(x) \tag{2.8}$$

where λ is an eigenvalue of $\mathcal{L}_{\mu,\varepsilon}$ with associated eigenvector $v(x) \in L^2(\mathbb{T})$. If λ is unstable, i.e. has positive real part, then the solution (2.8) grows exponentially fast in time.

We remark that the spectrum $\sigma_{L^2(\mathbb{T})}(\mathcal{L}_{\mu,\varepsilon})$ is a set which is 1-periodic in μ, and thus it is sufficient to consider μ in the first zone of Brillouin $[-\frac{1}{2}, \frac{1}{2})$. Furthermore by the reality of $\mathcal{L}_\varepsilon$ the spectrum $\sigma_{L^2(\mathbb{T})}(\mathcal{L}_{-\mu,\varepsilon})$ is the complex conjugated of $\sigma_{L^2(\mathbb{T})}(\mathcal{L}_{\mu,\varepsilon})$ and we can restrict $\mu \in [0, \frac{1}{2})$. From a physical point of view, the spectrum of the operator $\mathcal{L}_{\mu,\varepsilon}$ encodes the linear dynamics of any long-wave perturbation of the Stokes waves of amplitude ε evolving inside a tank of length $2\pi\mu^{-1}$.

The Hamiltonian nature of the operator $\mathcal{L}_{\mu,\varepsilon}$ constrains the eigenvalues with nonzero real part to arise, for small $\varepsilon > 0$, as perturbation of *multiple* eigenvalues of $\mathcal{L}_{\mu,0}(\mathtt{h})$. Indeed the Hamiltonian structure of $\mathcal{L}_{\mu,\varepsilon}$ implies that if λ is an eigenvalue of $\mathcal{L}_{\mu,\varepsilon}$ then also $-\overline{\lambda}$ is. In particular simple purely imaginary eigenvalues of $\mathcal{L}_{\mu,0}$ remain on the imaginary axis under perturbation.

The spectrum of $\mathcal{L}_{\mu,0}$. The spectrum of the Fourier multiplier matrix operator

$$\mathcal{L}_{\mu,0} = \begin{bmatrix} \mathtt{c_h}(\partial_x + \mathrm{i}\,\mu) & |D+\mu| \tanh(\mathtt{h}|D+\mu|) \\ -1 & \mathtt{c_h}(\partial_x + \mathrm{i}\,\mu) \end{bmatrix} \tag{2.9}$$

on $L^2(\mathbb{T}, \mathbb{C}^2)$ is given by the purely imaginary numbers

$$\lambda_j^\sigma(\mu,\mathtt{h}) := \mathrm{i}\,\omega^\sigma(j+\mu,\mathtt{h}) = \mathtt{c_h}(j+\mu) - \sigma\Omega(j+\mu,\mathtt{h})\,,\ \forall \sigma = \pm\,,\ j \in \mathbb{Z}\,,$$

where

$$\omega^\sigma(\varphi,\mathtt{h}) := \mathtt{c_h}\varphi - \sigma\Omega(\varphi,\mathtt{h})\,, \qquad \Omega(\varphi,\mathtt{h}) := \sqrt{\varphi \tanh(\mathtt{h}\varphi)}\,, \qquad \varphi \in \mathbb{R}\,, \quad \sigma = \pm\,.$$

Note that $\omega^+(-\varphi,\mathtt{h}) = -\omega^-(\varphi,\mathtt{h})$. Sometimes σ is called the Krein signature of the eigenvalue $\lambda_j^\sigma(\mu,\mathtt{h})$.

Restricted to $\varphi>0$, the function $\Omega(\cdot,\mathtt{h}):(0,+\infty)\to(0,+\infty)$ is, for any $\mathtt{h}>0$, increasing and concave namely $(\partial_\varphi\Omega)(\varphi,\mathtt{h})>0$ and $(\partial_{\varphi\varphi}\Omega)(\varphi,\mathtt{h})<0$.

For any $j\in\mathbb{Z}$, $\mu\in\mathbb{R}$ such that $j+\mu\neq 0$ we have $\Omega(j+\mu,\mathtt{h})\neq 0$ and we associate to the eigenvalue $\lambda_j^\sigma(\mu,\mathtt{h})$ the eigenvector

$$f_j^\sigma := f_j^\sigma(\mu,\mathtt{h}) := \frac{1}{\sqrt{2\Omega(j+\mu,\mathtt{h})}}\begin{bmatrix} -\sqrt{\sigma}\,\Omega(j+\mu,\mathtt{h}) \\ \sqrt{-\sigma}\end{bmatrix} e^{\mathrm{i}\,jx}\,, \quad \mathcal{L}_{\mu,0} f_j^\sigma(\mu,\mathtt{h}) = \lambda_j^\sigma(\mu,\mathtt{h}) f_j^\sigma(\mu,\mathtt{h})\,, \tag{2.10}$$

which satisfy, recalling (2.7), the reversibility property

$$\overline{\rho} f_j^+(\mu,\mathtt{h}) = f_j^+(\mu,\mathtt{h})\,, \quad \overline{\rho} f_j^-(\mu,\mathtt{h}) = -f_j^-(\mu,\mathtt{h})\,.$$

For any $\mu\notin\mathbb{Z}$ we have $\Omega(j+\mu,\mathtt{h})\neq 0$ for any $j\in\mathbb{Z}$ and the family of eigenvectors $\{f_j^\sigma(\mu)\}_{j\in\mathbb{Z}}$ in (2.10) forms a *complex symplectic basis* of $L^2(\mathbb{T},\mathbb{C}^2)$, with respect to the complex symplectic form $\mathcal{W}_c$ in (2.6), namely its elements are linearly independent, span densely $L^2(\mathbb{T},\mathbb{C}^2)$ and satisfy, for any pairs of integers $j,j'\in\mathbb{Z}$, and any pair of signs $\sigma,\sigma'\in\{\pm\}$,

$$\mathcal{W}_c\big(f_j^\sigma, f_{j'}^{\sigma'}\big) = \begin{cases} -\mathrm{i} & \text{if } j=j' \text{ and } \sigma=\sigma'=+\,, \\ \ \ \mathrm{i} & \text{if } j=j' \text{ and } \sigma=\sigma'=-\,, \\ \ \ 0 & \text{otherwise}\,. \end{cases} \tag{2.11}$$

Multiple eigenvalues of $\mathcal{L}_{\mu,0}(\mathtt{h})$. In order to describe the multiple nonzero eigenvalues of $\mathcal{L}_{\mu,0}$ we need the following result.

Lemma 2.4 (**Spectral collisions**). *For any depth $\mathtt{h}>0$ and any integer $\mathtt{p}\geq 2$, the following holds.*

i) For any $\sigma\in\{\pm\}$ the equations $\omega^\sigma\big(\varphi,\mathtt{h}\big)=\omega^\sigma\big(\varphi+\mathtt{p},\mathtt{h}\big)$ have no solutions, see Figure 2.

ii) The equation $\omega^-\big(\varphi,\mathtt{h}\big)=\omega^+\big(\varphi+\mathtt{p},\mathtt{h}\big)$ has a unique positive solution $\underline{\varphi}(\mathtt{p},\mathtt{h})>0$,

$$\omega^-\big(\underline{\varphi}(\mathtt{p},\mathtt{h}),\mathtt{h}\big) = \omega^+\big(\underline{\varphi}(\mathtt{p},\mathtt{h})+\mathtt{p},\mathtt{h}\big) = \mathtt{c_h}\underline{\varphi}(\mathtt{p},\mathtt{h}) + \Omega(\underline{\varphi}(\mathtt{p},\mathtt{h})) =: \omega_*^{(\mathtt{p})}(\mathtt{h}) > 0\,, \tag{2.12}$$

see Figure 3, and the negative solution $-\underline{\varphi}(\mathtt{p},\mathtt{h})-\mathtt{p}$,

$$\omega^-\big(-\underline{\varphi}(\mathtt{p},\mathtt{h})-\mathtt{p},\mathtt{h}\big) = \omega^+\big(-\underline{\varphi}(\mathtt{p},\mathtt{h}),\mathtt{h}\big) = -\omega_*^{(\mathtt{p})}(\mathtt{h}) < 0\,,$$

which is unique for any $\mathtt{p}\geq 3$. For $\mathtt{p}=2$ the equation $\omega^-\big(\varphi,\mathtt{h}\big)=\omega^+\big(\varphi+2,\mathtt{h}\big)$ possesses also the negative solution $\varphi=-1$ and $\omega^-\big(-1,\mathtt{h}\big)=\omega^+\big(1,\mathtt{h}\big)=0$.

iii) The equation $\omega^+(\varphi,\mathtt{h})=\omega^-(\varphi+\mathtt{p},\mathtt{h})$ has no solutions.

The positive solution $\underline{\varphi}(\mathtt{p},\mathtt{h})$ of (2.12) satisfies the following properties: the function $\underline{\varphi}(\mathtt{p},\cdot):(0,+\infty)\to(0,+\infty)$, $\mathtt{h}\mapsto\underline{\varphi}(\mathtt{p},\mathtt{h})$, is analytic. For any $\mathtt{h}>0$ the sequence $\mathtt{p}\mapsto\underline{\varphi}(\mathtt{p},\mathtt{h})$ is increasing (see Figure 4) and $\lim_{\mathtt{p}\to+\infty}\underline{\varphi}(\mathtt{p},\mathtt{h})=+\infty$. Furthermore $\lim_{\mathtt{h}\to0^+}\underline{\varphi}(\mathtt{p},\mathtt{h})=0$ and $\lim_{\mathtt{h}\to+\infty}\underline{\varphi}(\mathtt{p},\mathtt{h})=\frac14(\mathtt{p}-1)^2$.

Remark 2.5. For $\mathtt{p}=0,1$, the equations $\omega^\sigma\big(\varphi,\mathtt{h}\big)=\omega^{\sigma'}\big(\varphi+\mathtt{p},\mathtt{h}\big)$ admit other solutions *all* at the common value 0.

As a corollary we describe *all* the multiple *nonzero* eigenvalues of $\mathcal{L}_{\mu,0}$.

Lemma 2.6 (**Multiple eigenvalues of $\mathcal{L}_{\mu,0}$ away from 0**). *For any depth $\mathtt{h}>0$ and any $\mu\in\mathbb{R}$, the spectrum of $\mathcal{L}_{\mu,0}$ away from zero contains only simple or double eigenvalues:*

1. *for any $\mu \in \mathbb{R}$ and any integer $\mathtt{p} \geq 2$ such that $\underline{\varphi}(\mathtt{p},\mathtt{h}) - \mu \notin \mathbb{Z}$, all the eigenvalues of the operator $\mathcal{L}_{\mu,0}$ in* (2.9) *are simple;*

2. *For any $\mu \in \mathbb{R}$ and any integer $\mathtt{p} \geq 2$ such that $\underline{\varphi}(\mathtt{p},\mathtt{h}) - \mu \in \mathbb{Z}$, the operator $\mathcal{L}_{\mu,0}$ has the non-zero double eigenvalue*
$$\mathrm{i}\,\omega_*^{(\mathtt{p})}(\mathtt{h}) = \mathrm{i}\,\omega^-\big(\underline{\varphi}(\mathtt{p},\mathtt{h}),\mathtt{h}\big) = \mathrm{i}\,\omega^+\big(\underline{\varphi}(\mathtt{p},\mathtt{h}) + \mathtt{p},\mathtt{h}\big)\,, \tag{2.13}$$
and the operator $\mathcal{L}_{-\mu,0}$ has the complex conjugated non-zero double eigenvalue $-\mathrm{i}\,\omega_*^{(\mathtt{p})}(\mathtt{h})$. The sequence $\mathtt{p} \mapsto \omega_*^{(\mathtt{p})}(\mathtt{h})$ is increasing $0 < \omega_*^{(2)}(\mathtt{h}) < \ldots < \omega_*^{(\mathtt{p})}(\mathtt{h}) < \ldots$ and $\lim_{\mathtt{p}\to+\infty} \omega_*^{(\mathtt{p})}(\mathtt{h}) = +\infty$. The eigenspace of $\mathcal{L}_{\mu,0}$ associated with the double eigenvalue $\mathrm{i}\,\omega_*^{(\mathtt{p})}(\mathtt{h})$ in (2.13) *is spanned by the two eigenvectors*
$$\begin{aligned} f_k^-(\mu,\mathtt{h}) &= \frac{1}{\sqrt{2\Omega(\underline{\varphi}(\mathtt{p},\mathtt{h}),\mathtt{h})}} \begin{bmatrix} -\mathrm{i}\ \Omega(\underline{\varphi}(\mathtt{p},\mathtt{h}),\mathtt{h}) \\ 1 \end{bmatrix} e^{\mathrm{i}\,kx}\,, \\ f_{k'}^+(\mu,\mathtt{h}) &= \frac{1}{\sqrt{2\Omega(\underline{\varphi}(\mathtt{p},\mathtt{h}) + \mathtt{p},\mathtt{h})}} \begin{bmatrix} -\Omega(\underline{\varphi}(\mathtt{p},\mathtt{h}) + \mathtt{p},\mathtt{h}) \\ \mathrm{i} \end{bmatrix} e^{\mathrm{i}\,k'x}\,, \end{aligned} \tag{2.14}$$
where $k := \underline{\varphi}(\mathtt{p},\mathtt{h}) - \mu \in \mathbb{Z}$ and $k' := k + \mathtt{p}$.

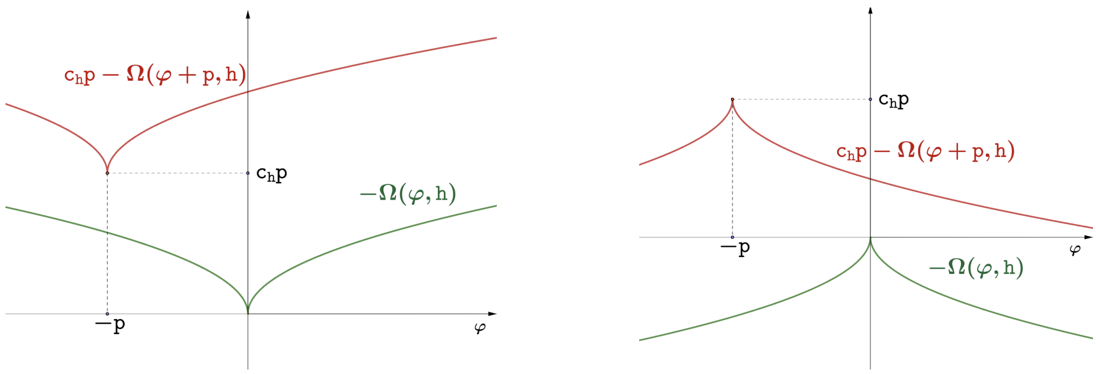

Figure 2: Absence of collisions for eigenvalues with same Krein signature for any $\mathtt{p} \geq 2$.

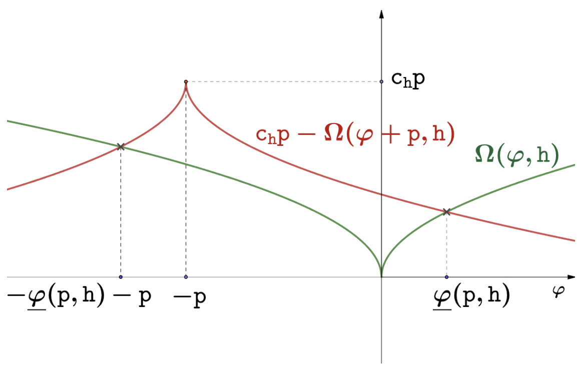

Figure 3: Collision of eigenvalues with opposite Krein signature for $\mathtt{p} \geq 2$. For $\mathtt{p} = 2$ actually there is another solution associated with the eigenvalue $\omega^-(-1,\mathtt{h}) = \omega^+(1,\mathtt{h}) = 0$.

The values $\pm\mathrm{i}\,\omega_*^{(\mathtt{p})}(\mathtt{h})$ are the branching points of the $\mathtt{p}$-th isolas stated in Theorem 1.1.
In order to state the next result we introduce a notation.

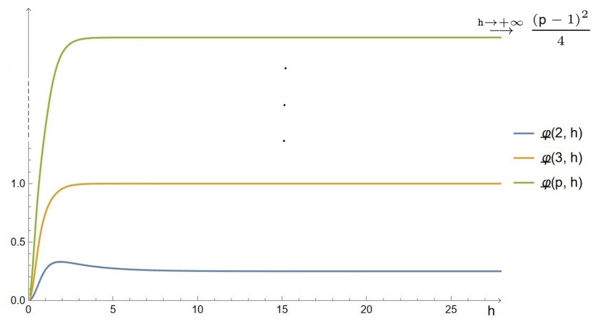

Figure 4: Plot of the functions $\underline{\varphi}(\mathtt{p},\mathtt{h})$ as $\mathtt{p}\geq 2$ varies.

- **Notation**. We denote by $r(\nu^{m_1}\varepsilon^{n_1},\ldots,\nu^{m_q}\varepsilon^{n_q})$ a real analytic scalar function that satisfies, for some $C>0$ and (ν,ε) small, the bound $|r(\nu^{m_1}\varepsilon^{n_1},\ldots,\nu^{m_q}\varepsilon^{n_q})|\leq C\sum_{j=1}^{q}|\nu|^{m_j}|\varepsilon|^{n_j}$. For any $\delta>0$, we denote by $B_\delta(x)$ the real interval $(x-\delta,x+\delta)$ centered at x.

The next result carefully describes the spectrum near $\mathrm{i}\,\omega_*^{(\mathtt{p})}(\mathtt{h})$ of the operator $\mathcal{L}_{\mu,\varepsilon}(\mathtt{h})$, for any (μ,ε) sufficiently close to $(\underline{\mu},0)$, value at which $\mathcal{L}_{\underline{\mu},0}(\mathtt{h})$ has a double eigenvalue.

Theorem 2.7. ***(Unstable spectrum)*** *[11] For any integer $\mathtt{p}\in\mathbb{N}$, $\mathtt{p}\geq 2$, for any $\mathtt{h}>0$ let $\underline{\mu}=\underline{\varphi}(\mathtt{p},\mathtt{h})>0$ (given by Lemma 2.4) such that the operator $\mathcal{L}_{\underline{\mu},0}(\mathtt{h})$ in (2.9) has a double eigenvalue $\mathrm{i}\,\omega_*^{(\mathtt{p})}(\mathtt{h})$. Then there exist $\varepsilon_1^{(\mathtt{p})},\delta_0^{(\mathtt{p})}>0$ and real analytic functions $\mu_0^{(\mathtt{p})},\mu_\vee^{(\mathtt{p})},\mu_\wedge^{(\mathtt{p})}\colon[0,\varepsilon_1^{(\mathtt{p})})\to B_{\delta_0^{(\mathtt{p})}}(\underline{\mu})$ satisfying*

$$\mu_0^{(\mathtt{p})}(0)=\mu_\wedge^{(\mathtt{p})}(0)=\mu_\vee^{(\mathtt{p})}(0)=\underline{\mu}$$

and, for any $0<\varepsilon<\varepsilon_1^{(\mathtt{p})}$,

$$\mu_\wedge^{(\mathtt{p})}(\varepsilon)<\mu_0^{(\mathtt{p})}(\varepsilon)<\mu_\vee^{(\mathtt{p})}(\varepsilon)\,,\quad |\mu_{\wedge,\vee}^{(\mathtt{p})}(\varepsilon)-\mu_0^{(\mathtt{p})}(\varepsilon)|=\frac{2|\beta_1^{(\mathtt{p})}(\mathtt{h})|}{T_1^{(\mathtt{p})}(\mathtt{h})}\varepsilon^{\mathtt{p}}+r(\varepsilon^{\mathtt{p}+1})\,, \tag{2.15}$$

(see Figure 5) where $T_1^{(\mathtt{p})}(\mathtt{h})>0$ and $\beta_1^{(\mathtt{p})}(\mathtt{h})\not\equiv 0$ are real analytic functions defined for any $\mathtt{h}>0$, such that the following holds true:

for any $(\mu,\varepsilon)\in B_{\delta_0^{(\mathtt{p})}}(\underline{\mu})\times B_{\varepsilon_1^{(\mathtt{p})}}(0)$ the operator $\mathcal{L}_{\mu,\varepsilon}(\mathtt{h})$ possesses a pair of eigenvalues of the form

$$\lambda_{\mathtt{p}}^{\pm}(\mu,\varepsilon)=\begin{cases}\mathrm{i}\,\omega_*^{(\mathtt{p})}+\mathrm{i}\,s^{(\mathtt{p})}(\mu,\varepsilon)\pm\frac{1}{2}\sqrt{D^{(\mathtt{p})}(\mu,\varepsilon)} & \text{if } \mu\in\big(\mu_\wedge^{(\mathtt{p})}(\varepsilon),\mu_\vee^{(\mathtt{p})}(\varepsilon)\big)\,,\\ \mathrm{i}\,\omega_*^{(\mathtt{p})}+\mathrm{i}\,s^{(\mathtt{p})}(\mu,\varepsilon)\pm\mathrm{i}\sqrt{|D^{(\mathtt{p})}(\mu,\varepsilon)|} & \text{if } \mu\notin\big(\mu_\wedge^{(\mathtt{p})}(\varepsilon),\mu_\vee^{(\mathtt{p})}(\varepsilon)\big)\,,\end{cases} \tag{2.16}$$

where $s^{(\mathtt{p})}(\mu,\varepsilon)$, $D^{(\mathtt{p})}(\mu,\varepsilon)$ are real-analytic functions of the form

$$\begin{aligned}&s^{(\mathtt{p})}(\mu_0^{(\mathtt{p})}(\varepsilon)+\nu,\varepsilon)=r(\varepsilon^2,\nu)\,,\\&D^{(\mathtt{p})}(\mu_0^{(\mathtt{p})}(\varepsilon)+\nu,\varepsilon)=4(\beta_1^{(\mathtt{p})}(\mathtt{h}))^2\varepsilon^{2\mathtt{p}}-(T_1^{(\mathtt{p})}(\mathtt{h}))^2\nu^2+r(\varepsilon^{2\mathtt{p}+1},\nu\varepsilon^{2\mathtt{p}},\nu^2\varepsilon,\nu^3)\,.\end{aligned} \tag{2.17}$$

For any $\mathtt{h}>0$ such that $\beta_1^{(\mathtt{p})}(\mathtt{h})\neq 0$ the function $D^{(\mathtt{p})}(\mu_0^{(\mathtt{p})}(\varepsilon)+\nu,\varepsilon)$ is positive in $\big(\mu_\wedge^{(\mathtt{p})}(\varepsilon),\mu_\vee^{(\mathtt{p})}(\varepsilon)\big)$, vanishes at $\mu=\mu_{\wedge,\vee}^{(\mathtt{p})}(\varepsilon)$ and turns negative outside $\big(\mu_\wedge^{(\mathtt{p})}(\varepsilon),\mu_\vee^{(\mathtt{p})}(\varepsilon)\big)$. For any fixed $\varepsilon\in(0,\varepsilon_1^{(\mathtt{p})})$ as μ varies in $\big(\mu_\wedge^{(\mathtt{p})}(\varepsilon),\mu_\vee^{(\mathtt{p})}(\varepsilon)\big)$ the pair of unstable eigenvalues $\lambda_{\mathtt{p}}^{\pm}(\mu,\varepsilon)$ in (2.16) describes a closed analytic curve in *the complex plane $\lambda = x + \mathrm{i}\, y$ that intersects orthogonally the imaginary axis, encircles a convex region, and it is symmetric with respect to y-axis. Such isola is approximated by an ellipse*

$$x^2 + E^{(\mathtt{p})}(\mathtt{h})^2(1 + r(\varepsilon^2))(y - y_0^{(\mathtt{p})}(\varepsilon))^2 = \beta_1^{(\mathtt{p})}(\mathtt{h})^2\varepsilon^{2\mathtt{p}}(1 + r(\varepsilon))$$

where $E^{(\mathtt{p})}(\mathtt{h}) \in (0, 1)$ is a real analytic function for any $\mathtt{h} > 0$ and $y_0^{(\mathtt{p})}(\varepsilon)$ is $O(\varepsilon^2)$-close to $\omega_*^{(\mathtt{p})}(\mathtt{h})$.

Let us make some comments.

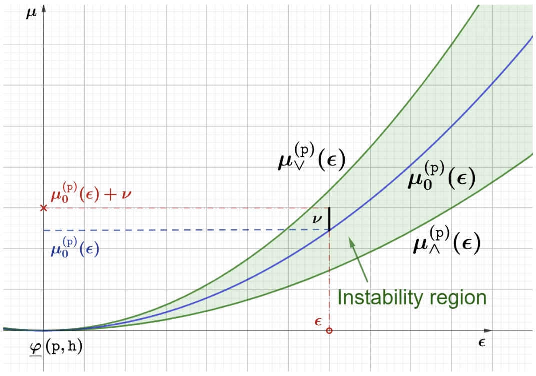

Figure 5: The instability region around the curve $\mu_0^{(\mathtt{p})}(\varepsilon)$ delimited by the curves $\mu_\wedge^{(\mathtt{p})}(\varepsilon)$ and $\mu_\vee^{(\mathtt{p})}(\varepsilon)$.

1. **Upper bounds:** In view of (2.17) and (2.15), for any $\mathtt{p} \geq 2$ and any depth $\mathtt{h} > 0$, the real part of the eigenvalues (2.16) is at most of size $O(\varepsilon^{\mathtt{p}})$ implying that the isolas, if ever exist, shrink exponentially fast as $\mathtt{p} \to +\infty$.

2. **Lower bounds:** For any $\mathtt{p} \geq 2$ and any depth $\mathtt{h} > 0$, a sufficient condition for the existence of the $\mathtt{p}$-th instability isola is that $\beta_1^{(\mathtt{p})}(\mathtt{h}) \neq 0$. In such a case the real part of the eigenvalues (2.16) is of size $|\beta_1^{(\mathtt{p})}(\mathtt{h})|\varepsilon^{\mathtt{p}} + O(\varepsilon^{\mathtt{p}+1})$. By (2.15), the portion of the unstable band is parametrized by Floquet exponents in the interval $\big(\mu_\wedge^{(\mathtt{p})}(\varepsilon), \mu_\vee^{(\mathtt{p})}(\varepsilon)\big)$ which has exponentially small width $\sim \frac{4|\beta_1^{(\mathtt{p})}(\mathtt{h})|}{T_1^{(\mathtt{p})}}\varepsilon^{\mathtt{p}}$.

3. **The function $\beta_1^{(\mathtt{p})}(\mathtt{h})$:** In [11] we analytically *prove* that, for *any* $\mathtt{p} \geq 2$, the map $\mathtt{h} \mapsto \beta_1^{(\mathtt{p})}(\mathtt{h})$ is *real analytic* and

$$\lim_{\mathtt{h}\to 0^+} \beta_1^{(\mathtt{p})}(\mathtt{h}) = -\infty \qquad \text{and} \qquad \lim_{\mathtt{h}\to +\infty} \beta_1^{(\mathtt{p})}(\mathtt{h}) = 0\,, \tag{2.18}$$

implying, in particular, that $\beta_1^{(\mathtt{p})}(\mathtt{h})$ is not identically zero. For $\mathtt{p} = 2, 3, 4$ the graphs of $\beta_1^{(\mathtt{p})}(\mathtt{h})$ are numerically plotted in Figures 6-8.

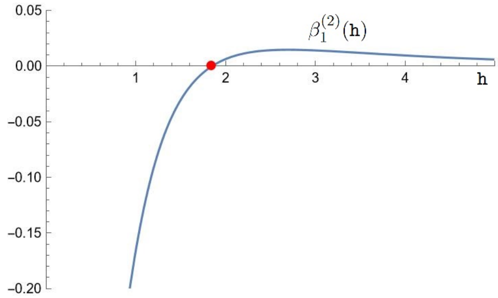

Figure 6: The function $\beta_1^{(2)}(\mathtt{h})$ vanishes only at the red point $\mathtt{h}_* \approx 1.84940$ coherently with [16, 21].

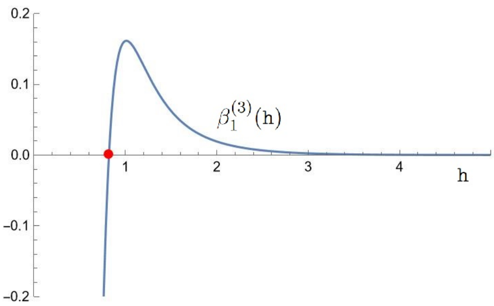

Figure 7: The function $\beta_1^{(3)}(\mathtt{h})$ vanishes only at the red point $\mathtt{h}_* \approx 0.82064$ coherently with [16].

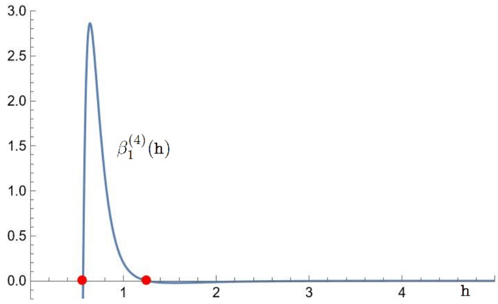

Figure 8: The function $\beta_1^{(4)}(\mathtt{h})$ vanishes only at the red points $\mathtt{h}_* \approx 0.566633$ and $\mathtt{h}_{**} \approx 1.255969$.

4. **The set of critical depths $\mathcal{S}^{(\mathtt{p})}$:** The above properties imply that each $\beta_1^{(\mathtt{p})}(\mathtt{h})$ vanishes only at a closed set of isolated points. For any depth $\mathtt{h}$ outside $(\beta_1^{(\ell)})^{-1}(0)$, where $\ell = 2, \ldots, \mathtt{p}$, the ℓ-th isola of instability exists and has elliptical shape. The set $\mathcal{S}^{(\mathtt{p})}$ in Theorem 1.1 is the union

$$\mathcal{S}^{(\mathtt{p})} := \bigcup_{\ell=2}^{\mathtt{p}} (\beta_1^{(\ell)})^{-1}(0)\,.$$

When $\beta_1^{(\mathtt{p})}(\mathtt{h}) = 0$ it is necessary to further expand the discriminant $D^{(\mathtt{p})}(\mu, \varepsilon)$ in (2.17) to determine its sign. The latter degeneracy occurs at $\mathtt{h} = +\infty$ for *every* $\mathtt{p} \geq 2$ since $\lim_{\mathtt{h}\to+\infty} \beta_1^{(\mathtt{p})}(\mathtt{h}) = 0$. Such degeneration ultimately descends from the structural property (3.4) of the Stokes waves. For the isola $\mathtt{p} = 2$ the higher order expansion to detect the unstable isola if $\mathtt{h} = +\infty$ is performed in [10].

3 Ideas of proof

We now comment on some key steps of the proof and of its main difficulties. In view of Lemma 2.6, **given an integer $\mathtt{p} \geq 2$, we fix**

$$\underline{\mu} := \underline{\varphi}(\mathtt{p}, \mathtt{h})\,, \quad k := 0\,, \quad k' := \mathtt{p}\,, \quad \omega_*^{(\mathtt{p})}(\mathtt{h}) := \omega^-(\underline{\varphi}(\mathtt{p}, \mathtt{h}), \mathtt{h}) = \omega^+(\mathtt{p} + \underline{\varphi}(\mathtt{p}, \mathtt{h}), \mathtt{h})\,.$$

The overall goal is to compute the eigenvalues of the operator $\mathcal{L}_{\mu,\varepsilon}$ in (2.5) that branch off from the $\mathtt{p}$-th double eigenvalue $\mathrm{i}\,\omega_*^{(\mathtt{p})}(\mathtt{h})$ of $\mathcal{L}_{\underline{\mu},0}(\mathtt{h})$ for (μ, ε) close to $(\underline{\mu}, 0)$.

We restrict to values $\mathtt{h} > 0$ such that $\underline{\mu} = \underline{\varphi}(\mathtt{p}, \mathtt{h}) \notin \mathbb{Z}$ so that the eigenvectors $\{f_j^\sigma(\underline{\mu})\}_{j\in\mathbb{Z}}$ in (2.10) form a complex symplectic basis of $L^2(\mathbb{T}, \mathbb{C}^2)$. Since the function $\underline{\varphi}(\mathtt{p}, \mathtt{h})$ is bounded and real-analytic, these depths form a dense set in $(0, +\infty)$. The spectrum of $\mathcal{L}_{\underline{\mu},0}$ decomposes into two disjoint parts

$$\sigma(\mathcal{L}_{\underline{\mu},0}) = \sigma'_{\mathtt{p}}(\mathcal{L}_{\underline{\mu},0}) \cup \sigma''_{\mathtt{p}}(\mathcal{L}_{\underline{\mu},0}) \qquad \text{where} \qquad \sigma'_{\mathtt{p}}(\mathcal{L}_{\underline{\mu},0}) := \left\{\mathrm{i}\,\omega_*^{(\mathtt{p})}(\mathtt{h})\right\}.$$

For any (μ,ε) sufficiently close to $(\underline{\mu},0)$ the perturbed spectrum $\sigma(\mathcal{L}_{\mu,\varepsilon})$ still admits a disjoint decomposition

$$\sigma(\mathcal{L}_{\mu,\varepsilon}) = \sigma'_{\mathtt{p}}(\mathcal{L}_{\mu,\varepsilon}) \cup \sigma''_{\mathtt{p}}(\mathcal{L}_{\mu,\varepsilon}) \, ,$$

where $\sigma'_{\mathtt{p}}(\mathcal{L}_{\mu,\varepsilon})$ consists of 2 eigenvalues close to the double eigenvalue $\mathrm{i}\,\omega_*^{(\mathtt{p})}(\mathtt{h})$ of $\mathcal{L}_{\underline{\mu},0}$. Actually, following the approach in [6, 8, 9, 10] the spectral problem is reduced to determine the eigenvalues of a 2×2 symplectic and reversible matrix, i.e. of the form

$$\begin{pmatrix} -\mathrm{i}\,\alpha^{(\mathtt{p})}(\mu,\varepsilon) & \beta^{(\mathtt{p})}(\mu,\varepsilon) \\ \beta^{(\mathtt{p})}(\mu,\varepsilon) & \mathrm{i}\,\gamma^{(\mathtt{p})}(\mu,\varepsilon) \end{pmatrix} \tag{3.1}$$

where $\alpha^{(\mathtt{p})}(\mu,\varepsilon)$, $\gamma^{(\mathtt{p})}(\mu,\varepsilon)$ and $\beta^{(\mathtt{p})}(\mu,\varepsilon)$ are real analytic functions depending on the Stokes wave. In particular the function $\beta^{(\mathtt{p})}(\mu,\varepsilon)$ is given by the scalar product

$$\beta^{(\mathtt{p})}(\mu,\varepsilon) = \frac{1}{\mathrm{i}}(\mathfrak{B}(\mu,\varepsilon)f_0^-, f_{\mathtt{p}}^+) \tag{3.2}$$

where $\mathfrak{B}(\mu,\varepsilon)$ is a linear operator obtained by the Kato similarity approach, see [11][Lemma 4.2], and $f_0^-, f_{\mathtt{p}}^+$ are the eigenvectors of $\mathcal{L}_{\underline{\mu},0}$ associated to the double eigenvalue $\mathrm{i}\,\omega_*^{(\mathtt{p})}(\mathtt{h})$ (see (2.14) with $k=0$ and $k'=\mathtt{p}$).

Clearly, for the matrix (3.1) to possess eigenvalues with non zero real part it is necessary that the off-diagonal entry $\beta^{(\mathtt{p})}(\mu,\varepsilon)$ *is not zero.* Such a function turns out to have a Taylor expansion at $(\underline{\mu},0)$ as, see [11][Theorem 5.3],

$$\beta^{(\mathtt{p})}(\underline{\mu}+\delta,\varepsilon) = \beta_1^{(\mathtt{p})}(\mathtt{h})\varepsilon^{\mathtt{p}} + r(\varepsilon^{\mathtt{p}+1}, \delta\varepsilon^{\mathtt{p}}, \ldots, \delta^{\mathtt{p}}\varepsilon) \, . \tag{3.3}$$

The coefficient $\beta_1^{(\mathtt{p})}(\mathtt{h})$ is an analytic function of the depth $\mathtt{h}>0$ which depends, via a very complicated expression, on the Taylor-Fourier coefficients of the Stokes wave up to order $\mathtt{p}$, see [11][formulas (5.7)-(5.8)]. The exact computation of $\beta_1^{(\mathtt{p})}(\mathtt{h})$ is a key step of [11]. It turns out to be expressed in terms of *only* the leading Taylor-Fourier coefficients $a_\ell^{[\ell]}$, $p_\ell^{[\ell]}$, $\ell=1,\ldots,\mathtt{p}$ of the functions $a_\varepsilon(x)$, $p_\varepsilon(x)$ in (2.5), where

$$a_\varepsilon(x) = \sum_{\ell\geq 1}\varepsilon^\ell a_\ell(x) \, , \quad a_\ell^{[\ell]} := \frac{1}{\pi}\int_{\mathbb{T}} a_\ell(x)\cos(\ell x)\mathrm{d}x \, ,$$

and similarly for $p_\ell^{[\ell]}$. This is ultimately due to the property (2.3) of the Taylor coefficients $\eta_\ell(x)$, $\psi_\ell(x)$ of the Stokes wave $\eta_\varepsilon(x), \psi_\varepsilon(x)$, that, as far as we know, has never been exploited before. Similar properties are inherited by the operator $\mathcal{L}_{\mu,\varepsilon}$ in (2.5) as well as the operator $\mathfrak{B}(\mu,\varepsilon)$ in (3.2). Consequences of this property are that the scalar product (3.2) starts with an $\varepsilon^{\mathtt{p}}$-term, as stated in (3.3) (recall that f_0^- has only the 0-harmonic and $f_{\mathtt{p}}^+$ has only the $\mathtt{p}$-harmonic, see (2.14)) and that $\beta_1^{(\mathtt{p})}(\mathtt{h})$ depends on *only* the leading Taylor-Fourier coefficients $a_\ell^{[\ell]}, p_\ell^{[\ell]}$ for $\ell=1,\ldots,\mathtt{p}$.

As discussed above, a *sufficient* condition for the matrix (3.1) to possess eigenvalues with non zero real part is that $\beta_1^{(\mathtt{p})}(\mathtt{h})\neq 0$. To analitically verify this property for a given depth $\mathtt{h}>0$ looks very complicated because the function $\beta_1^{(\mathtt{p})}(\mathtt{h})$ is the sum of $3^{\mathtt{p}-1}$ intricate addends, possibly canceling one another, that depend on the Taylor-Fourier coefficients of the Stokes wave and on the function $\underline{\varphi}(\mathtt{p},\mathtt{h})$. Furthermore the latter is defined only implicitly in (2.12).

Thus we focus in proving that $\beta_1^{(\mathtt{p})}(\mathtt{h})$ is a non zero analytic function. For *any* $\mathtt{p}\geq 2$, the limit of the function $\beta_1^{(\mathtt{p})}(\mathtt{h})$ as $\mathtt{h}\to+\infty$ is equal to zero. This degeneracy is ultimately due to cancellations of the Stokes wave in deep water, in particular of the identity, see [11][Proposition 3.16],

$$\lim_{\mathtt{h}\to+\infty}\eta_\ell^{[\ell]}(\mathtt{h}) = \lim_{\mathtt{h}\to+\infty}\psi_\ell^{[\ell]}(\mathtt{h}) \, , \quad \forall \ell\in\mathbb{N} \, . \tag{3.4}$$

Thus the limit as $\mathtt{h} \to +\infty$ does not allow to conclude that $\beta_1^{(\mathtt{p})}(\mathtt{h})$ is nonzero. Then we prove that

$$\lim_{\mathtt{h}\to 0^+} \beta_1^{(\mathtt{p})}(\mathtt{h}) = -\infty\,.$$

This is also a challenging task. We exhibit the *second order* asymptotic expansion of $\beta_1^{(\mathtt{p})}(\mathtt{h})$ as $\mathtt{h} \to 0^+$, showing that

$$\beta_1^{(\mathtt{p})}(\mathtt{h}) = \underbrace{\widetilde{A}^{(\mathtt{p})}}_{=0}\mathtt{h}^{\frac{7}{2}-3\mathtt{p}} + \underbrace{\widetilde{C}^{(\mathtt{p})}}_{<0}\mathtt{h}^{\frac{11}{2}-3\mathtt{p}} + O(\mathtt{h}^{\frac{15}{2}-3\mathtt{p}})\,.$$

The first order term of such expansion *vanishes* and we need to explicitly compute the Laurent expansion one order further. This ultimately requires the knowledge of the asymptotic expansion, as $\mathtt{h} \to 0^+$, of the leading Fourier coefficients of the Stokes waves at any order ℓ. We are able to prove in [11][Proposition 3.14] that, for any $\ell \in \mathbb{N}$, as $\mathtt{h} \to 0^+$,

$$\eta_\ell^{[\ell]} = \ell\, x_\ell \mathtt{h}^{3-3\ell} + z_\ell \mathtt{h}^{5-3\ell} + O(\mathtt{h}^{7-3\ell})\,, \quad \psi_\ell^{[\ell]} = x_\ell \mathtt{h}^{3-3\ell-\frac12} + y_\ell \mathtt{h}^{5-3\ell-\frac12} + O(\mathtt{h}^{7-3\ell-\frac12})\,,$$

where

$$x_\ell := \Big(\frac{3}{8}\Big)^{\ell-1}, \quad y_\ell := \frac{5\ell^2+3\ell-5}{18}\Big(\frac{3}{8}\Big)^{\ell-1}, \quad z_\ell := \frac{\ell(\ell-1)(\ell+2)}{6}\Big(\frac{3}{8}\Big)^{\ell-1}.$$

The resulting leading Laurent coefficient $\widetilde{C}^{(\mathtt{p})}$ is expressed by a complicated sum involving rational functions of the summation indices. To our surprise, as guessed by Mathematica, this term equals $\frac{\mathtt{p}(\mathtt{p}+1)^2}{3}$ for any value of $\mathtt{p}$. This is proved rigorously by Zeilberger-van Hoeij-Koutschan in [31] using methods of computer algebra, starting from our conjectured value. The final outcome is that, see [11][Theorem 6.1],

$$\beta_1^{(\mathtt{p})}(\mathtt{h}) = -\sqrt{\frac{\mathtt{p}^2-1}{3}}\Big(\frac{3}{8}\Big)^{\mathtt{p}-1}\frac{\mathtt{p}^2(\mathtt{p}+1)^2}{24}\mathtt{h}^{\frac{11}{2}-3\mathtt{p}} + O\big(\mathtt{h}^{\frac{15}{2}-3\mathtt{p}}\big) \qquad \text{as } \mathtt{h}\to 0^+\,.$$

This allows to prove Theorems 1.1 and 2.7.

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