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ON THICK SPRAY EQUATIONS

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On thick spray equations

Lucas Ertzbischoff

ABSTRACT. This note is based on a presentation done at the *Séminaire Laurent Schwartz* in November 2023. The goal is to review our recent result [**20**], obtained in collaboration with Daniel Han-Kwan.

We consider the equations of thick sprays, modeling the evolution of a gas-particles mixture through a Vlasov-Boltzmann equation for the particles and compressible Navier-Stokes equations for the gas. In the so-called *thick spray* regime, the volume occupied by the cloud of particles is not negligible in front of that of the gas.

Unlike other fluid-kinetic models, the mathematical analysis of such dense sprays is still in its infancy and raises several challenges, the Cauchy theory being one of the first. Inspired by recent works on singular Vlasov equations, we show that the thick spray equations are locally well-posed in Sobolev regularity, provided that the initial data satisfies a suitable Penrose stability condition.

1. Introduction

The study of sprays holds significant importance, due to its diverse applications in industrial process and physical or environmental sciences. Here, the broad term of *spray* (also called particle-laiden flows) refers to a general mixture between a phase made of small particles (e.g. droplets or dust specks) and a phase made of a fluid (e.g. a carrying gas). Among others, let us mention that sprays play a major role in combustion process, medical aerosols, sedimentation phenomena or the description of particles in the atmosphere (see [**30, 19**] for more references and examples).

The complexity and the interest of these systems arises from the fluid flow itself, from the coupling between the dispersed phase and from the fluid where particles evolve, and the interaction between the particles themselves. The study of such models has resulted in a substantial body of literature from a physical perspective [**34, 17, 25, 47**]. Additionally, it poses significant mathematical challenges, as spray systems often entail intricate analytical questions.

In the realm of multiphase flows, there exist different points of view to model such mixtures [**15**]. Here, we consider the following **fluid-kinetic approach**:

- the surrounding gas is described by macroscopic hydrodynamic quantities, such as its density $\varrho(t,x) \in \mathbb{R}^+$, velocity $u(t,x) \in \mathbb{R}^d$ or internal energy $\mathfrak{e}(t,x) \in \mathbb{R}^+$, solutions to Euler or Navier-Stokes equations;
- the evolution of the cloud of particles is seen through kinetic theory: we describe it by a distribution function $f(t,x,v) \in \mathbb{R}^+$ on the phase space, which typically solves a Vlasov-Boltzmann type equation.

An advantage of such formulation is that it allows to easily take into account the variability in size of the particles, even though we will not delve into such refinements. One of its specifity is that it couples the evolution of unkwowns that does not depend on the same set of variables. In what follows, we will consider the d-dimensional phase-space $\{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d\}$, where $\mathbb{T}^d = \mathbb{R}/(2\pi\mathbb{Z})^d$ is the flat torus endowed with the renormalized Lebesgue measure (typically, $d = 3$).

Fluid-kinetic models can give birth to a constellation of couplings, depending on the underlying physics. Here, we adopt the classification of O'Rourke [**41**] (see also [**15, 45**]) made through the fluid volumic fraction

$$\alpha(t,x) := 1 - \frac{4\pi r_p^3}{3} \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \tag{1}$$

where $r_p > 0$ is the radius of one droplet. Two main regimes are the followings.

- In the **thin spray** regime, we have $\alpha \sim 1$ (dilute regime), meaning that the volume occupied by the particles is negligible in the mixture. This is one of the most common regime among the many models of sprays, at least in the mathematical community. A prototype of great importance is the so-called Vlasov-(Navier-)Stokes system, where the main coupling effect between the two phases comes from a drag term. This set of equations has received a lot of attention over the past two decades but we refrain for dealing with the whole literature about it, which is too large. Let us just mention that the Cauchy theory, large time behavior and hydrodynamic limits are now quite well-understood features of such model. However, its rigorous derivation from first principles is still missing. We refer to [**40, 20, 19, 31**] for more details and references.
- In the **thick spray** regime, the volume fraction of the cloud particles is not negligible compared to that of the gas: it amounts to say that the quantity α is not assumed to be close to 1 and that it should explicitly appear in the system. Furthermore, collisions between particles are taken into account and the gas pressure arises as a new force acting on the cloud of particles (see the formal derivation of [**18, 41**]).

In this note, we focus on the latter thick spray regime, which has been much less studied in the mathematical literature[1]. It typically arises in the region where droplets are injected in the surrounding gas. A quite complete model for thick sprays can be written as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \left[f(-\nabla_x p(\varrho) + u - v) \right] = \kappa Q(f,f), \\ \partial_t(\alpha\varrho) + \mathrm{div}_x(\alpha\varrho u) = 0, \\ \partial_t(\alpha\varrho u) + \mathrm{div}_x(\alpha\varrho u \otimes u) + \alpha\nabla_x p - \nu\mathcal{A}u = \displaystyle\int_{\mathbb{R}^d} (v-u) f\,\mathrm{d}v, \\ \partial_t(\alpha\varrho\mathfrak{e}) + \mathrm{div}_x(\alpha\varrho\mathfrak{e}u) + p(\varrho,\mathfrak{e})\left(\partial_t\alpha + \mathrm{div}_x(\alpha u)\right) = \displaystyle\int_{\mathbb{R}^d} |v-u|^2 f\,\mathrm{d}v. \end{cases} \tag{2}$$

The first equation is a Vlasov-Boltzmann equation on f. It involves a transport part over the phase space with the force fields $u - v$ (drag term) and $-\nabla_x p$ (force of the fluid pressure), and the action of collisions[2] between the particles through an Boltzmann operator $Q(f,f)$ (with intensity $\kappa \geq 0$). The three other equations are the ones for the evolution of the gas, here of compressible type, where α now appears in the advection operators. The gas has a constant viscosity $\nu \geq 0$ and the operator featuring such an effect is the standard Lamé operator $\mathcal{A} = \Delta_x u + \nabla_x \mathrm{div}_x u$, while the pressure $p = p(\varrho,\mathfrak{e})$ depends on the density and energy through thermodynamics. The exchange of momentum and energy with the particles are finally encoded in the forces in the right hand-side, with moments in velocity of f.

[1] It is important to acknowledge the lack of standardized terminology for the thick spray regime, which varies across different scientific communities. This regime is alternatively described as "dense sprays" and can be part of the so-called *Williams-Boltzmann* equations (see [**35, 42**]).

[2] Which are typically of inelastic types, that is with conservation of mass and momentum, *but* without conservation of energy.

As we shall explain below, the mathematical theory of thick spray couplings is quite complex and has almost remained void until very recently. The purpose of this note is to present the result of [**20**] about the local well-posedness of (2) in the case $\nu > 0$.

To simplify the presentation, we consider the case $\kappa = 0$ and $\nu = 1$. The gas is assumed to be barotropic, the pressure $p = p(\varrho)$ being given as a function of the fluid density. A common example is $p(\varrho) = \varrho^\gamma$ $(\gamma > 1)$. In what follows, we will thus focus on the following system, obtained after renormalisation:

$$\text{(TS)} \qquad \begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \left[f(-\nabla_x p(\varrho) + u - v) \right] = 0, \\ \partial_t((1-\rho_f)\varrho) + \mathrm{div}_x((1-\rho_f)\varrho u) = 0, \\ (1-\rho_f)\varrho \left[\partial_t u + (u \cdot \nabla_x) u \right] + (1-\rho_f)\nabla_x p(\varrho) - \mathcal{A}u = j_f - \rho_f u, \end{cases}$$

where

$$\rho_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \quad j_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v) v\,\mathrm{d}v.$$

Let us now explain why the rigorous study of (TS) (and (2) as well) appears as challenging. First, a main feature of such systems is the presence of energy estimates involving a loss. To understand why, assume that we have a smooth solution (f, ϱ, u) with compact support to the system (TS). By invoking the parabolic regularity coming from the equation on u, it is easy to see that the fluid velocity involves lower order terms compared to (f, ϱ) and we can formally assume that this quantity is better-controlled. Because of the pressure gradient $\nabla_x p(\varrho)$, a standard energy estimate in the equation of f first provides

$$\text{(3)} \qquad \|f(t)\|_{\mathrm{H}^k_{x,v}} \lesssim \|\nabla_x p(\varrho)\|_{\mathrm{L}^\infty_t \mathrm{H}^k_x} \lesssim \|\varrho\|_{\mathrm{L}^\infty_t \mathrm{H}^{k+1}_x}, \quad k \geq 0.$$

At the same time, the presence of the term $1 - \rho_f$ in the equation on ϱ formally yields an estimate of the type

$$\text{(4)} \qquad \|\varrho(t)\|_{\mathrm{H}^k_{x,v}} \lesssim \|\partial_x(\rho_f, j_f)\|_{\mathrm{L}^\infty_t \mathrm{H}^k_x} \lesssim \|f\|_{\mathrm{L}^\infty_t \mathrm{H}^{k+1}_{x,v}}, \quad k \geq 0.$$

Combining (3)–(4), we end up with a **loss of derivative** displayed by the energy estimates, that is

$$\text{(5)} \qquad \|(f(t), \varrho(t))\|_{\mathrm{H}^k_{x,v} \times \mathrm{H}^k_x} \lesssim \|(f, \varrho)\|_{\mathrm{L}^\infty_t \mathrm{H}^{k+1}_{x,v} \times \mathrm{L}^\infty_t \mathrm{H}^{k+1}_x}, \quad k \geq 0.$$

The previous estimate rules out the possibility of using a classical fixed point procedure to construct a solution with Sobolev regularity. It indicates that a standard Cauchy theory for (TS), even local in time, cannot be easily obtained (appart from analytic regularity in a Cauchy-Kovalevskaya framework). This phenomemon is for instance well-kwown in Prandtl or hydrostatic equations of hydrodynamics (see [**36**]), as well as in singular Vlasov equations such as the kinetic Euler and Vlasov-Benney equations (see [**28**]). The link between the last one and (TS) will be our starting point in order to build a Cauchy theory for thick spray models.

Remark 1.1. *Several remarks are in order about the previous loss of derivatives coming from the energy estimates.*

- *One could argue that the estimate (4) is quite rough since the use of averaging lemma, well-known in kinetic theory (see [**23, 46**]), actually yields a better control on the moments ρ_f and j_f. However, to our knowledge, it only allows to recover half a derivative at most.*
- *By considering the new unknown $(f, \mathfrak{m}, u)$ with $\mathfrak{m} := (1-\rho_f)\varrho$, we can observe that $(\mathfrak{m}, u)$ displays estimates without loss and we now have*

  $$\|f(t)\|_{\mathrm{H}^k_{x,v}} \lesssim \left\| \nabla_x \frac{\mathfrak{m}}{(1-\rho_f)} \right\|_{\mathrm{L}^\infty_t \mathrm{H}^k_x} \lesssim \|\nabla_x \mathfrak{m}\|_{\mathrm{L}^\infty_t \mathrm{H}^k_x} + \|\nabla_x \rho_f\|_{\mathrm{L}^\infty_t \mathrm{H}^k_x} \lesssim \|f\|_{\mathrm{L}^\infty_t \mathrm{H}^{k+1}_x},$$

  which seems to be more favorable than the previous (5). This directly connects (TS) to the so-called Vlasov-Benney equation (see (VB) below) in plasma physics. However, this point of view does not seem robust enough to treat more general systems like (2).

- *In the case of an inviscid gas, that is without the elliptic operator $\mathcal{A}$ in the equation for the fluid velocity, there is an additional loss of derivative on u. The situation is thus even more intricate because it gathers a Vlasov-type equation with an hyperbolic-type system.*

On top of the previous analytical challenge, thick spray models are also known to be connected with the so-called *two-phase flow* equations. Broadly speaking, these are macroscopic systems describing a gas-liquid mixture (see [**32**]). In the case without viscosity, these bifluid equations of non-conservative form can display domains of non-hyperbolicity, making their rigorous study reputedly difficult[3]. In [**16**], Desvillettes and Matthiaud have shown that there exists a formal link between (2) and standard bifluid equations with common pressure (related to the term $-\nabla_x p$ in the kinetic equation of (2)). It is based on a hydrodynamic limit, in a dominant inelastic collisions regime (namely, the limit $\kappa \to +\infty$ in (2)).

For all of the former reasons, it has been argued that the study of (TS) is, at least formally, quite involved (even though some numerics are available [**9, 8**]). The construction of strong solutions, even locally in time, has for instance been raised as a conjecture by Baranger and Desvillettes [**3**]. There has recently been a (re)gain of interest about the thick sprays equations from the mathematical point of view. In [**11**], Buet, Després and Desvillettes has obtained the linear stability of (2) (with $\nu = 0$ or $\nu > 0$) around kinetic profiles that are of Penrose type (see below for a precise definition). In [**21, 12**], Buet, Després and Fournet have studied a regularized-averaged version of (TS) without viscosity (falling in the framework of symetrizable systems like [**3, 37**]), as well as some linear damping effects (but without the drag term $u - v$ and its retroaction).

2. Main result

Our main result for (TS), on the d-dimensional phase space $\mathbb{T}^d \times \mathbb{R}^d$, yields the local in time well-posedness in Sobolev regularity for (TS), in the class of **Penrose stable initial data**. This notion is defined as follows.

Definition 2.1. *We say that the couple* $(\mathrm{f}(x,v), \rho(x))$ *satisfies the c-**Penrose stability condition** (for the thick spray equations (TS)) if there exists $c > 0$ such that for all $x \in \mathbb{T}^d$*

$$(\mathbf{P}) \qquad \inf_{(\gamma,\tau,k)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\setminus\{0\}} \left| 1 - \frac{p'(\rho(x))\rho(x)}{1-\rho_{\mathrm{f}}(x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v \mathrm{f})(x, ks)\, \mathrm{d}s \right| > c,$$

where $\mathcal{F}_v$ is the Fourier transform in velocity on $\mathbb{R}^d$.

The Penrose condition (**P**) (from the pionnering work [**43**]) is related to stability issues in plasma physics, especially for the famous Vlasov-Poisson equations from kinetic theory. It has been introduced in the former generalized form (removing the prefactor before the integral) by Mouhot and Villani in their important contribution [**39**] on the Landau damping for Vlasov-Poisson on the torus.

Being essentially an assumption on the form of the velocity profile $\mathrm{f}(\cdot, v)$, it is satisfied by a broad class of data. First, any sufficiently small profile in size is stable and it is also the case for any bump profile (first increasing then decreasing) in dimension 1. In dimension $d \geq 1$, any decreasing radial profile satisfies the condition. This includes, in particular, the important case of Maxwellians in velocity. Finally, any sufficiently small and smooth perturbation of a profile satisfying (**P**) remains in this class.

For us, this condition shall arise in the analysis of the Cauchy theory for degenerate Vlasov equations. As a guideline, we consider the following **Vlasov-Benney equation** (following Bardos [**4**])

[3] When viscosity is (partially) added, the analysis seems a bit more favorable (see [**44, 10**]).

that describes the dynamics of ions (for a linearized thermalization of electrons) after considering the so-called **quasineutral limit**[4] from Vlasov-Poisson equations:

$$\text{(VB)} \qquad \partial_t f + v \cdot \nabla_x f - \nabla_x \rho_f \cdot \nabla_v f = 0, \quad \rho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v.$$

A key observation regarding (VB) is that the force field $-\nabla_x \rho_f$ loses one derivative with respect to f. It is expected that the mathematical analysis of (VB) is significantly different from that of the standard Vlasov-Poisson systems. We can now appreciate the parallel between the Vlasov-Benney equation (VB) and the thick spray system (TS): both involve a **singular kinetic coupling**.

Let us provide a brief overview of existing results on the well-posedness of (VB). First, it has been observed in [**7**] that there exist unstable stationary solutions $\mu(v)$ around which the linearized equations have an unbounded unstable spectrum. This leads to the following consequence: the system (VB) is generically ill-posed in the sense of Hadamard in Sobolev spaces, even with an arbitrary loss of derivatives and over an arbitrarily short time, as proven by Han-Kwan and Nguyen in [**27**] (a precise quantitative result refining the argument of [**7**]). Also noteworthy is the result by Baradat [**1**], which addresses instabilities around very irregular profiles (of measure type). The ill-posed nature of the Vlasov-Benney equation stands in sharp contrast to the well-established Cauchy theory of Vlasov-Poisson systems.

However, there are situations where the equation (VB) is well-posed. Given the preceding discussion, it seems that assumptions of very strong regularity or structure on the data or solutions are necessary to avoid potential instabilities. For instance, analytic data (see [**33, 39**]) or initial Sobolev data with a velocity profile with one bump for $d = 1$ (see [**5, 6**]) lead to a local-in-time solution for (VB).

In any dimension (and on the torus), the existence and uniqueness of local in time solutions for (VB), in a Sobolev framework, was obtained by Han-Kwan and Rousset in [**28**], under the assumption that that a generalized Penrose stability condition is initially satisfied. The last one is essentially of the form (**P**), and corresponds to the Penrose stability assumption for the Vlasov-Poisson system[5]. We also refer to [**13**] where optimal Penrose conditions are enforced to construct a unique solution to singular Vlasov equations.

Because of the analogy between (VB) and (TS), we will actually take our inspiration from the important contribution [**28**] to treat the thick spray equations. Let us mention that the linear stability analysis performed for thick spray equations in [**11**] takes places around radially decreasing profiles in velocity, which naturally echoes the Penrose condition (**P**).

Let us now state our result for the thick spray equations. For $(m, r) \in \mathbb{N}\times\mathbb{R}^+$ and $\mathrm{f} : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^+$, we define the weighted (in velocity) Sobolev norms as

$$\|\mathrm{f}\|_{\mathcal{H}^m_r} := \left(\sum_{|\alpha|+|\beta|\le m} \int_{\mathbb{T}^d}\int_{\mathbb{R}^d} \langle v\rangle^{2r} |\partial_x^\alpha \partial_v^\beta \mathrm{f}(x,v)|^2 \,\mathrm{d}x\,\mathrm{d}v \right)^{1/2}, \quad \langle v\rangle := (1+|v|^2)^{1/2}.$$

The main theorem from [**20**] is the following.

Theorem 2.2. *There exist* $(m_0, r_0) \in \mathbb{N}\times\mathbb{R}^+$*, depending only on the dimension, such that the following holds for all* $m > m_0$ *and* $r > r_0$*. Let*

$$f^{\mathrm{in}} \in \mathcal{H}^m_r, \quad \varrho^{\mathrm{in}} \in \mathrm{H}^{m+1}, \quad u^{\mathrm{in}} \in \mathrm{H}^m,$$

[4] We refer to the recent review [**24**] for more details and references on this very active subject of research.

[5] A heuristic way to understand this assumption is the following: the Vlasov-Benney equations arises in the quasineutral limit of Vlasov-Poisson systems. By a hyperbolic rescaling, the validity or not of this singular limit is actually connected with stability issues in time of Vlasov-Poisson itself, explaining the Penrose condition that is at stake.

such that $(f^{\mathrm{in}}, \varrho^{\mathrm{in}})$ satisfies the c-Penrose stability condition (**P**) *for some $c > 0$, and such that there is no void. Then there exist $T > 0$ and a solution (f, ϱ, u) to* (TS) *with initial condition $(f^{\mathrm{in}}, \varrho^{\mathrm{in}}, u^{\mathrm{in}})$ such that*

$$f \in \mathscr{C}\left([0,T]; \mathcal{H}_r^{m-1}\right), \quad \varrho \in \mathrm{L}^2(0,T;\mathrm{H}^m), \quad u \in \mathscr{C}([0,T];\mathrm{H}^m) \cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1}),$$

and with $(f(t), \varrho(t))$ satisfying the $c/2$-Penrose stability condition (**P**) *for all $t \in [0,T]$. In addition, this solution is unique in this regularity class.*

Let us insist on the fact that we build solutions of high but finite regularity. We do not appeal to a framework of analytic solutions, with some Cauchy-Kovalevskaya theorem ruling out potential instabilities. Let us also mention that our approach is robust enough to treat the full model (2) rather than (TS), under similar stability assumptions. However, we truly need the (reasonable) presence of viscosity[6].

To our knowledge, this is the first time that such a Penrose stability condition explicitly appears for fluid-kinetic systems (see nevertheless [**11**]) and that solutions to thick spray equations are built (appart from the modified case of [**21**]). Note that, similarly to the result of [**28**] for (VB), it is actually not clear to know if the breakdown of the Penrose condition (**P**) is a blow-up criterion.

What about (TS) if the Penrose condition (**P**) is not satisfied? The thick spray system is actually **ill-posed in the sense of Hadamard** in that case: roughly speaking, it means that a loss of derivative is indeed at stake and that there cannot exists a nice flow map for the equation, even for arbitrary short times and an arbitrary loss of derivatives. This fact, reminiscent of the mecanism driving instabilities in (VB) (see [**27, 1**]), will be proven in a forthcoming collaboration with Aymeric Baradat and Daniel Han-Kwan in [**2**].

Therefore, the Penrose condition (**P**) is essentially a necessary and sufficient condition ensuring the local well-posedness for thick sprays in Sobolev regularity. This answers the question raised by Baranger and Desvillettes in [**3**].

3. Strategy of proof

In the rest of this note, we provide a sketch of the proof for Theorem 2.2, focusing on the existence part. Even if we volontarily skip a lot of technical details, we hope that it can still offer a flexible framework being potentially useful in other situations. We refer to [**20**] for the complete details.

Inspired by the quasineutral limit problem from [**28**], we start by introducing a regularized thick spray system where we smooth the pressure gradient in the kinetic equation:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[f(u-v)] - p'(\varrho)\nabla_x\left[\mathrm{J}_\varepsilon \varrho\right] \cdot \nabla_v f = 0, \quad \mathrm{J}_\varepsilon := (\mathrm{Id} - \varepsilon^2 \Delta_x)^{-1}, \tag{6}$$

for a parameter $\varepsilon \in (0,1)$ (the other equations being left unchanged). By standard methods (for instance, an iterative scheme), we can obtain the existence of a solution $(f_\varepsilon, \rho_\varepsilon, u_\varepsilon)$ with a lifespan $T_\varepsilon > 0$ for this nonsingular coupling. Unfortunately, this lifespan may degenerate to 0 as $\varepsilon \to 0$.

Step 1: Boostrap procedure. We set up a boostrap argument aiming at uniformly bounding (in ε) the quantity

$$\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T) := \|f_\varepsilon\|_{\mathrm{L}^\infty\left(0,T;\mathcal{H}_r^{m-1}\right)} + \|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)} + \|u_\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^m)\cap \mathrm{L}^2(0,T;\mathrm{H}^{m+1})},$$

over a certain (short) time interval independent of ε. The loss of derivative from the kinetic equation is accounted for in this quantity via the shift of one derivative between the first two terms. The last term is reminiscent of the standard regularity for strong solutions of Navier-Stokes

[6] One could reasonably argue that the presence of (slightly) viscous effects can also be helpful for numerics perpectives. We insist on the fact that, when there is no viscosity for the gas, the rigorous Cauchy theory for thick spray models remains open.

equations. A uniform bound on $\mathcal{N}_{m,r}(f_\varepsilon, \varrho_\varepsilon, u_\varepsilon, T)$ will enable passing to the limit as $\varepsilon \to 0$ through a compactness argument. By estimates similar to (3), it follows that the key quantity to control is the term $\|\varrho_\varepsilon\|_{\mathrm{L}^2(0,T;\mathrm{H}^m)}$.

Step 2: Semi-Lagrangian procedure. The essential observation is that the density ϱ_ε satisfies, modulo a lower-order term S (with respect to the number of derivatives we control), the following twisted transport equation

$$\partial_t \varrho_\varepsilon + u_\varepsilon \cdot \nabla_x \varrho_\varepsilon + \frac{\varrho_\varepsilon}{1-\rho_{f_\varepsilon}} \mathrm{div}_x \left[j_{f_\varepsilon} - \rho_{f_\varepsilon} u_\varepsilon \right] = S, \tag{7}$$

depending on f_ε only through its velocity moments ρ_{f_ε} and j_{f_ε}. The Penrose condition (**P**) will provide an estimate without loss on ϱ_ε.

Hence, we first work on the kinetic moments themselves in order to obtain a closed equation on ϱ_ε. By taking m spatial derivatives in the kinetic equation satisfied by f_ε, commutators involving terms of the form

$$\partial_x \nabla_x \varrho_\varepsilon \cdot \partial_x^{m-1} \nabla_v f_\varepsilon \quad \text{or} \quad \partial_x^2 \nabla_x \varrho_\varepsilon \cdot \partial_x^{m-2} \nabla_v f_\varepsilon$$

appear. As we only control $m-1$ derivatives of f_ε, these expressions are problematic due to the divergence term on the moments in (7). Following [**28**], the idea is to introduce an augmented unknown $\mathcal{F} = (\partial_{x,v}^I f_\varepsilon)_{|I|=m-1,m}$, which satisfies a system of coupled Vlasov equations where the dominant term is now of the form $\partial_x^I (p'(\varrho_\varepsilon) \nabla_x \left[\mathrm{J}_\varepsilon \varrho_\varepsilon \right]) \cdot \nabla_v f_\varepsilon$ for $|I| = m-1, m$.

By adopting a Lagrangian point of view, we can simplify the structure of this system. Starting from the dynamics of the characteristics curves

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{X}^{s;t}(x,v) = \mathrm{V}^{s;t}(x,v), \\ \dfrac{\mathrm{d}}{\mathrm{d}s} \mathrm{V}^{s;t}(x,v) = -\mathrm{V}^{s;t}(x,v) + u(s, \mathrm{X}^{s;t}(x,v)) - p'(\varrho(s, \mathrm{X}^{s;t}(x,v))) \nabla_x \mathrm{J}_\varepsilon \varrho(s, \mathrm{X}^{s;t}(x,v)), \\ \quad \mathrm{X}^{t;t}(x,v) = x, \quad \mathrm{V}^{t;t}(x,v) = v, \end{cases}$$

we use a change in variable in velocity that straightens the trajectories in space: for small times, there exists a diffeomorphism $\psi_x^{s,t} : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\mathrm{X}^{s;t}(x, \psi_x^{s,t}(v)) = x + (1 - e^{t-s})v.$$

This mainly brings the dynamics of the Vlasov equation (6) to that of the free transport with friction

$$\partial_t g + v \cdot \nabla_x g - v \cdot \nabla_v g = 0.$$

By a Duhamel formula along trajectories for $\mathcal{F}$, we finally aim at obtaining a decomposition of the form

$$\partial_x^m \rho_{f_\varepsilon} = \mathrm{Main}_0[\mathrm{J}_\varepsilon \partial_x^m \varrho_\varepsilon] + R, \quad \partial_x^m j_{f_\varepsilon} = \mathrm{Main}_1[\mathrm{J}_\varepsilon \partial_x^m \varrho_\varepsilon] + R, \tag{8}$$

where the two terms $\mathrm{Main}[\mathrm{J}_\varepsilon \partial_x^m \varrho_\varepsilon]$ are leading terms in ϱ_ε and where R should be a remainder controlled in $\mathrm{L}_t^2 \mathrm{H}_x^1$.

Step 3: Smoothing estimates and averaging. Reaching (8) actually requires to rely on some smoothing integro-differential operators. More precisely, we want to prove that

$$\partial_x^m \rho_{f_\varepsilon} = \mathrm{K}_G^{\mathrm{free}}[\mathrm{J}_\varepsilon \partial_x^m \varrho_\varepsilon] + R, \quad \partial_x^m j_{f_\varepsilon} = \mathrm{K}_{vG}^{\mathrm{free}}[\mathrm{J}_\varepsilon \partial_x^m \varrho_\varepsilon] + R. \tag{9}$$

Here, given a kernel $G(t,s,x,v)$ (or $G(t,x,v)$), the operator $\mathrm{K}_G^{\mathrm{free}}$ is an integral operator acting on $F(t,x)$ by

$$\mathrm{K}_G^{\mathrm{free}}[F](t,x) = \int_0^t \int_{\mathbb{R}^d} [\nabla_x F](s, x-(t-s)v) \cdot G(t,s,x,v) \, \mathrm{d}v \, \mathrm{d}s.$$

In view of the strategy devised in the previous step, we also need to consider the same operator with the dynamics with friction, namely

$$\mathrm{K}_G^{\mathrm{fric}}[F](t,x) = \int_0^t \int_{\mathbb{R}^d} [\nabla_x F](s, x + (1-e^{t-s})v) \cdot G(t,s,x,v)\, \mathrm{d}v\, \mathrm{d}s.$$

We can show that the kernel $G(t,x,v)$ in (9) is given by $G = p'(\varrho_\varepsilon)\nabla_v f_\varepsilon$.

Note that the operators $\mathrm{K}_G^{\mathrm{free}}$ and $\mathrm{K}_G^{\mathrm{fric}}$ explicitly feature a loss of derivative in space, reminiscent of (3). To obtain (9), we use the following key results about smoothing estimates.

Proposition 3.1. *If the kernel G is sufficiently smooth and decaying in velocity then*

- *the operator $\mathrm{K}_G^{\mathrm{free}}$ and $\mathrm{K}_G^{\mathrm{fric}}$ are bounded on $\mathrm{L}^2(0,T;\mathrm{L}^2_x)$;*
- *if $G(t,t,x,v) = 0$, then the operators $\mathrm{K}_G^{\mathrm{free}}$ and $\mathrm{K}_G^{\mathrm{fric}}$ are bounded from $\mathrm{L}^2(0,T;\mathrm{L}^2_x)$ to $\mathrm{L}^2(0,T;\mathrm{H}^1_x)$;*
- *the operator $\mathrm{K}_G^{\mathrm{free}} - \mathrm{K}_G^{\mathrm{fric}}$ is bounded from $\mathrm{L}^2(0,T;\mathrm{L}^2_x)$ to $\mathrm{L}^2(0,T;\mathrm{H}^1_x)$.*

The first point concerning $\mathrm{K}_G^{\mathrm{free}}$ is already proven by Han-Kwan and Rousset in [**28**] (we also refer to [**14**] for an extension). Note that the second point has also been observed in [**29**].

These results are reminiscent of the famous averaging lemmas in kinetic theory [**22**]. Note that they allow for a gain of (at least) one derivative, which is not directly provided by this theory (see the detailed discussion by Han-Kwan in [**26**, Section 6.2] with references therein). Here, it is strongly based on the interplay between time and velocity averages, and on the (fixed) regularity/decay of the kernel. We refer to [**20**] for a complete proof.

Step 4: Pseudodifferential estimates *via* (P). The final step of the proof is to insert (9) into the equation (7), after taking m derivatives in it. Recall that the goal is to obtain a control on $h := \partial_x^m \varrho_\varepsilon$ in $\mathrm{L}^2_{t,x}$. Through a commutation between div_x and the integral operators in (9), it can be shown that h satisfies

$$\left(\mathrm{Id} - \frac{\varrho_\varepsilon}{1-\rho_{f_\varepsilon}} \mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon\right)\left[\partial_t h + u_\varepsilon \cdot \nabla_x h\right] = \text{lower-order terms}, \tag{10}$$

where $G(t,x,v) = p'(\varrho_\varepsilon(t,x))\nabla_v f_\varepsilon(t,x,v)$. This is an explicit factorization between a transport operator along u_ε and an integro-differential operator depending on ϱ_ε and f_ε. The Penrose condition (**P**) will ensure estimates in $\mathrm{L}^2_{t,x}$ on the solutions $H = \partial_t h + u_\varepsilon \cdot \nabla_x h$ of the previous equation.

Following [**28**], one can relate (modulo time conjugation) the previous integro-differential operator to a semiclassical pseudodifferential operator in time-space, with symbol

$$\mathscr{P}_{f,\varrho}(t,x,\gamma,\tau,k) := \frac{p'(\varrho(t,x))\varrho(t,x)}{1-\rho_f(t,x)} \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{ik}{1+|k|^2} \cdot (\mathcal{F}_v \nabla_v f)(t,x,ks)\, \mathrm{d}s.$$

This symbol fully depends on time-space and on the dual variables, forcing us to consider suitable extensions in time for the solutions (that we do not detail here). Note that because of the transport part in (10), we cannot come down to a symbol at time $t=0$ as in [**28**] (that would produce another loss of derivative). Here, the variable $\gamma > 0$ should be understood as a (large) parameter. The Penrose stability condition (**P**), extended to short times for the solutions themselves, now appears and can be written as

$$\inf_{(t,x,\gamma,\tau,k)} |1 - \mathscr{P}_{f,\varrho}(t,x,\gamma,\tau,k)| > c.$$

It means that the integro-differential operator $\mathrm{Id} - \frac{\varrho_\varepsilon}{1-\rho_{f_\varepsilon}} \mathrm{K}_G^{\mathrm{free}} \circ \mathrm{J}_\varepsilon$ in (10) is *elliptic*. A suitable semiclassical pseudodifferential calculus with large parameter γ (inspired by Métivier's framework [**38**]) then allows to derive $\mathrm{L}^2_{t,x}$ estimates on H. Similar estimates are obtained on h via the transport operator.

The stability condition (**P**) thus mainly ensures that the fluid density equation can be factorized into an *elliptic* part and a *hyperbolic* part, providing estimates without loss for short times. It eventually allows to close the bootstrap argument.

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