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STRONG HARNACK INEQUALITY FOR THE BOLTZMANN EQUATION

AMÉLIE LOHER

Abstract. We review local regularity properties of the non-cutoff Boltzmann equation for moderately soft potentials. We explain how to view the Boltzmann equation as a non-local hypoelliptic equation. We show that despite its non-locality, we can derive a Strong Harnack inequality. To this end, we establish a linear First De Giorgi Lemma, which relates the local supremum to the local L^2 norm without non-local tail terms.

Contents

1. Introduction

1.1. **Boltzmann equation.** The Boltzmann equation models the dynamics of a dilute gas. It is given by

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ encodes the density of the gas particles, which at any given time $t \in \mathbb{R}$ have location $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. The left hand side of (1.1) is the transport operator, which describes the trajectories of the particles. The right hand side takes into account the fluctuations in velocity that result from particle interactions. We denote with Q the Boltzmann collision operator, whose explicit form is given by

$$Q(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \left[f(v'_*)f(v') - f(v_*)f(v)\right] B(|v - v_*|, \cos\theta) \,\mathrm{d}\sigma \,\mathrm{d}v_*, \tag{1.2}$$

where v, v_* are the post-collisional velocities, and v', v'_* the pre-collisional velocities, so that

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

The rate of change in velocities is determined through the cross-section B, which reads

$$B(r, \cos\theta) = r^\gamma b(\cos\theta), \quad b(\cos\theta) \approx |\sin(\theta/2)|^{-(d-1)-2s},$$

with parameters $\gamma \in (-d, 1]$ and $s \in (0, 1)$, and where $\cos\theta$ is defined as

$$\cos\theta := \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad \sin(\theta/2) := \frac{v' - v}{|v' - v|} \cdot \sigma.$$

Note that here and in the sequel we will often omit to write out the dependency of f and of K_f on time and space, if there is no risk of confusion. By a change of coordinates to so-called Carleman coordinates, and using the well-known cancellation lemma [16], we can rephrase the collision operator as

$$Q(f, f) = \int \big[f(v') - f(v)\big] K_f(v, v')\,\mathrm{d}v' + c\,(f * |\cdot|^\gamma)\,f, \tag{1.3}$$

where $c > 0$ (see [15, Lemma 5.1]) and where the kernel K_f depends implicitly on the solution f and is given by

$$K_f(v, v') = 2^{d-1}\,|v' - v|^{-1} \int_{w \perp v' - v} f(v + w) B(r, \cos\theta) r^{-d+2}\,\mathrm{d}w, \tag{1.4}$$

with $r^2 = |v' - v|^2 + |w|^2$ and $\cos\theta = \frac{w - (v - v')}{|w - (v - v')|} \cdot \frac{w + (v' - v)}{|w + (v' - v)|}$. Our aim is to derive interior regularity properties on the solution of the Boltzmann equation (1.1). In particular, we are interested in a bound on the local supremum in terms of the local infimum, an estimate known as the Strong Harnack inequality. From this, one can potentially deduce further consequences, such as heat kernel estimates, a powerful tool to understand the long time behaviour of the solution.

In order to discuss the behaviour of solutions to the Boltzmann equation (1.1), we need to classify the kernel K_f. We would like to introduce a notion of ellipticity suitable to the non-local coefficients encoded through K_f. It seems out of reach to treat the quasi-linearity in K_f without further restrictions. A meaningful setting is reached, however, by working in a regime conditional to certain hydrodynamic bounds. Thus we define the following macroscopic quantities associated to (1.1)

$$\begin{aligned} M(t, x) &:= \int f(t, x, v)\,\mathrm{d}v, \\ E(t, x) &:= \int f(t, x, v)\,|v|^2\,\mathrm{d}v, \\ H(t, x) &:= \int f(t, x, v) \ln f(t, x, v)\,\mathrm{d}v. \end{aligned} \tag{1.5}$$

They describe the mass density, the energy and the entropy respectively. In 2016, Silvestre [15] determined a conditional regime, in which he was able to derive a-priori L^∞ bounds on the solution to (1.1) in case of moderately soft potentials, that is $0 \le \gamma + 2s \le 2$. He showed that it suffices to assume that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, there is m_0, M_0, E_0, H_0 such that

$$0 < m_0 \le M(t, x) \le M_0, \qquad E(t, x) \le E_0, \qquad H(t, x) \le H_0. \tag{1.6}$$

Then $f \in L^\infty(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$, see [15, Theorem 7.3]. Therefore, the quasi-linearity in the kernel K_f is treated through this a-priori boundedness of f, and in particular, a suitable notion of ellipticity can be introduced. Moreover, the second term of the right hand side in (1.3) can be understood as a lower order term compared to the first term, as the a-priori boundedness of the solution permits to view it as a non-negative source term in L^∞, [15, Section 5]. Thus we focus the next section on the classification of the ellipticity of K_f.

1.2. **Ellipticity class.** The conditional setting prescribed by (1.5) implies conditions on the kernel K_f. We refer the reader to Section 4 of [15, Section 4] and [12, Appendix A]. One can prove that there exists $0 < \lambda_0 < \Lambda_0$ and $\gamma_0 > 0$ depending on s, d, m_0, M_0, E_0, H_0, such that for any $\overline{R} > 0$ and any $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ supported in $B_{\overline{R}}$ there holds:

i. the kernel K_f is coercive:

$$\int_{B_{2\overline{R}}}\int_{B_{2\overline{R}}} \big(\varphi(v) - \varphi(v')\big)^2 K_f(v, v')\,\mathrm{d}v'\,\mathrm{d}v \geq \lambda_0 \int_{B_{\overline{R}}}\int_{B_{\overline{R}}} \frac{|\varphi(v) - \varphi(v')|^2}{|v - v'|^{d+2s}}\,\mathrm{d}v\,\mathrm{d}v'. \tag{1.7}$$

ii. it satisfies an upper bound: for any $r > 0$

$$\forall v \in \mathbb{R}^d \quad \int_{B_r(v)} K_f(v, v')\,|v - v'|^2\,\mathrm{d}w \leq \Lambda_0 r^{2-2s}. \tag{1.8}$$

iii. the cancellation

$$\forall v \in \mathbb{R}^d \quad \left|\mathrm{PV}\int_{\mathbb{R}^d} \big(K_f(v, v') - K_f(v', v)\big)\,\mathrm{d}w\right| \leq \Lambda_0, \tag{1.9}$$

and if $s \geq 1/2$ there holds for all $r > 0$

$$\forall v \in \mathbb{R}^d \quad \left|\mathrm{PV}\int_{B_r(v)} (v - v')\big(K_f(v, v') - K_f(v', v)\big)\,\mathrm{d}v'\right| \leq \Lambda_0 r^{1-2s}. \tag{1.10}$$

iv. a bound on the oscillation of the jumps: for any $r > 0$

$$\sup_{v \in B_{r/2}} \int_{\mathbb{R}^d \setminus B_r} K_f(v, v')\,\mathrm{d}v' \leq \gamma_0 \fint_{B_{r/2}}\int_{\mathbb{R}^d \setminus B_r} K_f(v, v')\,\mathrm{d}v'\,\mathrm{d}v. \tag{1.11}$$

We emphasise that the implicit dependency of K_f on the solution f itself is absorbed in the constants λ_0, Λ_0 and γ_0 through the a-priori boundedness of f. Therefore, we may omit the subscript in the sequel and write instead $K = K_f$.

We thus justified the viewpoint on the Boltzmann equation (1.1) for moderately soft potentials as a kinetic integro-differential equation

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = \int_{\mathbb{R}^d} \big[f(t, x, v') - f(t, x, v)\big] K(t, x, v, v')\,\mathrm{d}v' + h(t, x, v), \tag{1.12}$$

where $K : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is a non-negative kernel satisfying (1.7)-(1.11), and $h \in L^\infty(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$ is a bounded non-negative source term. We understand the regularity of (1.1) if we understand it for (1.12). Since we are interested in deriving *local* properties for (1.12), we introduce a notion of solution domains that respects the invariances of (1.12).

1.3. **Local domain.** On the one hand, equation (1.12) is scaling-invariant. Specifically, for any $r \in [0, 1]$ the scaled function $f_r(t, x, v) = f(r^{2s}t, r^{1+2s}x, rv)$ satisfies

$$\partial_t f_r + v \cdot \nabla_x f_r = \int \big[f_r(v') - f_r(v)\big] K_r(v, v')\,\mathrm{d}v' + h_r,$$

where the kernel and the source scale as

$$K_r(t, x, v, v') = r^{d+2s} K(r^{2s}t, r^{1+2s}x, rv, rv'), \qquad h_r(t, x, v) = r^{2s} h(r^{2s}t, r^{1+2s}x, rv).$$

For $r \in [0,1]$ the scaled kernel K_r satisfies (1.7)-(1.11) in the larger radius $\overline{R}/r$ instead of $\overline{R}$. Moreover, h_r is bounded provided that h is.

On the other hand, (1.12) is invariant under Galilean transformations

$$z \to z_0 \circ z = (t_0 + t, x_0 + x + tv_0, v_0 + v)$$

with $z_0 = (t_0, x_0, v_0) \in \mathbb{R}^{1+2d}$. If f is a solution of (1.12), then its Galilean transformation $f_{z_0}(z) = f(z_0 \circ z)$ solves

$$\partial_t f_{z_0} + v \cdot \nabla_x f_{z_0} = \int \big[f_{z_0}(v') - f_{z_0}(v)\big] K_{z_0}(v, v') \,\mathrm{d}v' + h_{z_0},$$

where the translated kernel and source are given by

$$K_{z_0}(t, x, v, v') = K(z_0 \circ z, v_0 + v'), \qquad h_{z_0}(t, x, v) = h(z_0 \circ z).$$

Again the modified kernel K_{z_0} satisfies (1.7)-(1.11) provided that K does, and h_{z_0} is bounded provided that h is.

In view of these invariances we define kinetic cylinders

$$Q_r(z_0) := \left\{(t, x, v) : -r^{2s} \leq t - t_0 \leq 0, |v - v_0| < r, |x - x_0 - (t - t_0)v_0| < r^{1+2s}\right\},$$

for $r > 0$ and $z_0 = (t_0, x_0, v_0) \in \mathbb{R}^{1+2d}$. For later reference, we also introduce the cylinder shifted to the past

$$Q_r^-(z_0) := Q_r\big(z_0 - (2r^{2s}, 2r^{2s}v_0, 0)\big),$$

and shifted to the future

$$Q_r^+(z_0) := Q_r\big(z_0 + (2r^{2s}, 2r^{2s}v_0, 0)\big).$$

In particular for $z_0 = 0$

$$Q_r^- := Q_r(-2r^{2s}, 0, 0) = (-3r^{2s}, -2r^{2s}] \times B_{r^{1+2s}} \times B_r.$$

and

$$Q_r^+ := Q_r(2r^{2s}, 0, 0) = (r^{2s}, 2r^{2s}] \times B_{r^{1+2s}} \times B_r.$$

Figure 1 illustrates these domains.

Given a solution of (1.12) in some kinetic cylinder $Q_R(z_0)$, what can we say about its behaviour in the interior?

2. Harnack inequalities

In a series of works by Imbert-Silvestre(-Mouhot) [6–9], they showed first that any solution of (1.12) satisfies the Weak Harnack inequality, and is thus Hölder continuous [7], then that the regularity of the kernel K can be transferred onto the solution in the sense of Schauder estimates [9], and finally that these local results can be globalised via a change of variables [8, Section 5] due to the decay properties of the solution [6], so that eventually the Schauder estimates can be bootstrapped to obtain smooth solutions [8]. Later a constructive proof of the Weak Harnack inequality and the Schauder estimates appeared in [11, 13]. One of the difficulties posed by (1.12) is its non-local operator, which means that the behaviour of the solution inside a given domain is affected by the values attained in the whole velocity space, especially also outside the domain in which we assume the equation to be satisfied. This is the reason why we have to assume that the solution is essentially bounded for almost every $v \in \mathbb{R}^d$ to make sense of the equation. Then we can deduce Hölder continuity with a constant depending on the essential bound of f in the whole domain.

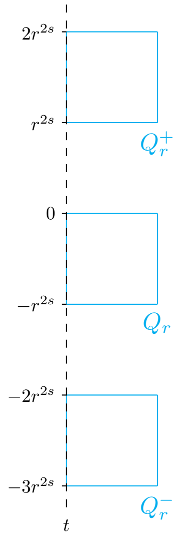

FIGURE 1. The kinetic cylinder Q_r, its future cylinder Q_r^+ and its past cylinder Q_r^- for some $r > 0$. The dashed vertical line represents the timeline.

However, in contrast to equations with a local diffusion operator, we cannot deduce from these Hölder estimates a bound on the local supremum of the solution in terms of the local infimum. Such a bound is known as the Strong Harnack inequality. The problem is that the behaviour of the tail is encoded in the constant depending on the essential bound of f in the whole domain. Intuitively, it might not even be clear that we should expect a fully local bound on a solution to a non-local equation. However, from the literature on parabolic non-local equations, it is known that such a local bound does hold, a result that was derived via probabilistic methods [2, 17], and only recently has been proven analytically [10]. The key is to capture the behaviour of the tail.

We will therefore first try to understand the part of the proof of the Weak Harnack inequality that makes the appearance of the tail explicit. Then we try to relate the tail to local quantities in order to derive the Strong Harnack inequality.

2.1. **Previous result.** The derivation of the Strong Harnack inequality relies on the Weak Harnack inequality.

Theorem 2.1 (Weak Harnack Inequality [7, Thm. 1.6], [11, Thm. 1.1])**.** *Let f be a non-negative super-solution to* (1.12) *in* $[-3, 0] \times Q_1^t := [-3, 0] \times B_1 \times B_1$ *with a non-negative kernel K satisfying* (1.7)-(1.10) *for* $\overline{R} = 2$*. Then there is $C > 0$ and $\zeta > 0$ depending on s, d, λ, Λ such that for any* $0 < r_0 < 1/3$ *the Weak Harnack Inequality is satisfied:*

$$\left(\int_{\tilde{Q}^-_{r_0/2}} f^\zeta \, \mathrm{d}z \right)^{1/\zeta} \leq C \Big(\inf_{Q_{r_0/2}} f + \|h\|_{L^\infty(Q_1)} \Big), \tag{2.1}$$

where $\tilde{Q}^-_{r_0/2} := Q_{r_0/2}\big((-\frac{5}{2} r_0^{2s} + \frac{1}{2}(r_0/2)^{2s}, 0, 0)\big)$*, see Figure 2.*

The proof of the Weak Harnack Inequality follows the De Giorgi method: in a first step one shows the local gain of regularity from L^2 to L^∞, then in a second step from L^∞ to C^α. The Weak Harnack Inequality follows in the end from the C^α regularity by using a covering argument. Figuratively, this can be visualised as

$$\underbrace{L^2 \xrightarrow{\text{First De Giorgi Lemma}} L^\infty \xrightarrow{\text{Second De Giorgi Lemma}} C^\alpha}_{\implies \text{Weak Harnack (2.1).}}.$$

To obtain the Strong Harnack inequality, one could be led to think that it now suffices to combine the Weak Harnack (2.1) with the First De Giorgi Lemma, since

$$\underbrace{\inf \xrightarrow{\text{Weak Harnack (2.1)}} L^\zeta \xrightarrow{\text{Young's}} L^2 \xrightarrow{\text{First De Giorgi Lemma}} L^\infty}_{\implies \text{Strong Harnack.}}. \tag{2.2}$$

This is also true for *local* equations. However, we have to take into account the tail term arising from the non-local operator. It is hidden in the fact that the First De Giorgi Lemma, as derived in [11, Lemma 4.1], states a non-linear L^2-L^∞ bound, which can be linearised at the cost of a non-local tail term [11, Remark 4.2]. So in reality, what has been done in [7,11] was

$$\underbrace{L^2 + \text{Tail} \xrightarrow{\text{First De Giorgi Lemma}} L^\infty \xrightarrow{\text{Second De Giorgi Lemma}} C^\alpha}_{\implies \text{Weak Harnack (2.1).}},$$

so that reapplying the First De Giorgi Lemma gives a Harnack inequality with tail term

$$\underbrace{\inf + \text{Tail} \xrightarrow{\text{Weak Harnack (2.1)}} L^\zeta + \text{Tail} \xrightarrow{\text{Young's}} L^2 + \text{Tail} \xrightarrow{\text{First De Giorgi Lemma}} L^\infty}_{\implies \text{"Not-so-Strong" Harnack.}},$$

denoted as "Not-so-Strong Harnack" in [11, Theorem 1.3].

Thus our aim is to linearise the L^2-L^∞ bound from [11, Lemma 4.1] by capturing the behaviour of the tail. This yields the Strong Harnack inequality for the Boltzmann equation with moderately soft potentials.

2.2. Main result.

Theorem 2.2 (Strong Harnack Inequality for Boltzmann)**.** *Let $s \in (0,1)$ and $\gamma \in (-d,1]$ be such that $0 < \gamma + 2s \le 2$. Let f be a non-negative solution of the Boltzmann equation (1.1) in $[-3,0]\times B_1 \times B_1$. Assume for all $(t,x) \in [-3,0]\times B_1$ there exists $m_0, M_0, E_0, H_0 > 0$ such that (1.6) is satisfied. Then, there is $C > 0$ depending on s, d, m_0, M_0, E_0, H_0, such that for any $0 < r_0 < 1/6$ the Strong Harnack inequality is satisfied:*

$$\sup_{\tilde{Q}^-_{r_0/4}} f \le C \inf_{Q_{r_0/4}} f, \tag{2.3}$$

where

$$\tilde{Q}^-_{r_0/4} := \left[-\frac{5}{2}r_0^{2s} + \frac{1}{2}\left(\frac{r_0}{2}\right)^{2s} - \left(\frac{r_0}{4}\right)^{2s}, -\frac{5}{2}r_0^{2s} + \frac{1}{2}\left(\frac{r_0}{2}\right)^{2s}\right] \times B_{(r_0/4)^{1+2s}} \times B_{r_0/4}$$

and

$$Q_{r_0/4} := \left[-(r_0/4)^{2s}, 0\right] \times B_{(r_0/4)^{1+2s}} \times B_{r_0/4},$$

see Figure 2.

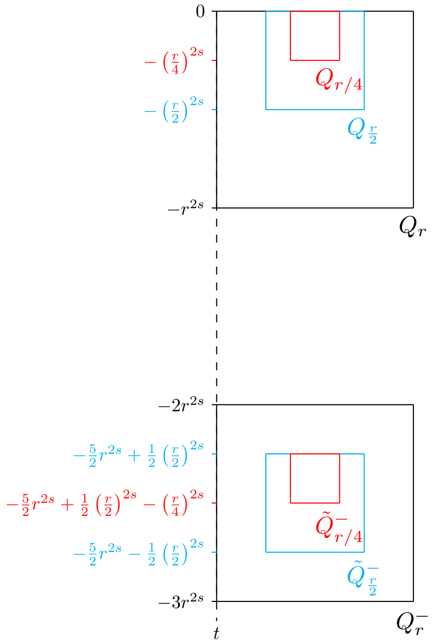

FIGURE 2. The kinetic cylinder Q_r and its past cylinder Q_r^- for some $r > 0$. The dashed vertical line represents the timeline. The blue cylinder $Q_{\frac{r}{2}}$ and its corresponding past $\tilde{Q}^-_{\frac{r}{2}}$ represent the domains appearing in the Weak Harnack inequality, Theorem 2.1. The red cylinder $Q_{r/4}$ and its correpsonding past $\tilde{Q}^-_{r/4}$ illustrates the domains appearing in the Strong Harnack inequality, Theorem 2.2.

3. Conceptual proof ideas

The proof relies on the linearisation of the L^2-L^∞ bound, resulting in the following proposition.

Proposition 3.1 (First De Giorgi Lemma)**.** *Let $s \in (0,1)$ and $\gamma \in (-d, 1]$ be such that $0 < \gamma + 2s \leq 2$. Let $0 < r < R$ and let f be a non-negative solution of the Boltzmann equation* (1.1) *in Q_R. Assume for all $(t,x) \in [-R^{2s}, 0] \times B_{R^{1+2s}}$ there exists $m_0, M_0, E_0, H_0 > 0$ such that* (1.6) *is satisfied. Then there is $C > 0$ depending on s, d, m_0, M_0, E_0, H_0 such that*

$$\|f\|_{L^\infty(Q_r)} \leq C(R-r)^{-(2d(1+s)+2s)} \|f\|_{L^2(Q_R)}.$$

Visually, this corresponds to showing

$$L^2 \; \text{~~+ Tail~~} \xrightarrow{\text{First De Giorgi Lemma}} L^\infty$$

for a non-local equation, from which we derive Theorem 2.2 as a consequence of the chain of relations depicted in (2.2).

Concretely, we set up a De Giorgi iteration in the same vein as originally conceived by De Giorgi [4]. We define increasing levels l_k and decreasing cylinders Q_k by introducing for some $L > 0$

$$l_k = L(1 - 2^{-k}), \qquad r_k = r + 2^{-k}(R - r), \qquad t_k = -r^{2s} - 2^{-k}\left(R^{2s} - r^{2s}\right),$$

so that

$$Q_k = (t_k, 0] \times B_{r_k^{1+2s}} \times B_{r_k}.$$

Then we take the L^2 norm as control quantity

$$A_k = \int_{Q_k} (f - l_k)_+^2 \, \mathrm{d}z. \tag{3.1}$$

The aim is to show that $A_k \to 0$ as $k \to \infty$, for this implies $f \leq L$ almost everywhere in Q_r. To this end, it suffices to derive a nonlinear recurrence relation

$$A_{k+1} \leq C A_k^{1+\beta}, \tag{3.2}$$

for some $\beta > 0$. De Giorgi's argument relies on three steps. First, we use the equation to relate the energy to the control quantity, here the L^2 norm, via an energy estimate. This is done by testing (1.1) with the solution itself and some suitable cutoff. Second, we extract some gain of integrability. For elliptic or parabolic equations, one would use Sobolev's embedding. However, for kinetic equations, since we deal with a degenerate diffusion operator, in the sense that the diffusion only acts on velocity, we have to combine Sobolev's embedding with averaging lemmas, as was done in [5]. Alternatively, we can compare our equation to the fractional Kolmogorov equation, and exploit the gain of integrability known for this constant coefficient equation for (1.1). The gain of integrability relates the L^p norm for some $p > 2$ to the energy of (1.1). Due to the non-locality of (1.1), we naturally have a tail quantity that appears not only in the energy estimate, but also in the gain of integrability. Schematically, these two steps can be visualised as

$$L^2 + \text{Tail} \xrightarrow{\text{Energy estimate}} \text{Energy} + \text{Tail} \xrightarrow{\text{Gain of integrability}} L^p.$$

Third, to extract a nonlinear relation from a linear equation, we use Chebyshev's inequality that relates the measure of a level set function to the control quantity on the next level set.

So far the setup for the De Giorgi argument is rather standard. What remains to be understood, is how to relate the tail to a local quantity. If we manage to derive a local tail bound, then we can treat the tail as we would treat a bounded source term. The fact that we work with level set functions means that any source or tail quantity is multiplied by an indicator function, which by Chebyshev is related to the control quantity. We obtain such a local tail bound by testing the equation with the solution itself *to some inverse power*. The non-local operator is then estimated by separating the singular from the non-singular part. The non-singular part gives on the one hand the tail, on the other hand some local Lebesgue norm of the solution. The singular part is localised due to the cutoff function. Moreover, we split the singular part into the symmetric and the anti-symmetric part of our kernel. The symmetric part has a good sign, and the anti-symmetric part is bounded by (1.9), so that it gives again a local Lebesgue norm on the solution. The transport term is dealt with using integration by parts. Combined with the fact that f solves (1.1), we end up with a bound in $L^q_{t,x}$ on the tail for any $1 \leq q < \infty$.

Incorporating the local bound tail bound in the De Giorgi iteration yields a linear L^2-L^∞ bound for (1.1). The next section demonstrates the local tail bound. Then we perform the De Giorgi iteration.

3.1. **Local tail bound.** For any $0 < r < R$ and $v_0 \in \mathbb{R}^d$ we define the non-local tail term, a quantity that measures the non-singular part of the non-local operator in (1.1), which takes into account function values that lie beyond the solution domain. It is given by

$$\mathbf{T}(f; r, R, v_0) := \sup_{v \in B_r(v_0)} \int_{\mathbb{R}^d \setminus B_R(v_0)} f(w) K(v, w) \, \mathrm{d}w. \tag{3.3}$$

Proposition 3.2. *For given $0 < r < R/2$ and any non-negative super-solution f of* (1.12) *in Q_R with a non-negative, essentially bounded source term $h \geq 0$, and such that the non-negative kernel satisfies* (1.8)-(1.11)*, there holds for any $1 \leq q < +\infty$ and any $\delta > 0$*

$$\left\{ ⨍_{-r^{2s}}^{0} ⨍_{B_{r^{1+2s}}} \Big[\mathbf{T}(f; r, R, 0)\Big]^q \, \mathrm{d}x \, \mathrm{d}t \right\}^{1/q} \leq C R^{-2s+d} \sup_{Q_R} f. \tag{3.4}$$

For the details of the proof we refer the reader to [12, Proposition 3.1]. The main idea is to test against some inverse power of the solution itself.

Proof Sketch. Let $\eta : \mathbb{R}^d \to [0, 1]$ be a smooth cutoff such that $\eta = 1$ in $B_{R/2}$ and $\eta = 0$ outside $B_{3R/4}$. We consider $f_\varepsilon := f + \varepsilon$ for some $\varepsilon > 0$, and test (1.12) with $f_\varepsilon^{-1/2}\eta^2$. Then we split the non-local operator into its singular and non-singular part for fixed $(t, x) \in [-R^{2s}, 0] \times B_{R^{1+2s}}$

$$\begin{aligned}\mathcal{E}(f, \eta f_\varepsilon^{-1/2}) &:= \int_{B_R} \int_{\mathbb{R}^d} \big(f(v) - f(w)\big) \eta^2(v) f_\varepsilon^{-1/2}(v) K(v, w) \, \mathrm{d}w \, \mathrm{d}v \\ &= \underbrace{\int_{B_R} \int_{\mathbb{R}^d \setminus B_R} \dots \, \mathrm{d}w \, \mathrm{d}v}_{=:\mathcal{E}_1} + \underbrace{\int_{B_R} \int_{B_R} \dots \, \mathrm{d}w \, \mathrm{d}v}_{=:\mathcal{E}_2}.\end{aligned} \tag{3.5}$$

Step 1. It is easy to see that

$$\begin{aligned}\mathcal{E}_1\left(f, \eta^2 f_\varepsilon^{-1/2}\right) &= \int_{B_R} \int_{\mathbb{R}^d \setminus B_R} \big(f(v) - f(w)\big) \eta^2(v) f_\varepsilon^{-1/2}(v) K(v, w) \, \mathrm{d}w \, \mathrm{d}v \\ &\leq C R^{-2s} \left\| f_\varepsilon^{1/2} \right\|_{L^1_v(B_R)} - \int_{B_{R/2}} \int_{\mathbb{R}^d \setminus B_R} f_\varepsilon^{-1/2}(v) f(w) K(v, w) \, \mathrm{d}w \, \mathrm{d}v,\end{aligned} \tag{3.6}$$

due to the support of η and the upper bound (1.8).

Step 2. On the other hand, we claim

$$\begin{aligned}\mathcal{E}_2(f, \eta^2 f_\varepsilon^{-1/2}) \leq &C R^{-2s} \left\| f_\varepsilon^{1/2} \right\|_{L^1(B_R)} \\ &- C \int_{B_R} \int_{B_R} \Big[\big(\eta f^{1/4}\big)(w) - \big(\eta f^{1/4}\big)(v)\Big]^2 K(v, w) \, \mathrm{d}w \, \mathrm{d}v.\end{aligned} \tag{3.7}$$

This claim relies on first rewriting $\mathcal{E}_2$ in such a way as to be able to distinguish between the symmetric and the anti-symmetric part of K. The symmetric part has a sign. The anti-symmetric part is of lower order due to the cancellation assumptions (1.9), (1.10), and thus yields the L^1 norm of $f^{1/2}$.

Step 3. We then combine (3.5) with (3.6) from Step 1 and (3.7) from Step 2, so that for every $(t,x) \in [-R^{2s},0] \times B_{R^{1+2s}}$:

$$-\mathcal{E}(f,\eta^2 f_\varepsilon^{-1/2}) \geq -CR^{-2s} \left\| f_\varepsilon^{1/2} \right\|_{L^1(B_R)} + C \int_{B_{R/2}} f_\varepsilon^{-1/2}(v) \Big(\int_{\mathbb{R}^d \setminus B_R} f(w) K(v,w)\,\mathrm{d}w \Big)\,\mathrm{d}v. \tag{3.8}$$

Step 4. In a last step we use the equation (1.12). We take a cutoff $\phi : \mathbb{R}^{1+d} \to [0,1]$ in time and space, such that $\phi = 1$ in a compact subset of $[-(R/2)^{2s},0] \times B_{(R/2)^{1+2s}}$, $\phi = 0$ outside $[-R^{2s},0] \times B_{R^{1+2s}}$, and $\phi(0,\cdot) = 0$, as well as $|\partial_t \phi| \sim R^{-2s}$ and $|v \cdot \nabla_x \phi| \sim R^{-2s}$. Then the support of ϕ coincides with supp $\phi = [-R^{2s},0] \times B_{R^{1+2s}}$, and there holds for almost every t,x

$$-\int_{\mathbb{R}^d} \mathcal{T} f f_\varepsilon^{-1/2} \eta^2 \phi^2 \,\mathrm{d}v - \mathcal{E}\left(f, f_\varepsilon^{-1/2}\eta^2\right)\phi^2 \leq 0,$$

since f is a super-solution of (1.12) and $h \geq 0$. We then take the supremum over time and space, and we use that $\sup(-g) = -\inf g$ and $\inf \leq \fint$, so that

$$0 \geq -\fint_{\mathrm{supp}\,\phi} \int_{\mathbb{R}^d} \mathcal{T} f f_\varepsilon^{-1/2} \eta^2 \phi^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t - \mathcal{E}\left(f, f_\varepsilon^{-1/2}\eta^2\right)\phi^2.$$

For the transport term we integrate by parts, and for the non-local operator we use (3.8), to obtain for almost every t,x

$$\begin{aligned}\Big[\int_{B_{R/2}} f_\varepsilon^{-1/2}(v) \Big(\int_{\mathbb{R}^d \setminus B_R} f(w) K(v,w)\,\mathrm{d}w \Big)\,\mathrm{d}v\Big]\phi^2(t,x) \\ \leq CR^{-2s} \left\| f_\varepsilon^{1/2} \right\|_{L^1(B_R)} \phi^2(t,x) + CR^{-2s} \fint_{\mathrm{supp}\,\phi} \int_{B_R} f_\varepsilon^{1/2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

This implies (3.3) after we integrate over time and space, and take out the infimum of $f_\varepsilon^{-1/2}$ on the left hand side so that using $1/\inf g = \sup g^{-1}$ for any $g > 0$

$$\begin{aligned}&\left\{ \int_{-(R/2)^{2s}}^0 \int_{B_{(R/2)^{1+2s}}} \Big[\int_{B_{R/2}} \Big(\int_{\mathbb{R}^d \setminus B_R} f(w) K(v,w)\,\mathrm{d}w \Big)\,\mathrm{d}v \Big]^q \mathrm{d}x\,\mathrm{d}t \right\}^{1/q} \\ &\leq CR^{-2s+d(1-1/q)} \Big(\sup_{Q_{R/2}} f_\varepsilon^{1/2} \Big) \left\| f_\varepsilon^{1/2} \right\|_{L^q(Q_R)} \\ &\qquad + CR^{-2s+((2s+1)d+2s)((1/q)-1)} \Big(\sup_{Q_{R/2}} f_\varepsilon^{1/2} \Big) \left\| f_\varepsilon^{1/2} \right\|_{L^1(Q_R)} \\ &\leq CR^{-2s+d} \sup_{Q_R} f. \end{aligned}$$

□

3.2. **De Giorgi iteration.** Let $0 < r < R/4$. Define a cutoff in time and space $\psi \in C_c^\infty(\mathbb{R}^{1+d})$ and a cutoff in velocity $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \psi, \eta \leq 1$, with $\psi(t,x) = 1$, $\eta(v) = 1$ for $(t,x,v) \in Q_{r/2}$, and $\psi(t,x) = 0$, $\eta(v) = 0$ for $z = (t,x,v) \in \mathbb{R}^{1+2d} \setminus Q_{R/4}$.

3.2.1. *Energy estimate.* Let $l \in \mathbb{R}$. We test (1.12) with $(f-l)_+(z)\psi^2(t,x)\eta^2(v)$. Writing $z = (t,x,v)$, we get

$$\begin{aligned}\int_{\mathbb{R}^{1+2d}} & h\psi^2(t,x)\eta^2(v)(f-l)_+(z)\,\mathrm{d}z \\ &\geq \int_{\mathbb{R}^{1+2d}} \mathcal{T} f(z)\psi^2(t,x)\eta^2(v)(f-l)_+(z)\,\mathrm{d}z \\ &\quad + \int_{\mathbb{R}^{1+2d}}\int_{\mathbb{R}^d} \big[f(z)-f(t,x,w)\big]K(z,w)(f-l)_+(z)\eta^2(v)\psi^2(t,x)\,\mathrm{d}w\,\mathrm{d}z \\ &\geq \frac{1}{2}\int_{\mathbb{R}^{1+2d}} \mathcal{T}(f-l)_+^2(z)\psi^2(t,x)\eta^2(v)\,\mathrm{d}z \\ &\quad + \int_{\mathbb{R}^{1+2d}}\int_{\mathbb{R}^d} \big[f(z)-f(t,x,w)\big]K(z,w)(f-l)_+(z)\eta^2(v)\psi^2(t,x)\,\mathrm{d}w\,\mathrm{d}z.\end{aligned} \tag{3.9}$$

The core of the energy estimate rests in extracting the relation of the local energy of the equation to the non-local tail term. By local energy we refer to the symmetric part of the non-local operator, which due to the coercivity assumption (1.7) is related to the H^s_v norm of the solution, and in particular has a sign. It is rather standard in the non-local literature to check that there holds the following relation:

$$\begin{aligned}\int_{B_r}\int_{B_r} & \Big[\big(f_{l+}\eta\big)(v)-\big(f_{l+}\eta\big)(w)\Big]^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &\leq 4\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big[f(v)-f(w)\big]f_{l+}(v)\eta^2(v)K(v,w)\,\mathrm{d}w\,\mathrm{d}v \\ &\quad + C(R-r)^{-2s}\left\|f_{l+}^2\right\|_{L^1(B_{R/4})} + C_1\left\|f_{l+}\right\|_{L^1(B_{R/4})}\mathbf{T}\big(f_{l+};r/2,R/4,0\big),\end{aligned} \tag{3.10}$$

where $f_{l+} = (f-l)_+$ and $\mathbf{T}$ is the non-local tail term defined in (3.3). All details can be found in [12, Lemma 4.1]. Thus we obtain

$$\begin{aligned}\sup_{t\in[-r^{2s},0]}\int_{Q_r^t} & (f-l)_+^2(z)\,\mathrm{d}v\,\mathrm{d}x + \int_{Q_r}\int_{B_r}\Big[\big((f-l)_+\eta\big)(v)-\big((f-l)_+\eta\big)(w)\Big]^2 K(v,w)\,\mathrm{d}w\,\mathrm{d}z \\ &\leq C(R-r)^{-2s}\int_{Q_{R/4}}(f-l)_+^2(z)\,\mathrm{d}z + C(R-r)^{2s}\int_{Q_{R/4}} h^2\chi_{f>l}\,\mathrm{d}z \\ &\quad + C(R-r)^{2s}\int_{Q_{R/4}}\mathbf{T}^2(f;r/2,R/4,0)\chi_{f>l}\,\mathrm{d}z.\end{aligned} \tag{3.11}$$

3.2.2. *Gain of integrability.* We obtain the gain of integrability by comparing a solution of (1.12) to the fractional Kolmogorov equation, which is written as

$$(\partial_t + v\cdot\nabla_x)\,f = (-\Delta_v)^s f + h := \int_{\mathbb{R}^d}\frac{|f(w)-f(v)|}{|v-w|^{d+2s}}\,\mathrm{d}w + h, \qquad (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d, \tag{3.12}$$

with source term $h \in L^2(\mathbb{R}_+\times\mathbb{R}^d, H^{-s}(\mathbb{R}^d))$ and initial datum $f_0 \in L^2(\mathbb{R}^{2d})$. This equation admits a fundamental solution, implicitly given through the Fourier transform. In particular, one can extract a gain of integrability from the fundamental solution: any sub-solution f of (3.12) satisfies for any $2 \leq p < 2 + 2s/d(1+s)$

$$\|f\|_{L^p(\mathbb{R}_+\times\mathbb{R}^{2d})} \leq C\left(\|f_0\|_{L^2(\mathbb{R}^{2d})} + \|h\|_{L^2(\mathbb{R}_+\times\mathbb{R}^d;H^{-s}(\mathbb{R}^d))}\right), \tag{3.13}$$

with a constant depending only on s, d. This results from [7, Proposition 2.2] or [11, Proposition 3.4].

For our purposes, we can check that for any $0 < \rho$ and for $\tilde\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\tilde\eta = 0$ outside $B_{r+2\rho}$ and $\tilde\eta = 1$ in $B_{r+\rho}$ there holds on $\mathbb{R}^{1+2d}$

$$
\begin{aligned}
\mathcal{T}\big[(f-l)_+\eta\psi\big] &+ (-\Delta_v)^s\big[(f-l)_+\eta\psi\big] \\
&\le \mathcal{L}\big[(f-l)_+\eta\psi\big] + (-\Delta_v)^s\big[(f-l)_+\eta\psi\big] + h\chi_{f>l}\eta\psi + (f-l)_+\eta\mathcal{T}\psi \\
&\quad + \Big(\int_{\mathbb{R}^d\setminus B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\,\mathrm{d}w\Big)\chi_{f>l}\psi\eta \\
&\quad + \Big(\int_{\mathbb{R}^d\setminus B_\rho(v)} \big(f(w)-l\big)_+ K(v,w)\,\mathrm{d}w\Big)\chi_{f>l}\psi\big(1-\tilde\eta\big) \\
&\quad + \Big(\int_{B_\rho(v)} (f(w)-l)_+ K(v,w)\big(\eta(v)-\eta(w)\big)\,\mathrm{d}w\Big)\chi_{f>l}\tilde\eta\psi \\
&=: H,
\end{aligned} \tag{3.14}
$$

that is $\tilde f := (f-l)_+\eta\psi$ is a sub-solution of (3.12) with source H. In particular, due to the boundedness result of the non-local operator $\mathcal{L} \in H_v^{-s}(\mathbb{R}^d)$, as derived in [7, Theorem 4.1] and [11, Theorem 2.1], we see that $H \in L^2(\mathbb{R}_+ \times \mathbb{R}^d; H^{-s}(\mathbb{R}^d))$, and the first two terms in (3.14) are bounded by the $L^2_{t,x}H^s_v$ norm of $(f-l)_+$, localised due to the cutoffs. Moreover, the term involving the source h, and the transport operator $\mathcal{T}\psi$ are bounded by the localised $L^2_{t,x,v}$ norm of h and of $(f-l)_+$, respectively. Finally, from the three integral quantities appearing in (3.14), the only truly non-local one is the first one, on the third line from above. This is indeed nothing else but the localised $L^2_{t,x,v}$ norm of the tail *on the set where $f > l$.* The other two integral quantities can be shown to be bounded by the localised $L^2_{t,x,v}$ and $L^2_{t,x}H^s_v$ norm of $(f-l)_+$, by using a non-local analogue of the product rule, as obtained in [7, Lemma 4.10, 4.11] and [12, Lemma 2.3]. Overall, if we choose $\rho \sim R - r$, we obtain for any $0 < r \le R/4$

$$
\begin{aligned}
\|H\|_{L^2(R_+\times\mathbb{R}^d, H_v^{-s}(\mathbb{R}^d))} &\lesssim \|(f-l)_+\|_{L^2_{t,x}H^s_v(Q_{R/4})} + (R-r)^{-s}\,\|(f-l)_+\|_{L^2(Q_{R/2})} + \|h\chi_{f>l}\|_{L^2(Q_{R/4})} \\
&\quad + \Big\|\Big(\int_{\mathbb{R}^d\setminus B_\rho(v)} f(w)K(v,w)\,\mathrm{d}w\Big)\chi_{f>l}\eta\Big\|_{L^2(Q_{r+2\rho})} + \rho^{-2s}\,\|(f-l)_+\|_{L^2(Q_{R/4})} \\
&\quad + \rho^{-2s}\,\|(f-l)_+\|_{L^2(Q_{r+2\rho})} + \rho^{-s}\,\|(f-l)_+\|_{L^2_{t,x}H^s_v(Q_{r+2\rho})} \\
&\lesssim (R-r)^{-2s}\,\|(f-l)_+\|_{L^2(Q_R)} + \|h\chi_{f>l}\|_{L^2(Q_R)} + \Big\|\mathbf{T}(f;r,R/2,0)\chi_{f>l}\Big\|_{L^2(Q_R)}.
\end{aligned}
$$

where the second inequality uses the energy estimate (3.11) from the previous step to relate the H^s_v norm to the L^2_v norm up to the tail. Combined with the gain of integrability (3.13) stemming from the fractional Kolmogorov equation (3.12), we thus find for any p such that $2 \le p < 2 + 2s/d(1+s)$

$$
\begin{aligned}
\|(f-l)_+\eta\psi\|_{L^p([0,T]\times\mathbb{R}^{2d})} &\lesssim \big\|\big[(f-l)_+\eta\psi\big](0,\cdot,\cdot)\big\|_{L^2(\mathbb{R}^{2d})} + (R-r)^{-2s}\,\|(f-l)_+\|_{L^2(Q_R)} \\
&\quad + \|h\chi_{f>l}\|_{L^2(Q_R)} + \Big\|\mathbf{T}(f;r,R/2,0)\chi_{f>l}\Big\|_{L^2(Q_R)}.
\end{aligned} \tag{3.15}
$$

3.2.3. *Non-linear recurrence relation.* This is the final step of the De Giorgi iteration. We recall that our goal is to show that the control quantity A_k, defined in (3.1), converges to zero as $k \to \infty$.

We achieve our aim by deriving a non-linear recurrence relation of the form given in (3.2) for some $\beta > 0$. To obtain such a non-linear relation from a linear equation, we use Chebyshev's inequality, which relates the measure of the set $\{f > l_k\} \cap Q_k$ to the control quantity A_k: since $Q_{k+1} \subset Q_k$, there holds

$$|\{f > l_{k+1}\} \cap Q_{k+1}| = \left|\left\{(f - l_k)_+ > 2^{-k-2}L\right\} \cap Q_{k+1}\right| \leq 2^{2k+4}L^{-2}A_k. \tag{3.16}$$

We now use Hölder's inequality for some $2 \leq p \leq \infty$

$$A_{k+1} \leq \left(\int_{Q_{k+1}} (f - l_{k+1})_+^p \,\mathrm{d}z\right)^{2/p} |\{f > l_{k+1}\} \cap Q_{k+1}|^{1-2/p}.$$

We then bound the measure of the indicator function with Chebyshev (3.16), so that for some $t_{k+1/2} \in [t_k, t_{k+1}]$ to be determined, there holds

$$A_{k+1} \leq \left(2^{2k+4}L^{-2}A_k\right)^{(p-2)/p} \left(\int_{[t_{k+1/2},0]\times Q^t_{k+1})} (f - l_{k+1})_+^p \,\mathrm{d}z\right)^{2/p}. \tag{3.17}$$

The L^p norm is then bounded using the gain of integrability from the previous step. We pick $p \in \big[2, 2s/d(1+s)\big)$, so that (3.15) implies

$$\begin{aligned}\|(f - l_{k+1})_+\|_{L^p([t_{k+1/2},0]\times Q^t_{k+1})} &\lesssim \left\|\left[(f - l_{k+1})_+\right](t_{k+1/2}, \cdot, \cdot)\right\|_{L^2(Q^t_k)} \\ &\quad + 2^{2s(k+1)}(R - r)^{-2s} \|(f - l_{k+1})_+\|_{L^2(Q_k)} \\ &\quad + 2^{s(k+1)} \left\|h\chi_{f>l_{k+1}}\right\|_{L^2(Q_k)} \\ &\quad + 2^{s(k+1)}\left\|\mathbf{T}(f; r, R/2, 0)\chi_{f>l_{k+1}}\right\|_{L^2(Q_k)}.\end{aligned} \tag{3.18}$$

We now bound term by term. First, we pick $t_{k+1/2} \in [t_k, t_{k+1}]$ such that

$$\left\|\left[(f - l_{k+1})_+\right](t_{k+1/2}, \cdot, \cdot)\right\|^2_{L^2(Q^t_k)} \leq \frac{1}{t_{k+1} - t_k} \int_{Q_k} (f - l_{k+1})_+^2 \,\mathrm{d}z \leq 2^k(R^{2s} - r^{2s})^{-1}A_k. \tag{3.19}$$

Second, we notice that $f - l_{k+1} \leq f - l_k$ so that

$$\|(f - l_{k+1})_+\|^2_{L^2(Q_k)} \leq A_k. \tag{3.20}$$

Third, we use $h \in L^\infty(Q_R)$, so that by Chebyshev (3.16)

$$\left\|h\chi_{f>l_{k+1}}\right\|^2_{L^2(Q_k)} \leq 2^{2k+4}L^{-2} \|h\|^2_{L^\infty(Q_R)} A_k. \tag{3.21}$$

Fourth, for the tail, we use Hölder's inequality for some $1 \leq q < \infty$, Chebyshev (3.16) and we exploit the local tail bound derived in Proposition 3.2, to find

$$\begin{aligned}\left\|\mathbf{T}(f; r, R, 0)\chi_{f>l_{k+1}}\right\|^2_{L^2(Q_k)} &\leq \left\{\int_{Q_k} \Big(\mathbf{T}(f; r, R, 0)\Big)^{2q} \mathrm{d}z\right\}^{1/q}\left\{\int_{Q_k} \chi_{f>l_{k+1}} \,\mathrm{d}z\right\}^{1-1/q} \\ &\leq C(R - r)^{-4s} \|f\|^2_{L^\infty(Q_R)} \left(2^{2k+4}L^{-2}A_k\right)^{1-1/q}.\end{aligned} \tag{3.22}$$

We combine (3.19), (3.20), (3.21), (3.22) with (3.18) so that

$$\begin{aligned}\|(f - l_{k+1})_+\|^2_{L^p([t_{k+1/2},0]\times Q^t_{k+1})} &\lesssim_k (R - r)^{-4s}A_k + L^{-2} \|h\|^2_{L^\infty(Q_R)} A_k \\ &\quad + (R - r)^{-4s} \|f\|^2_{L^\infty(Q_R)} L^{-(2q-2)/q}A_k^{(q-1)/q}.\end{aligned}$$

Then (3.17) implies for any p such that $2 < p < 2 + 2s/d(1+s)$ and some $1 \le q < \infty$

$$A_{k+1} \lesssim 2^{4k} L^{-(2p-4)/p} A_k^{(p-2)/p} \cdot \Big[(R-r)^{-4s} A_k + L^{-2} \|h\|^2_{L^\infty(Q_R)} A_k + (R-r)^{-4s} \|f\|^2_{L^\infty(Q_R)} L^{-(2q-2)/q} A_k^{(q-1)/q} \Big] .$$

In order to obtain the non-linear recurrence (3.2), we need to ensure that $(q-1)/q > 2/p$. Thus, for any $0 < \varepsilon_0 < (p/2) - 1 < s/d(1+s)$, we pick $q = (1+\varepsilon_0)/\varepsilon_0$.

Finally, we determine how large we can pick $L > 0$ using a barrier argument. We consider $A_k^* := A_0 Q^{-k}$ for $k \ge 0$ and for some $Q > 1$ to be determined. Then, we enforce that A_k^* satisfies the reverse recurrence, that is

$$\begin{aligned} A_{k+1}^* \gtrsim & 2^{4k} L^{-(2p-4)/p} (A_k^*)^{1+(p-2)/p} \Big[(R-r)^{-4s} A_k^* + L^{-2} \|h\|^2_{L^\infty(Q_R)} A_k^* \Big] \\ & + 2^{4k} (R-r)^{-4s} L^{-(2p-4)/p-(q-2)/q} (A_k^*)^{2-1/q-2/p} \|f\|^2_{L^\infty(Q_R)} . \end{aligned} \tag{3.23}$$

We choose Q sufficiently large, and for any $\delta_0 \in (0,1)$, we pick L as

$$\begin{aligned} L = & \delta_0 \|f\|_{L^\infty(Q_R)} + C(\delta_0, Q)(R-r)^{-2spq/(pq-p-2q)} A_0^{1/2} \\ & + C(Q)(R-r)^{-2sp/(p-2)} A_0^{1/2} + \delta_0 \|h\|_{L^\infty(Q_R)} , \end{aligned} \tag{3.24}$$

so that, using Young's inequality, one can check that (3.23) is satisfied. Since, moreover,

$$A_0 \le (R-r)^{4sp/(p-2)} Q^{-p/(p-2)} L^2,$$

we deduce inductively

$$A_k \le Q^{-kp/(p-2)} A_0,$$

so that $A_k \to 0$ as $k \to +\infty$. In particular, for almost every $z \in Q_r$ there holds

$$\begin{aligned} f \le L = & \delta_0 \|f\|_{L^\infty(Q_R)} + C(\delta_0, Q)(R-r)^{-2spq/(pq-p-2q)} A_0^{1/2} \\ & + C(Q)(R-r)^{-2sp/(p-2)} A_0^{1/2} + \delta_0 \|h\|_{L^\infty(Q_R)} . \end{aligned}$$

Recalling that for (1.1), the source is bounded by $\|h\|_{L^\infty(Q_R)} \le C(m_0, M_0, E_0, H_0) \|f\|_{L^\infty(Q_R)}$, we can use a standard covering argument to absorb the source, as well as the first term, on the left hand side, and we recall that $A_0 = \|f\|^2_{L^2(Q_R)}$, so that since the result holds for any ε_0 and p satisfying $0 < \varepsilon_0 < (p/2) - 1 < s/d(1+s)$, we conclude the proof of Proposition 3.1.

3.3. **Proof of the Strong Harnack inequality.** Theorem 2.2 follows from the chain of inequalities depicted in (2.2). Concretely, as a consequence of Proposition 3.1, Young's inequality, the non-negativity of f and the Weak Harnack (2.1) for (1.1), where we recall that, due to the non-negativity of h, the function f is a super-solution of (1.1) with zero source term, so that we obtain for any $\delta_1 \in (0,1)$

$$\sup_{\tilde Q^-_{r_0/4}} f \le C \|f\|_{L^2(\tilde Q^-_{r_0/2})} \le \delta_1 \|f\|_{L^\infty(\tilde Q^-_{r_0/2})} + C(\delta_1) \|f\|_{L^\zeta(\tilde Q^-_{r_0/2})} \le \delta_1 \|f\|_{L^\infty(\tilde Q^-_{r_0/2})} + C(\delta_1) \inf_{Q_{r_0/4}} f.$$

A standard covering argument permits to absorb the first term on the right hand side, and we conclude

$$\sup_{\tilde Q^-_{r_0/4}} f \le C \inf_{Q_{r_0/4}} f.$$

3.3.1. *Consequences.* As a consequence of the Strong Harnack inequality one can derive heat kernel estimates, which in turn give information on the long time behaviour and decay properties of solutions to the equation. The connection between Harnack inequalities and heat kernel bounds has first been discovered by Nash [14] for parabolic equations, and further been developed by Fabes-Stroock [3], and Aronson [1]. For kinetic integro-differential equations, a first result is discussed in [12].

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