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INVARIANT DISTRIBUTIONS IN PARABOLIC DYNAMICS

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ABSTRACT. In the past two decades L. Schwartz theory of distributions has become an essential tool in the study of smooth dynamical systems. In this talk we have focused on two kinds of applications to parabolic systems: smooth rigidity and effective equidistribution. Parabolic systems are zero entropy systems characterized by polynomial divergence of nearby orbits with time. Examples include unipotent flows on quotient of semisimple groups, in particular horocycle flows for hyperbolic surfaces, nilflows and translation flows on surfaces. The rigidity problem asks whether systems which are topologically conjugated are necessarily smoothly conjugated. This problem is motivated by the results of M. Herman and J.-C. Yoccoz on circle diffeomorphisms, but is wide open in higher dimensions. We review results and related conjectures on cohomological equations for flows and propose a general conjecture that states that the structure of the space of its *invariant distributions* determines whether a smoothly stable system is rigid. Effective equidistribution consists in bounds on the speed of convergence of ergodic averages (of smooth functions). We present two results of effective equidistribution for homogenous flows: the first on twisted horocycle flows, the second on a class of nilflows, with applications to questions in analytic number theory. Our method is based on the analysis of the scaling of Sobolev norms of invariant distributions under scaling of the homogeneous structure.

1. INTRODUCTION

In this talk we survey two kind of applications of L. Schwartz theory of distributions to smooth dynamical systems. By the Krylov-Bogolyubov theorem all continuous action of $\mathbb{Z}$ (iteration of invertible continuous maps) or $\mathbb{R}$ (continuous flows) on compact manifolds have probability invariant measures.

Ergodic theory establishes the existence of ergodic measures and their relevance for the long-time behavior of time averages of integrable functions (observables). In the past decades it has become clear that, to establish in smooth dynamics quantitative results on the asymptotic of ergodic averages or correlations of smooth observable, it is necessary to go beyond the space of Borel probability measures, and consider the larger space of distributions, which are not necessarily signed measures, invariant or normalized by the smooth dynamical system.

For *hyperbolic* dynamical systems, characterized by *exponential divergence* of nearby orbits with time, powerful analytical methods to prove exponential decay of correlations have been developed, starting with the seminal work of D. Dolgopyat [Do98] generalized by analytical methods in the work of C. Liverani [Li04], and more recently in the microlocal analysis framework of F. Faure, N. Roy and J. Sjöstrand [FRS08]. In this direction there is a vast and growing literature mostly focused on exponential decay of correlations and other statistical properties of Anosov flows.

In this talk we focus on *parabolic* dynamical systems, for which the divergence of nearby orbits is *at most polynomial*. Examples of parabolic systems arise as flows along the horospherical (stable or unstable) foliations of (partially) hyperbolic diffeomorphisms, and the quantitative ergodicity problem for the parabolic flow is essentially equivalent to the quantitative mixing problem for the hyperbolic diffeomorphism [GL19], [FGL19]. We call such parabolic flows *renormalizable* since they possess an exact self-similarity given by the action of the (partially) hyperbolic map (the renormalization map).

We illustrate, in Section 4 in a couple of examples, an approach to establish *quantitative ergodicity* beyond the renormalizable case. Since the renormalizable case (and even the more general case of approximate self-similarity) appears to be exceptional, it is crucial for the quantitative theory of parabolic flows to go beyond renormalization. This line of research is motivated by the important connections between quantitative ergodicity of parabolic flows (for instance unipotent flows and nilflows) and classical questions in analytic number theory.

In Section 2 we review basic definitions, results and conjectures related to *cohomological equations* and *invariant distributions* for smooth flows.

In Section 3 we examine the relevance of invariant distributions for a different classical problem in smooth dynamics; the problem of smooth rigidity.

A smooth dynamical system is called *smoothly rigid* if any other dynamical system which is topologically conjugate to it, is in fact smoothly conjugated (that is, there exists a C^∞ conjugacy). Smooth rigidity is know for circle diffeomorphisms under a Diophantine condition on the rotation number as a consequence of the celebrated theorem of M. Herman [He79] and J.-C.Yoccoz [Yo84].

Several conjectural examples or sufficient conditions for smooth rigidity have been proposed. R. Krikorian has conjectured that all Diophantine toral flows are at least locally smoothly rigid (that is, smoothly rigid for small perturbations).

In a joint recent work with A. Kanigowski [FK21] we have argued on the basis of examples that the structure of the space of invariant distributions has to be taken into account in the general problem of smooth rigidity.

In this talk we formulate a generalized rigidity conjecture (Conjecture 2.7) which implies that among non-toral homogeneous flows only flows isomorphic to nilflows (and in particular step 2 nilflows) are conjecturally (locally) smoothly rigid.

2. INVARIANT DISTRIBUTIONS

Let $\phi^X_{\mathbb{R}}$ a flow on a compact (finite-volume) manifold generated by a smooth vector field X on M.

Definition 2.1. The *cohomological equation* for the flow $\phi^X_{\mathbb{R}}$ is the linear partial differential equation (for the unknown function u given f smooth)

$$Xu = f\,.$$

Results on the existence of solutions of the cohomological equation have several applications to the dynamical properties of the flow, in particular in relation to time-changes (reparametrizations), linearized conjugacy problems, and bounds on ergodic integrals. *Invariant distributions* can be defined as the obstructions to the existence of smooth solutions:

Definition 2.2. A distribution $D \in \mathcal{E}'(M)$ is called $\phi^X_{\mathbb{R}}$-invariant if

$$XD = 0 \text{ in } \mathcal{E}'(M) \iff D(Xu) = 0\,, \text{ for all } u \in C^\infty(M)\,.$$

For *hyperbolic systems* (characterized by exponential divergence of trajectories), there is an abundance of invariant measures and periodic orbits.

In the uniformly hyperbolic case, the properties of the cohomological equation are described by *Livsic theory* for Hölder solutions [Liv71], and by the work of R. de la Llave, J. Marco, R.Moriyón [LMM86] for smooth solutions. Such results have been extended to the partially hyperbolic non-accessible case by A. Katok and A. Kononenko [KK96] and A. Wilkinson [Wil13]. In the accessible case there are fewer results (for instance [Vee86]).

For *parabolic/elliptic systems* (characterized by at most polynomial divergence of nearby trajectories), the dynamics is often *uniquely ergodic* or has few independent ergodic invariant measures. For homogeneous unipotent flows all ergodic invariant measures are classified by the theory of M. Ratner [Ra90], [Ra91a], [Ra91b]. It is an important problem in parabolic and unipotent dynamics to develop a quantitative version of Ratner theory. Invariant distributions, which are not signed measures, have been proved to exist in several examples. In those examples invariant distributions are closely related to quantitative equidistribution of smooth observables.

Examples 2.3 (Parabolic examples)**.** For a few examples of parabolic maps or flows the theory of cohomological equations and invariant distributions is rather well-understood. The main examples are

- unipotent flows on semi-simple quotients ([Hel70], [FlaFo03], [Wa15]),
- nilflows:homogeneous flows on nilpotent quotients ([Ka03], [FlaFo06]),
- translation flows on higher genus surfaces, rational polygonal billiards ([F97], [MMY05]).

In homogeneous dynamics, the analysis of the cohomological equation is based on methods of *non-Abelian harmonic analysis* (theory of unitary representations), while for translation flows is based on methods of classical harmonic analysis (results boundary behavior of holomorphic functions, see [F97]) or *renormalization* (see [MMY05], [FGL19]).

2.1. **Existence problem and Katok–Hurder conjectures.** For a continuous dynamical system on a compact manifold, a probability invariant measure always exists (by the Krylov–Bogolyubov theorem). Such a system is called *uniquely ergodic* when it has a unique probability invariant measure (hence ergodic averages of continuous functions converge uniformly).

Definition 2.4. A smooth dynamical system on a compact manifold is called *distributionally uniquely ergodic* (DUE) when it has a unique invariant distribution up to scalar multiplication, equal to the unique invariant probability measure.

For example, all (uniquely) ergodic *linear* toral flows are DUE. A. Avila and A. Kocsard [AK11] have generalized this property to all ergodic smooth diffeomorphisms of the circle, hence to all smooth flows on 2-tori without singular points.

Theorem 2.5. [AK11] *All smooth diffeomorphisms of the circle with irrational rotation number are DUE.*

Examples of non-toral DUE flows were constructed by Avila, B. Fayad and Kocsard [AFK15] by the approximation by conjugation method (Anosov-Katok).

Definition 2.6 (Stability)**.** (A. Katok [Ka01], [Ka03]) A flow $\phi^X_{\mathbb{R}}$ on M with generator a smooth vector filed X is *smoothly stable* if the range of the Lie derivative operator

$$L_X(C^\infty(M)) \text{ is closed in } C^\infty(M).$$

We recall that a vector $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ is called Diophantine of exponent $\nu \geq n-1$ if there exists a constant $C(\omega) > 0$ such that

$$|\langle \omega, k\rangle| \geq \frac{C(\omega)}{\|k\|^\nu}, \quad \text{for all } k = (k_1, \dots, k_n) \in \mathbb{Z}^n \setminus \{0\}.$$

Conjecture 2.7 (Cohomological Rigidity)**.** *All flows $\phi^X_{\mathbb{R}}$ which are DUE and stable are smoothly conjugate to Diophantine linear toral flows (hence the non-toral DUE examples mentioned above are not stable).*

The cohomological rigidity conjecture is immediate for flows in dimension 2 and it was proved for flows in dimension 3 independently by the author [F08], A. Kocsard [Ko09] and S. Matsumoto [Mt09]:

Theorem 2.8. *The cohomological rigidity conjecture holds for smooth flows on compact* 3*-manifolds.*

The above result is based on a general result of F. Rodriguez Hertz and J. Rodriguez Hertz:

Theorem 2.9. [RHRH06] *Let $\phi^X_{\mathbb{R}}$ a smooth DUE, smoothly stable flow on a compact manifold M. Then M fibers over a torus $\mathbb{T}^{\beta_1(M)}$ of dimension $\beta_1(M) := dim_{\mathbb{R}} H^1(M, \mathbb{R})$, the first Betti number of M, so that the flow $\phi^X_{\mathbb{R}}$ projects under the fibration to a Diophantine linear flow on $\mathbb{T}^{\beta_1(M)}$.*

After the above theorem, it appears that the most difficult case the proof of cohomological rigidity for flows in dimension 3 is that of flows on homology spheres. Indeed, the argument is a reduction to C. Taubes' proof [Tau07] of the Weinstein conjecture for Reeb flows on contact 3-manifolds (in fact, the Weinstein conjecture for the case of homology spheres suffices).

For homogeneous flows and affine maps on homogeneous spaces the cohomological rigidity conjecture is completely proved:

Theorem 2.10. [FlaFoRH16] *All homogeneous flows/affine maps which are not smoothly conjugate to linear toral flows/translations have an infinite dimensional space of invariant distributions, hence they are not DUE. As a consequence the cohomological rigidity conjecture holds for homogeneous flows and affine maps.*

The following conjecture on stability, proposed by A. Katok, is wide open:

Conjecture 2.11 (Stability)**.** *A smooth flow $\phi_{\mathbb{R}}^X$ is not stable if and only if it has a Liouvillean factor (fast periodic approximations).*

3. Rigidity conjectures

Question 3.1 (M. Herman)**.** *Does there exist a smooth flow which is smoothly stable under small perturbations? Do there exist non-toral flows which are smoothly stable with finite codimension under small perturbations?*

In the above question the notion of stability refers to the conjugacy problem.

Definition 3.2. Two flows $\phi_{\mathbb{R}}^X$ and $\psi_{\mathbb{R}}^Y$ are topologically [resp. C^k] conjugate if there exists a homeomorphism [resp. a C^k diffeomorphism] $h : M \to M$ such that

$$\psi_{\mathbb{R}}^Y = h \circ \psi_{\mathbb{R}}^X \circ h^{-1} .$$

Definition 3.3. A smooth flow $\phi_{\mathbb{R}}^X$ is locally smoothly stable if there exists a neighborhood $\mathcal{U}$ of $\phi_{\mathbb{R}}^X$ in the C^∞ topology such that all flows $\phi_{\mathbb{R}}^Y \in \mathcal{U}$ are C^∞ conjugate to $\phi_{\mathbb{R}}^X$; it is smoothly stable with finite codimension if the local C^∞ conjugacy class is a closed submanifold of finite codimension.

The above question and the theorem of M. Herman [He79] and J.-C. Yoccoz [Yo84] on global conjugacy of circle diffeomorphisms motivate the following

Definition 3.4. Let $\phi_{\mathbb{R}}^X$ be a smooth flow on M.

- The flow $\phi_{\mathbb{R}}^X$ is smoothly rigid if any smooth flow $\psi_{\mathbb{R}}^Y$ which is topologically (C^0) conjugate to $\phi_{\mathbb{R}}^X$ is also smoothly (C^∞) conjugate.
- The flow $\phi_{\mathbb{R}}^X$ has critical regularity $k \geq 0$ if the existence of a C^k conjugacy between any smooth flow $\psi_{\mathbb{R}}^Y$ and $\phi_{\mathbb{R}}^X$ implies the existence of a C^∞ conjugacy.

Motivated by the theorem Herman and Yoccoz, which implies the conjecture for 2-dimensional toral flows, R. Krikorian has proposed the following higher dimensional conjecture.

Conjecture 3.5 (Krikorian's Rigidity Conjecture)**.** *Higher dimensional Diophantine toral flows are smoothly rigid, that is, smooth flows C^0 conjugated to a Diophantine linear flow are C^∞ conjugated.*

K. Khanin's formulated in his ICM talk in 2018 [Kha18] *a generalization for any compact manifold* of Krikorian's conjecture, replacing the Diophantine condition by a lower bound on close returns (quantitative non-periodicity). However Khanin's condition fails even for 2-dimensional tori (see [FK21]).

We propose a different generalized rigidity conjecture which is not contradicted by examples so far. We introduce the following

Definition 3.6. A smooth flow $\phi_{\mathbb{R}}^X$ on M with generator the smooth vector field X is linearly rigid if the Lie derivative operator L_X is hypoelliptic, in the following sense: for every $u \in C^0(M)$, if $L_X u \in C^\infty(M)$ then $u \in C^\infty(M)$.

We remark that the linear rigidity property encodes the structure of the invariant distributions of the flow: a linearly rigid flow does not have invariant distributions of higher (Sobolev or Hölder) order. We can then formulate our conjecture:

Conjecture 3.7 (Generalized rigidity conjecture)**.** *A smooth flow $\phi^X_{\mathbb{R}}$ on M is smoothly rigid if it is stable and linearly rigid.*

The above conjecture seems completely out of reach. In particular, it reduces to Krikorian's conjecture in the case of Diophantine linear toral flows (which are known to be stable and DUE), which already seems extremely difficult.

Examples 3.8 (Non-toral flows)**.**
- The horocycle flow, that is an $SL(2,\mathbb{R})$-unipotent flow, is stable but not linearly rigid (see [FlaFo03]). It has smooth time-changes such that their time-T maps are C^0 but not C^1 conjugate (as well as smooth time-changes such that their time-T maps are C^k, but not C^{k+1} conjugated) so it is not smoothly rigid (see [FK21]).
- Ergodic Heisenberg nilflows (or linear-skew shifts of $\mathbb{T}^2$ defined as

$$(x,y) \to (x+\alpha, y+x) \quad mod.\, \mathbb{Z}^2$$

are stable and linearly rigid (see [Ka01], [Ka03], [FlaFo06]) under a Diophantine condition on the 'frequency' $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.

It has been proved recently [Wa] by a KAM scheme that Diophantine linear-skew shifts of $\mathbb{T}^2$ have critical regularity (at most $3/2$). According to the above generalized rigidity conjecture Diophantine Heisenberg nilflows (or linear-skew shifts of $\mathbb{T}^2$) should be smoothly rigid.

4. ERGODICITY, INVARIANT DISTRIBUTIONS, AND SCALING

In this section we illustrate a heuristic approach to prove effective (polynomial) ergodicity for parabolic flows. The method is based on the analysis of the *scaling of the Sobolev norms of invariant distributions* under appropriate scaling of a Riemannian structure on he phase space of the parabolic flow. Bounds on ergodic integrals are derived under the hypothesis of slow (sub-exponential) degeneration of the Riemannian geometry under the scaling.

This heuristic is motivated by the basic property that for stable flows (in the sense of A. Katok recalled above) smooth functions in the kernel of all invariant distributions are smooth coboundaries, hence have bounded ergodic integrals. Invariant distributions therefore carry deviations of ergodic averages from the mean. This approach is a generalization of the renormalization method, which has been proven to be extremely powerful in special cases.

Let $\phi^X_{\mathbb{R}}$ a smooth volume-preserving (parabolic) flow on a compact (finite volume) manifold M. Suppose $(X,\mathcal{Y}) := \{X, Y_1, \dots, Y_{d+1}\}$ is a frame of TM and consider a (one-parameter) scaling

$$X(t) := e^t X, \quad Y_1(t) := e^{-\rho_1 t} Y_1, \dots, Y_d(t) := e^{-\rho_d t} Y_d$$

(with $\rho_1, \dots \rho_d \geq 0$ and $\sum_{i=1}^d \rho_i = 1$).

Let $R_t(X,\mathcal{Y}) := R_{X(t),Y_1(t),\dots,Y_d(t)}$ be the unique Riemannian metric on M such that the frame $\{X(t),Y_1(t),\dots,Y_d(t)\}$ is orthonormal.

We propose the following heuristics and illustrate it below in several examples:

$$\phi^X_{\mathbb{R}} \text{ (quantitatively) ergodic } \longleftrightarrow R_t(X,\mathcal{Y}) \text{ `diverges slowly'.}$$

The divergence of the metric $R_t(X,\mathcal{Y})$ in the space of Riemannian metrics on M is measured by geometric invariants, such as the injectivity radius or the Cheeger constant [Ch70]. The speed of ergodicity depends on the scaling exponent of the norms of invariant distributions with respect to the Sobolev norms associated to the Riemannian metric $R_t(X,\mathcal{Y})$.

Examples 4.1. There are a few classical examples of the above heuristics.

- Self-similar parabolic flows: the foliation of $\phi^X_{\mathbb{R}}$ is the full unstable (horospherical) foliation of a hyperbolic map A or flow. In this case there is no divergence of the geometry and we have the following guiding principle:

  $$\text{Polynomial Ergodicity of } \phi^X_{\mathbb{R}} \longleftrightarrow \text{Exponential mixing for } A$$

- Renormalizable parabolic flows (linear flows on tori, translation flows, Heisenberg nilflows). In this case the scaling defines a recurrent (volume preserving) hyperbolic renormalization flow on a moduli space.

  The *deviation exponents* of ergodic integrals from the mean are given by the *Lyapunov exponents* of a cocycle on a bundle of *invariant distributions*. The geometry is controlled by a *Diophantine condition* (a more or less explicit bound on the speed of divergence of the geometry).

4.1. **Translation flows.** Translation flows are linear flows on translation surfaces. Translation surfaces can be defined as closed orientable surface endowed by an atlas whose coordinate changes (outside of a finite set of singularities) are given by translation of the plane, or alternatively as flat surfaces with trivial holonomy, or finally as Riemann surfaces endowed with a holomorphic 1-form (Abelian differential). It can be proved that all the above characterization are equivalent.

4.1.1. *Unique ergodicity.* For translation flows the crucial *Masur's unique ergodicity criterion* [Ma92] roughly states that *recurrence in moduli space under renormalization (Teichmüller flow) implies unique ergodicity*. The Masur's criterion implies that any Teichmüller invariant probability measure is supported on the set of Abelian differentials with uniquely ergodic vertical and horizontal foliations.

4.1.2. *Deviations of ergodic averages.* The quantitative unique ergodicity behavior of translations flows are expressed by the *Kontsevich–Zorich deviations*, discovered by A. Zorich in numerical experiments and conjecturally explained by A. Zorich and M. Kontsevich in terms of the Lyapunov exponents of a renormalization cocycle (now called the Kontsevich–Zorich cocycle).

The Zorich–Kontsevich conjectures were later proved, for translation flows and Interval Exchange Transformations, by the author [F02], [F06], [F11] and A. Avila and M. Viana [AV07]. The complete proof of the original conjectures for smooth area preserving flows with Morse saddles has been given only recently by K. Frączek and C. Ulcigrai [FrU21]).

The *Zorich deviation phenomenon* can be stated as follows (from [F02] or [FrU21]). Let $\phi^X_{\mathbb{R}}$ be a generic area-preserving flow on a surface of genus $g \geq 1$ with Morse saddles. There exist *invariant distributions* $D_1 = \text{area}, D_2, \dots, D_g$ and numbers $\lambda_1 = 1 > \lambda_2 > \cdots > \lambda_g > 0$ (*Zorich exponents*) such that, for all $f \in C^\infty(M)$ such that $f = 0$ at the singular set, and for almost all $x \in M$,

$$\int_0^T f \circ \phi^X_t(x) dt = D_1(f) T + \sum_{j=2}^{g} D_j(f) T^{\lambda_j + o_x(T)} + O_\varepsilon(T^\varepsilon)$$

4.2. **Non-renormalizable parabolic systems.** Most parabolic dynamical systems are not renormalizable (in the above sense). Such non-renormalizable examples include

- non-horospherical unipotent flows, that is, unipotent flows other than $SL(2,\mathbb{R})$ unipotents;
- nilflows on higher steps nilmanifolds and affine maps;
- billiards in general polygons (beyond the rational case).

Non-horospherical unipotent flows are normalized by a hyperbolic diagonal flow, but their orbit foliations does not coincide with the full unstable (horospherical) foliation of the hyperbolic flow, hence the terminology.

In recent breakthroughs effective (polynomial) equidistribution has been proved for non-horospherical unipotent flows in quotients of $\mathrm{SL}(2,\mathbb{R})^2$ and $\mathrm{SL}(2,\mathbb{C})$ by E. Lindenstrauss, A. Mohammadi, and Zhiren Wang [LMW23] and of $\mathrm{SL}(3,\mathbb{R})$ by Lei Yang [Ya23].

In all other cases mentioned above (in particular higher step nilflows and polygonal billiards flows), no hyperbolic dynamical systems which normalizes parabolic flow is known. We have considered two cases:

- *Twisted horocycle flows*: homogeneous flows $\phi^X_{\mathbb{R}}$ generated by vector fields of the form $X = U + \lambda \partial/\partial\theta$ on $M = \Gamma\backslash \mathrm{SL}(2,\mathbb{R}) \times (\mathbb{R}/\mathbb{Z})$ with $\lambda \in \mathbb{R} \setminus \{0\}$ and U the generator of a unipotent subgroup of $\mathrm{SL}(2,\mathbb{R})$ (horocycle flow);
- *Ergodic filiform nilflows*: ergodic nilpotent flows $\phi^X_{\mathbb{R}}$ on quotient of nilpotent Lie group with filiform Lie algebra

$$[X,Y_1] = Y_2, \dots [X,Y_i] = Y_{i+1}, \dots [X,Y_k] = 0.$$

This choice of examples is motivated by the remark that, on the one hand, in either case there is no natural hyperbolic dynamical systems which normalizes the parabolic flow and, on the other hand, independent results from analytic number theory give benchmarks to evaluate our scaling approach to effective ergodicity.

4.2.1. *Twisted horocycles and cusp forms.* The twisted (horocycle) flow is a product of a (horocycle) flow and a linear flow on the circle $\mathbb{R}/\mathbb{Z}$. We have proved by our scaling method the following effective equidistribution result.

Theorem 4.2. [FFT16] *Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be a lattice and let $M = \Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ be compact, or M be non-compact and let x belong to a (closed) cuspidal horocycle* *of length $T \geq 1$ such that $\lambda T \in 2\pi\mathbb{Z}$. There exists $C_s := C_s(M) > 0$ such that, if $|\lambda T| \geq e$,*

$$|\int_0^T e^{i\lambda t} f \circ \phi_t^U(x)dt| \leq C_s \|f\|_s (1 + \frac{1}{|\lambda|^{1/6}}) T^{5/6} \log^{1/2}(|\lambda T|) .$$

As a consequence, following a method of A. Venkatesh [Ven10] we derive the bounds on Fourier coefficients of cusp forms stated below.

We recall the definition of a *cusp form*, in the holomorphic and Maass cases.

Definition 4.3 (Cusp forms)**.** Let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ denote a non co-compact lattice. A cusp form for Γ is a function f on the upper half plane which is modular, that is, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$f\left(\frac{az+b}{cz+d}\right) = \begin{cases} (cz+d)^k f(z) & \text{(holomorphic case)} \\ (\frac{cz+d}{|cz+d|})^k f(z) & \text{(Maass case)} \end{cases} ,$$

satisfies a (cuspidal) growth conditions as the imaginary part of the argument diverges, and has vanishing constant coefficient in its Fourier expansion.

In fact, a cusp form f has a Fourier expansion of the form

$$f(z) = \begin{cases} \sum_{n>0} f_n \exp(2\pi i n z) & \text{(holomorphic case)} \\ \sum_{n \in \mathbb{Z}\setminus\{0\}} f_n W_{\frac{n}{|n|}\frac{k}{2}, i\eta}(4\pi|n|y) e^{2\pi i n x} & \text{(Maass case)} \end{cases}$$

where $W_{\frac{n}{|n|}\frac{k}{2}, i\eta}$ is a Whittaker function.

Corollary 4.4. *Let $\{f_n\} \subset \mathbb{C}$ denote the sequence of the Fourier coefficients of a holomorphic or Maass (non-holomorphic) cusp form f of integral weight k for a non co-compact lattice $\Gamma \subset SL(2,\mathbb{R})$. There is a constant $C_f > 0$ such that for all $n \in \mathbb{Z}\setminus\{0\}$ we have*

$$|f_n| \leq C_f |n|^{k/2-1/6} (1 + \log|n|)^{1/2} .$$

The above results are the best to date for *general lattices*: in the holomorphic case our result was first proved by A. Good, [Go81], without the logarithmic factor; in the Maass case, by J. Bernstein and A. Reznikov [BeRe99], up to a term $O_\varepsilon(n^\varepsilon)$.

We recall that for *congruence lattices*

$$|f_n| \leq C_f |n|^{k/2-1/2+\varepsilon} \quad \text{(Ramanujan-Petersson conjecture)}$$

(P. Deligne and J.-P. Serre [DS74], Deligne [De69], [De74], as a consequence of the Weil conjectures, in the holomorphic case).

4.2.2. *Other related questions.* Effective equidistribution results have applications to sparse equidistribution problems (again following Venkatesh [Ven10]).

For instance, from bounds on twisted ergodic integrals we have derived bounds on ergodic averages for horocycle maps, hence a (minor) improvement on a question of N. Shah:

Question 4.5 (N. Shah's question)**.** *For $\gamma \in (0,\gamma_0)$, does for any $f \in C(M)$*

$$\frac{1}{N}\sum_{n=0}^{N-1} f(h_{n^{1+\gamma}}(x)) \to \int_M f d\mathrm{vol}?$$

The state of the art on this question consists in efforts to improve on the threshold γ_0. Shah conjectured that one can take $\gamma_0 = +\infty$, but even $\gamma_0 = 2$ is beyond the current techniques. In fact, the best results to date are as follows:

(A. Venkatesh, [Ven10] : $\gamma_0 \in (0, 1/48)$ with a precise bound depending on the spectral gap of the regular representation; J. Tanis and P. Vishe [TaVi15]: $\gamma_0 = 1/26$ independent on the spectral gap; L. Flaminio, the author and J. Tanis [FFT16]: $\gamma_0 = 1/13$ independent of the spectral gap).

Another related, but harder, sparse equidistribution problem concerns the classical horocycle flow at prime-times (P. Sarnak and A. Ubis, [SU15])

4.2.3. *Nilflows and Weyl sums.* The following is an effective equidistribution result for a class of higher step nilflows (which are suspensions of linear skew-shifts maps on tori), based on the method of scaling of invariant distributions.

Let $DC(\nu) \subset \mathbb{R}$ denote the set of all Diophantine 2-dimensional vectors of exponent $\nu \geq 1$, that is,

$$DC(\nu) := \{\omega \in \mathbb{R}^2 | \inf_{k \in \mathbb{Z}^2} \|k\|^\nu |\langle k, \omega\rangle| > 0\}.$$

Theorem 4.6. [FlaFo23] *Let $\{\phi^X_t\}$ be a nilflow on a k-step filiform nilmanifold M which projects to a linear toral flow on the 2-dimensional torus $\mathbb{T}^2$ with Diophantine frequency in the set $\cap\{DC(\nu) | \nu > k/2\}$. Let $\sigma > k^2$. For every $\varepsilon > 0$ there exists a constant $K_{\sigma,\varepsilon} > 0$ such that for every function $f \in W^\sigma(M)$ and for all $(x,T) \in M \times \mathbb{R}^+$,*

$$\left|\frac{1}{T}\int_0^T f \circ \phi^X_s(x)\, ds - \int_M f \mathrm{vol}\right| \leq K_{\sigma,\varepsilon} T^{-\frac{1}{k(k-1)}+\varepsilon} \|f\|_\sigma.$$

A general bound for on ergodic integrals for polynomial sequences on nilmanifolds, with polynomial (power) rate was proved by B. Green and T. Tao [GT12] by a far reaching generalization of the Van der Corput method. Although their stated results does not include estimates on the deviation exponents, presumably their method cannot go beyond the Weyl exponent $-1/2^{k-1}$ (instead of $-1/k(k-1)$) which decays (to zero) exponentially with the step $k \geq 2$ of the nilmanifold.

From the above theorem we have derived the bound on *Weyl sums* stated below.

Definition 4.7 (Weyl sums)**.** Given a real polynomial $P_k(a_k, \ldots, a_0, X) = \sum_{j=0}^k a_j X^j$ of degree $k \geq 2$, the Weyl sums for P_k are the exponential sums

$$W_N(a_k, \ldots, a_0) = \sum_{n=0}^{N-1} \exp[2\pi i P_k(a_k, \ldots, a_0; n)], \quad \text{for all } N \in \mathbb{N}.$$

Corollary 4.8. [FlaFo23] *Let* $a_k \in \cap\{DC(\nu) | \nu > k/2\}$. *For every* $\varepsilon > 0$, *there exists a constant* $K_\varepsilon > 0$ *such that for all* $(a_0, \dots a_{k-1}) \in \mathbb{R}^k$ *and for all* $N \geq 1$,

$$|W_N(a_k, \dots, a_0)| \leq K_\varepsilon N^{1 - \frac{1}{k(k-1)} + \varepsilon}.$$

This bound follows from the proof of the *Vinogradov Main Conjecture* by J. Bourgain, C. Demeter and L. Guth [BDG16], and T. Wooley [Wo18] with the same exponent but under a weaker Diophantine condition of exponent $\nu \leq k-1$ (see Bourgain survey's article [Bou17]).

4.3. **Outline of the proofs: the twisted horocycle flow.** We outline below the main steps in our proof [FFT16] of effective equidistribution for the twisted horocycle flow. The argument relies on the analysis of the cohomological equation and invariant distributions for the flow and their scaling, carried out based on the theory of unitary representations of $\mathrm{SL}(2,\mathbb{R})$, and on bounds on the geometry, that is, bounds on a geometric invariant analogous to a point-wise injectivity radius of the scaled metric.

4.3.1. *Scaling of invariant distributions.* Let $M = \Gamma \backslash \mathrm{SL}(2,\mathbb{R})$ and let (G,U,V) denotes the frame of TM given by generators of the geodesic flow, unstable and stable horocycle flows respectively.

The analysis of the cohomological equation and invariant distributions for the generator $X := U + \lambda \frac{\partial}{\partial\theta}$, that is,

$$(U + \lambda \frac{\partial}{\partial\theta})u = f\,.$$

is carried out based on the theory of unitary representations of $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}$ which is well understood.

We then consider for $t > 0$ the scaled Riemannian metrics $R_t(X, \mathcal{Y})$ defined as the unique metrics such that the frame of $T(\Gamma \backslash \mathrm{SL}(2,\mathbb{R}) \times \mathbb{T})$

$$(e^t (U + \lambda \frac{\partial}{\partial\theta}), e^{-\rho_1 t} G, e^{-\rho_2} V, e^{\rho_3 t} \frac{\partial}{\partial\theta})$$

is orthonormal and we estimate the scaling exponents of the Sobolev norms of X-invariant distributions with respect to the Sobolev norms induced by the scaled Riemannian metrics $R_t(X, \mathcal{Y})$ as $t > 0$ diverges.

The *scaling exponent* of Sobolev norms of invariant distributions is equal to $1/6$ under the *optimal scaling*

$$(U + \lambda \frac{\partial}{\partial\theta}, G, V, \frac{\partial}{\partial\theta}) \to (e^t (U + \lambda \frac{\partial}{\partial\theta}), e^{-t/3} G, e^{-2t/3} V, \frac{\partial}{\partial\theta})$$

4.3.2. *Average width and Sobolev trace theorem.* Averages along orbit segments are estimated in terms of Sobolev norms by a dynamically adapted version of the Sobolev trace theorem. We introduce a notion of *average width* of an orbit segment with respect to the scaled metric.

Definition 4.9. For any orbit segment starting at $x \in M \times \mathbb{T}$ of length $T > 0$ for the flow generated by $X(t) := e^t(U + \lambda \partial/\partial\theta)$, let $\Omega_t := \Omega_t(x,T)$ be any embedded neighborhood and let $w_{\Omega_t}(\tau)$ be the volume of the transverse section of Ω_t along the orbit segment, with respect to transverse the frame $Y(t) := (e^{-t/3}X, e^{-2t/3}V, \partial/\partial\theta)$. The average width is the integral

$$w_t(x,T) := \sup_{\Omega_t}\Big(\frac{1}{T}\int_0^T \frac{1}{w_{\Omega_t}(\tau)}d\tau\Big)^{-1}.$$

A generalized Sobolev trace theorem states that (for $s > 2$)

$$\Big\|\frac{1}{T}\int_0^T f \circ \Phi_\tau^{e^t(U+\lambda\partial/\partial\theta)}(x)d\tau\Big\|_{C^0} \leq C_s T^{-1/2}\frac{1}{w_t(x,T)^{1/2}}\|f\|_{s,Y(t)}$$

4.3.3. *Bounds on the geometry.* The average width of uniformly bounded orbit segments is similar to an injectivity radius of the scaled metric. Bounds on the average width are therefore bounds on the geometry of the scaled metric at a given point. From a dynamical point of view, such bounds can be viewed as Diophantine-type bounds on close returns of a given orbit.

Theorem 4.10. *There exists a constant $K_\Gamma > 0$ and a function $c_\Gamma(x,T)$ encoding the 'Diophantine conditions' such that the following holds. For any $x \in M$, for any $T \geq 1$ and $t > 0$, there is an open embedded tubular neighborhood $\Omega_{t,T}(x)$ of the orbit-segment starting at x of length $T > 0$ such that the following estimate holds*

$$\frac{1}{T}\int_0^T \frac{1}{w_{\Omega_{t,T}}(\tau)}d\tau \leq K_\Gamma \cdot c_\Gamma^2(x, e^tT)T(1 + \log(e^{t/3}T)).$$

The proof of the above result is quite straightforward given the *linear* divergence of nearby orbits of the (twisted) horocycle flow.

4.4. **Outline of the proofs: filiform nilflows.** We conclude with an outline of the main steps in our proof [FlaFo23] of effective equidistribution for the twisted horocycle flow. In particular, we explain how the scaling factor is determined and how the control of the geometry is achieved in this case. The general outline of the argument is similar to the case of twisted horocycle.

4.4.1. *Scaling of invariant distributions.* The k-step filiform Lie algebra is determined by the commutation relations

$$[X, Y_1] = Y_2, \quad \ldots, \quad [X, Y_{k-1}] = Y_k, \quad [X, Y_k] = 0.$$

Fir the sake of clarity of exposition, we restrict below to unitary representations of filiform nilmanifolds are of the form (written on the Lie algebra)

$$X \to \frac{d}{dx}, \quad Y_1 \to i\frac{\lambda_k}{(k-1)!}x^{k-1}, \quad Y_j \to i\frac{\lambda_k}{(k-j)!}x^{k-j}, \quad Y_k \to i\lambda_k,$$

on the Hilbert space $L^2(\mathbb{R})$, hence

$$X(t) \to e^t\frac{d}{dx}, \quad \ldots, \quad Y_j \to i\frac{\lambda_k}{(k-j)!}(e^{-\frac{2t}{k(k-1)}}x)^{k-j}, \quad \ldots, \quad Y_k \to i\lambda_k.$$

In general, the vector fields $Y_1, \dots, Y_k$ are represented by arbitrary polynomials $p_1(x), \dots, p_k(x)$ of degrees $k-1, \dots, 0$ respectively, with leading coefficients

$$i\lambda_k/(k-1)!, \dots, i\lambda_k/(k-j)!, \dots, i\lambda_k .$$

The space of X-invariant distributions is one-dimensional generated by

$$D(\hat{f}) = \int_{\mathbb{R}} \hat{f}(x)dx .$$

For every $\sigma \in \mathbb{R}$, let $W^{\sigma}_{Y_1,\dots,Y_k}$ denote the L^2 Sobolev space with respect to the frame $\mathcal{Y} := \{Y_1, \dots, Y_k\}$, transverse to the flow direction. Let $\mathcal{Y}(t) := \{Y_1(t), \dots, Y_k(t)\}$ denote the rescaled frame. We want to compute

$$\|D\|_{W^{-\sigma}_{\mathcal{Y}(t)}} := \sup_{\hat{f}\neq 0} \frac{|D(\hat{f})|}{\|\hat{f}\|_{W^{\sigma}_{\mathcal{Y}(t)}}} .$$

Let

$$\hat{f}_t(x) = e^{-\frac{t}{k(k-1)}} \hat{f}(e^{-\frac{2t}{k(k-1)}} x) , \qquad \text{for all } x \in \mathbb{R} .$$

Then, by change of variables,

$$\|\hat{f}_t\|_{W^{\sigma}_{\mathcal{Y}(t)}} = \|\hat{f}\|_{W^{\sigma}_{\mathcal{Y}}}$$

and

$$D(\hat{f}_t) = e^{-\frac{t}{k(k-1)}} \int_{\mathbb{R}} \hat{f}(e^{-\frac{2t}{k(k-1)}} x)dx = e^{\frac{t}{k(k-1)}} D(\hat{f}) ,$$

so that

$$\sup_{\hat{f}_t\neq 0} \frac{|D(\hat{f}_t)|}{\|\hat{f}_t\|_{W^{\sigma}_{\mathcal{Y}(t)}}} = e^{\frac{t}{k(k-1)}} \sup_{\hat{f}\neq 0} \frac{|D(\hat{f})|}{\|\hat{f}\|_{W^{\sigma}_{\mathcal{Y}}}}$$

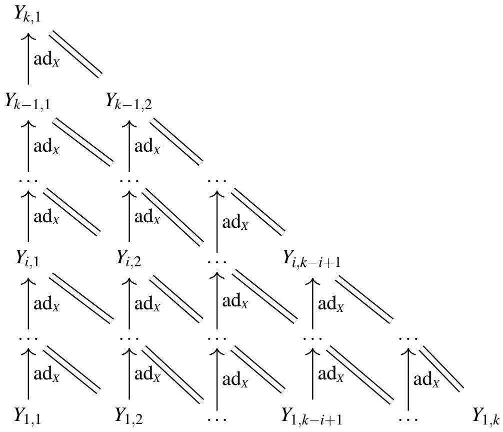
The following scaling optimizes the scaling of Sobolev norms of invariant distributions in all irreducible representations:

$$\rho_t \begin{pmatrix} X \\ Y_1 \\ \dots \\ Y_j \\ \dots \\ Y_k \end{pmatrix} = \begin{pmatrix} X(t) \\ Y_1(t) \\ \dots \\ Y_j(t) \\ \dots \\ Y_k(t) \end{pmatrix} := \begin{pmatrix} e^t X \\ e^{-\frac{2}{k}t} Y_1 \\ \dots \\ e^{-\frac{2(k-j)}{k(k-1)}t} Y_j \\ \dots \\ Y_k \end{pmatrix} .$$

The Sobolev norms of invariant distributions then scale by a factor $e^{-\frac{1}{k(k-1)}t}$ in the sense that

$$\|D\|_{W^{\sigma}_{\mathcal{Y}(t)}} \approx e^{\frac{1}{k(k-1)}t} \|D\|_{W^{\sigma}_{\mathcal{Y}}} .$$

4.4.2. *Bounds on the geometry for filiform nilmanifolds.* In this case the whole problem is lifted to a suitable covering nilmanifold $\hat{M}$ with Lie algebra:



The slanted symbol $=$ denotes the identifications induced by an ideal $\mathcal{I}$ of the Lie algebra. The quotient of the above Lie algebra by $\mathcal{I}$ is the filiform Lie algebra.

The ideal $\mathcal{I}$ is generated by

$$\{V_{i,j} := Y_{i,j} - Y_{i+1,j-1} | 1 \leq i \leq k-1, 2 \leq j \leq k-i+1\}$$

hence the Lie algebra is generated by $\{X, Y_{1,1}, \ldots, Y_{1,k}, V_{i,j}\}$. The scaling is

$$X \to e^t X\,, \quad \ldots, \quad Y_{1,j} \to e^{\frac{2(k-j)}{k-1}t} Y_{1,j}\,, \quad \ldots, \quad V_{i,j} \to V_{i,j}\,.$$

The Abelianized torus is generated by the projections of $\{X, Y_{1,1}, \ldots, Y_{1,k}\}$, hence *the geometry of the covering nilmanifold $\hat{M}$ is controlled by that of the Abelianized torus* (of dimension $k+1$). This gives a higher dimensional Diophantine condition from which a one-dimensional Diophantine condition is derived by coordinate projection in the space of lattices.

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