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GEOMETRIC OPTICS APPROXIMATION FOR THE EINSTEIN VACUUM
EQUATIONS

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Geometric optics approximation for the Einstein vacuum equations

Arthur Touati *

Abstract

We present recent works on the construction of high-frequency solutions to the Einstein vacuum equations in general relativity, based on [Tou22b, Tou23]. In these articles, the author shows the existence of a family of vacuum spacetimes $(g_\lambda)_{\lambda \in (0,1]}$ in generalised wave gauge oscillating at frequency λ^{-1} and defined by a geometric optics ansatz. In this survey, we first review the Cauchy theory for the Einstein vacuum equations in wave gauge, geometric optics and Burnett’s conjecture in general relativity. This will motivate our main result, for which we provide a sketch of proof high-lighting both quasi-linear and semi-linear challenges.

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1 The Einstein vacuum equations

In [Tou22b, Tou23], the author constructs oscillating solutions to the Einstein vacuum equations, which are at the heart of the theory of general relativity and were introduced by Einstein in [Ein15].

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1.1 General relativity

After Einstein introduced special relativity in his *annus mirabilis* 1905, he and his collaborators felt the need to include gravitation in their new description of reality. Despite its many successes, the (at the time) standard theory of gravitation from Newton failed to match special relativity's main requirement on the finiteness of any interaction's speed of propagation, including gravitational interaction. Years of computations and thought experiments led the team of physicists to a geometric description of both space and time, and thus gravitation. We now present some aspects of this theory as they are known today from mathematicians, and we refer to [CB09] for a very complete presentation.

In general relativity, the spacetime is modeled by a 4-dimensional manifold \mathcal{M} endowed with a Lorentzian metric g , that is a metric with signature $(-, +, +, +)$. In order to describe a physical spacetime with matter, the metric g is required to solve the Einstein equations

$$R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = T_{\mu\nu}. \quad (1.1)$$

In (1.1), $R_{\mu\nu}(g)$ is the Ricci tensor of g , $R(g)$ is the scalar curvature defined as the trace $g^{\alpha\beta}R_{\alpha\beta}(g)$ of the Ricci tensor, and $T_{\mu\nu}$ is the energy-momentum tensor describing the energy and matter in the spacetime. The Einstein equations thus describes how energy and matter are the source of curvature and of geometry. The most simple choice of energy-momentum tensor is the one describing vacuum, i.e., $T = 0$. In this case, (1.1) actually reduce to the following Einstein vacuum equations

$$R_{\mu\nu}(g) = 0. \quad (1.2)$$

Despite the absence of source term, solutions to (1.2) might have a very rich geometry. This goes against the second half of Wheeler's famous quote from [MTW73]

“Space tells matter how to move. Matter tells space how to curve.”

Obvious examples of this fact are the various black holes models, like Schwarzschild or Kerr, which are both explicit (families of) stationary solutions to (1.2). Another, much simpler, example of solution to (1.2) is the Minkowski spacetime (\mathbb{R}^{3+1}, m) from special relativity, where

$$m = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Another choice of energy-momentum tensor is

$$T_{\alpha\beta}^{\text{mV}} = \int_{g^{-1}(p,p)=0} f(x,p)p_\alpha p_\beta d\mu,$$

where $d\mu$ is the measure on the cotangent bundle. It depends on a density of massless particles interacting through gravity only. This interaction is described by the Vlasov equation

$$p^\alpha \partial_\alpha f - p^\alpha p^\beta \Gamma_{\alpha\beta}^\rho \partial_{p^\rho} f = 0, \quad (1.3)$$

coming from $\text{div}T^{\text{mV}} = 0$, which itself comes from the LHS of (1.1) being divergence free (a consequence of the contracted Bianchi identities). If we put (1.1) and (1.3) with $T = T^{\text{mV}}$ together, we obtain the massless Einstein-Vlasov system for the unknowns (\mathcal{M}, g, f) :

$$\begin{cases} R_{\alpha\beta}(g) = \int_{g^{-1}(p,p)=0} f(p)p_\alpha p_\beta d\mu, \\ p^\alpha \partial_\alpha f - p^\alpha p^\beta \Gamma_{\alpha\beta}^\rho \partial_{p^\rho} f = 0. \end{cases} \quad (1.4)$$

1.2 Cauchy problem in wave coordinates

From the generic expression of the Ricci tensor in terms of the metric, we see that (1.2) is a system of second order PDEs for the metric coefficients. However the nature of this system is not clear at all. To unveil it, we benefit from the gauge invariance of (1.2) and choose a particular coordinate system.

Lemma 1.1. *In any coordinate system $(x^\alpha)_{\alpha=0,\dots,3}$, we have*

$$2R_{\alpha\beta}(g) = - {}_g g_{\alpha\beta} + g_{\rho(\alpha} \partial_{\beta)} H^\rho + P_{\alpha\beta}(g)(\partial g, \partial g), \quad (1.5)$$

with

$$g = g^{\mu\nu} \mathbf{D}_\mu \mathbf{D}_\nu, \quad H^\rho = g^{\mu\nu} \Gamma_{\mu\nu}^\rho,$$

where \mathbf{D} is the covariant derivative and $P_{\alpha\beta}(g)(\partial g, \partial g)$ is a quadratic non-linearity.

This lemma shows that $R_{\alpha\beta}(g)$ is an hyperbolic operator for the metric coefficients, up to the term $g_{\rho(\alpha} \partial_{\beta)} H^\rho$ which also contains second order derivatives of some metric coefficients. One can check that $H^\rho = - {}_g x^\rho$. Choosing coordinates satisfying ${}_g x^\rho = 0$ thus transforms (1.2) into a system of quasi-linear wave equations for the metric coefficients with a quadratic non-linearity. This is good news since the analysis and the Cauchy problem for such systems is well understood (see for example [Sog95]). However, we need the solution g to define the gauge, and the gauge to define the solution g ...

To unravel this problem, let us first define the initial data for (1.2), taking inspiration from second order hyperbolic equations. An initial data set for (1.2) is composed of a 3-dimensional manifold Σ_0 endowed with a Riemannian metric \bar{g} and a symmetric 2-tensor K . A development of (Σ_0, \bar{g}, K) is defined as a Lorentzian manifold (\mathcal{M}, g) solving (1.2) and such that Σ_0 is a spacelike hypersurface of \mathcal{M} and \bar{g} and K are the first and second fundamental form respectively of the embedding $\Sigma_0 \hookrightarrow \mathcal{M}$. Together with (1.2), these last two requirements imply that (\bar{g}, K) must satisfy the constraint equations

$$\begin{cases} R(\bar{g}) + (\text{tr}_{\bar{g}} K)^2 - |K|_{\bar{g}}^2 = 0, \\ -\text{div}_{\bar{g}} K + \text{dtr}_{\bar{g}} K = 0. \end{cases} \quad (1.6)$$

The constraint equations form a system of non-linear elliptic equations on Σ_0 for (\bar{g}, K) . The procedure to construct a local solution to (1.2) in wave coordinates goes then as follows.

1. We fix Σ_0 (consider \mathbb{R}^3 for simplicity) and first obtain (\bar{g}, K) by solving (1.6). We will give more details on this step in Section 4.2.
2. We define initial data for the system

$${}_g g_{\alpha\beta} = P_{\alpha\beta}(g)(\partial g, \partial g), \quad (1.7)$$

obtained from $R_{\alpha\beta}(g) = 0$ as if $H^\rho = 0$, and for which we seek solutions on $[0, T] \times \mathbb{R}^3$ for $T > 0$. In the Minkowski coordinates (t, x^1, x^2, x^3) , we define the initial data for g as

$$g_{ij}|_{t=0} := \bar{g}_{ij}, \quad g_{00}|_{t=0} := -1, \quad g_{0i}|_{t=0} := 0,$$

which in particular imply that ∂_t will be the unit normal to Σ_0 in (\mathcal{M}, g) . This forces us to define $\partial_t g_{ij}|_{t=0} := -2K_{ij}$. Now, the equations $H^\rho|_{t=0} = 0$ rewrite as a invertible linear system for $\partial_t g_{0\alpha}|_{t=0}$, thus prescribing the remaining initial data. We can now solve (1.7) with energy methods in Sobolev spaces as in [Sog95].

3. At this point, the solution $(g_{\alpha\beta})_{\alpha,\beta=0,\dots,3}$ defines a Lorentzian metric on $[0, T] \times \mathbb{R}^3$ satisfying only

$$2R_{\alpha\beta}(g) = g_{\rho(\alpha} \partial_{\beta)} H^\rho, \quad (1.8)$$

where we used (1.5) and (1.7). In order to solve (1.2), we need to propagate the gauge condition $H^\rho = 0$, which for now only holds at $t = 0$. By plugging (1.8) into the contracted Bianchi identities $2\mathbf{D}^\mu R_{\mu\nu}(g) = \partial_\nu R(g)$, we show that H^ρ satisfy a linear homogeneous system of wave equations. By putting together (1.6) and (1.8), we obtain $\partial_t H^\rho|_{t=0} = 0$. Since we already have $H^\rho|_{t=0} = 0$, the wave gauge condition $H^\rho = 0$ holds on the whole spacetime, and (1.8) becomes (1.2).

The following fundamental theorem from Choquet-Bruhat summarizes this three steps program.

Theorem 1.1 ([CB52]). *Given an initial data set (Σ_0, \bar{g}, K) satisfying the constraint equations, there exists a globally hyperbolic development (\mathcal{M}, g) solving the Einstein vacuum equations.*

A geometric uniqueness statement came later with Geroch, see [CBG69].

Remark 1.1. The spacelike Cauchy problem presented here is not the only possible formulation. As for generic hyperbolic equations, one can also set initial data on characteristic hypersurface (null hypersurface in the language of geometry). We refer to [Ren90] and [Luk12] for the study of the characteristic initial value problem for the Einstein vacuum equations.

2 Geometric optics in general relativity

We now describe the main motivation of [Tou22b, Tou23], i.e., the justification of the geometric optics approximation for the Einstein vacuum equations (1.2). We start by giving an overview of geometric optics in the weakly non-linear regime. We refer to [Rau12] and [M 09] for two very rich presentations of the field of geometric optics.

2.1 Weakly non-linear geometric optics

Consider a generic first order non-linear hyperbolic system

$$A_0(u)\partial_t u + \sum_{i=1}^d A_i(u)\partial_i u = F(u), \quad (2.1)$$

where A_0 is positive definite and the A_i 's are symmetric. Consider also oscillating initial data of the form $u_\lambda(0, x) = \exp\left(i\frac{\varphi_0(x)}{\lambda}\right) f_0(x)$ where $\lambda > 0$ is a small wavelength. In [Lax57], Lax studied the linear case and showed that the solution u_λ takes the form of a WKB ansatz

$$u_\lambda = \exp\left(i\frac{\varphi}{\lambda}\right) \left(f^{(0)} + \lambda f^{(1)} + \lambda^2 f^{(2)} + \dots\right), \quad (2.2)$$

where the phase φ satisfies the eikonal equation $\det\left(A_0\partial_t\varphi + \sum_{i=1}^d A_i\partial_i\varphi\right) = 0$, and each $f^{(i)}$ satisfies a polarization condition and is transported along the rays of φ . In the non-linear case, we need a more refined ansatz than (2.2), i.e

$$u_\lambda \sim \lambda^p \sum_{n \geq 0} \lambda^n U_n \left(t, x, \frac{\varphi(t, x)}{\lambda}\right),$$

where the profiles $U_n(t, x, \theta)$ are periodic in θ . If p is large, the first profile is still linearly propagated. If one decreases p , we reach the regime of weakly non-linear geometric optics when non-linear terms enter the transport equation for the first profile. We refer to [HK83, JR92, JMR93] for standard results on this regime. For quadratic interactions as in (1.2), the threshold for this regime is $p = 1$.

2.2 Choquet-Bruhat's approximate solutions

The application of geometric optics to general relativity and the Einstein vacuum equations is motivated by a potential non-linear description of gravitational waves. See Chapter 35 of [MTW73] for a complete presentation of gravitational waves in the linearized gravity setting. As first shown in [CB69], geometric optics provides a mathematical framework that goes beyond linearized gravity. In this article, she considers WKB ansatz for the metric of the form

$$g_\lambda = g_0 + \lambda g^{(1)} \left(\frac{u_0}{\lambda}\right) + \lambda^2 g^{(2)} \left(\frac{u_0}{\lambda}\right). \quad (2.3)$$

The notation $g^{(i)} \left(\frac{u_0}{\lambda}\right)$ means that $g^{(i)}$ are symmetric 2-tensor depending on an extra phase argument $\frac{u_0}{\lambda}$. We have formally $R_{\mu\nu}(g_\lambda) = \frac{1}{\lambda} R_{\mu\nu}^{(-1)} + R_{\mu\nu}^{(0)} + O(\lambda)$. To ensure $R_{\mu\nu}^{(-1)} = 0$, one needs $g_0^{-1}(du_0, du_0) = 0$ and a polarization condition for $g^{(1)}$. To ensure $R_{\mu\nu}^{(0)} = 0$, one needs the following transport equation for $g^{(1)}$

$$2\partial^\alpha u_0 \mathbf{D}_\alpha g_{\mu\nu}^{(1)} + (\mathbf{D}^\alpha \mathbf{D}_\alpha u_0) g_{\mu\nu}^{(1)} = 0. \quad (2.4)$$

Moreover, one needs $R_{\mu\nu}(g_0) = \tau \partial_\mu u_0 \partial_\nu u_0$ with $\tau > 0$ depending quadratically on $g^{(1)}$ and $g^{(2)}$ to satisfy a polarization condition with a RHS also depending quadratically on $g^{(1)}$. Therefore, the metric g_λ constructed in [CB69] is an approximate solution of (1.2) since $R_{\mu\nu}(g_\lambda) = O(\lambda)$.

2.3 Transparency

As it is clear from (2.4), a particular cancellation in the quadratic terms in (1.2) occurs and the wave $g^{(1)}$ is linearly propagated, even in the weakly non-linear regime. This phenomenon is known as transparency in the geometric optics literature, see [JMR00]. As discussed in depth in [Lan13], transparency is closely related to the famous null condition for quadratic interaction introduced in the seminal articles [Chr86, Kla86] for the study of long time dynamics of initially small non-linear waves. As noted in [CB00], the Einstein vacuum equations (1.2) do not satisfy the null condition. In wave gauge, they satisfy the weak null condition of Lindblad and Rodnianski introduced in [LR03], allowing the same author to prove stability of the Minkowski spacetime in wave gauge in [LR10].

In [Tou22b], the structure of the quadratic terms in (1.2) plays a crucial role. As in Choquet-Bruhat's construction, it implies transparency, i.e., the linearity of the transport equation for the first profile. Moreover, it will also play a role in the control of high-high interactions in $R_{\mu\nu}^{(1)}$. Finally, note that the transparency phenomenon distinguishes (1.2) among the systems satisfying the weak null condition introduced in [LR03]. Indeed, the system

$$\begin{cases} u = (\partial_t v)^2, \\ v = (\partial_t u)^2 - |\nabla u|^2, \end{cases}$$

does satisfy the weak null condition but does not display transparency when geometric optics is performed (see Section 3.1.6 of [Tou22b]).

3 The Burnett conjecture in general relativity

A secondary motivation for the works [Tou22b, Tou23] is Burnett's conjecture in general relativity.

3.1 Backreaction

Burnett's conjecture describes backreaction for the Einstein vacuum equations. Backreaction is expected when small scale inhomogeneities in a vacuum spacetime interact and produce a non-zero large scale average, which one hopes to describe with a energy-momentum tensor. Different averaging scheme have been considered in [BH64] and in particular in [Isa68a, Isa68b], but Burnett is the first to consider weak limits in [Bur89]. The conjecture stated in this article can be schematically presented as follows:

$$\overline{\{\text{Solutions of (1.2)}\}}^{*H_{\text{loc}}^1} = \left\{ \begin{array}{l} \text{Solutions of the massless} \\ \text{Einstein-Vlasov system} \end{array} \right\}. \quad (3.1)$$

The topology defining the closure on the LHS of (3.1) is the following: a sequence $(g_\lambda)_\lambda$ converges to g_0 if $g_\lambda \rightarrow g_0$ strongly in L_{loc}^∞ and if $\partial g_\lambda \rightharpoonup \partial g_0$ weakly in L_{loc}^2 . Mathematically, (3.1) describes a double conjecture:

- Direct conjecture: consider a sequence $(g_\lambda)_\lambda$ of vacuum spacetimes converging in the weak sense described above and show that the limit g_0 satisfies the massless Einstein-Vlasov system for some density f .
- Indirect conjecture: fix a solution (g_0, f) of the massless Einstein-Vlasov system and construct a sequence $(g_\lambda)_\lambda$ of vacuum spacetimes converging weakly to g_0 .

The indirect conjecture is the only half of the conjecture where one solves (1.2). In order to obtain the weak convergence $\partial g_\lambda \rightharpoonup \partial g_0$, we expect unbounded second order derivatives $\partial^2 g_\lambda$. This shows that the indirect conjecture actually is a low regularity statement for the Einstein vacuum equations, in particular below the bounded L^2 curvature theorem of Klainerman, Rodnianski and Szeftel [KRS15], which requires H^2 bounds in order to control the time of existence of generic solutions to (1.2).

The first result on the direct conjecture is from Green and Wald in [GW11], where they show that the effective energy-momentum tensor $T_{\mu\nu}^{\text{eff}}$ at the RHS of the equation for the limit g_0 is traceless and

satisfies the weak energy condition. This agrees with the massless Einstein-Vlasov system but is still far from it. However, this already discards dark energy as a potential product of backreaction. This lead to debates in the cosmology community, see for instance [BCE⁺15] and [GW15].

3.2 State of the art

We review here the two main settings where Burnett's conjecture have been studied from a PDE perspective. First, Huneau and Luk worked under the $U(1)$ symmetry assumption, i.e., with a spacelike Killing field. Under this symmetry, the Einstein vacuum equations in $3 + 1$ dimensions reduce to the Einstein-wave map system in $2 + 1$ dimensions. This dimension reduction allows the use of the elliptic gauge, which rewrites the Einstein equations as a semi-linear elliptic system. Local well-posedness for the whole coupling in elliptic gauge is proved in [HL18a, Tou22a]. Based on the first of these articles, Huneau and Luk prove the indirect conjecture in [HL18b] for a class of "discretized" kinetic spacetime called null dusts (see Section 4.1). The direct conjecture in $U(1)$ symmetry is proved with microlocal defect measure in [HL19, GdC21].

The first result without any symmetry assumption is from Luk and Rodnianski in [LR20], where they prove both the direct and indirect conjecture. They work with a double null foliation, for which they proved a low regularity local existence result in [LR17]. The main feature of this result is that it allows derivatives of the metric in the two null directions of the foliation to live only in L^2 , fitting perfectly with Burnett's conjecture regime. Because of the L^2 curvature theorem, this low regularity is compensated by angular derivatives of the metric being very regular. As a consequence, Luk and Rodnianski prove the Burnett conjecture for a restricted class of kinetic spacetimes, namely 2 null dusts. Note however that [LR17, LR20] also shed new lights on the formation of trapped surfaces or null dust shell solutions.

Remark 3.1. At first glance, it might seem deceiving that mathematical works on a physically meaningful conjecture such as Burnett's rely so much on gauge choices. As explained above, this is due to the difficulty of solving the Einstein vacuum equations. While physicists always strive for a gauge-free description of reality, there is no such thing as gauge-free analysis of PDEs.

4 Statement of the result and sketch of proof

We now present the main contributions of the author, corresponding to the articles [Tou22b, Tou23]. We already highlight the two strengths of these articles, following the discussion of Sections 2 and 3:

- they give the first justification of the geometric optics approximation for (1.2) by constructing exact high-frequency solutions,
- they show how Burnett's conjecture could be studied in wave gauge.

4.1 Local existence in generalised wave gauge

The main result of [Tou22b] is the following.

Theorem 4.1. *Let (g_0, u_0, F_0) be a smooth solution of the Einstein-null dust system*

$$\begin{cases} R_{\mu\nu}(g_0) = F_0^2 \partial_\mu u_0 \partial_\nu u_0, \\ g_0^{-1}(du_0, du_0) = 0, \\ 2g_0^{\alpha\beta} \partial_\alpha u_0 \partial_\beta F_0 + (\text{ }_{g_0} u_0) F_0 = 0, \end{cases} \quad (4.1)$$

in wave coordinates on $[0, 1] \times \mathbb{R}^3$. There exists a family $(g_\lambda)_{\lambda \in (0,1]}$ of the form

$$g_\lambda = g_0 + \lambda \cos\left(\frac{u_0}{\lambda}\right) F^{(1)} + O(\lambda^2), \quad (4.2)$$

solving the Einstein vacuum equations in generalised wave coordinates on $[0, 1] \times \mathbb{R}^3$. Moreover, the symmetric 2-tensor $F^{(1)}$ satisfies

$$2g_0^{\alpha\beta} \partial_\alpha u_0 \mathbf{D}_\beta F^{(1)} + (\text{ }_{g_0} u_0) F^{(1)} = 0, \quad (4.3)$$

and $|F^{(1)}|_{g_0}^2 = 8F_0^2$, where \mathbf{D} is the covariant derivative of g_0 . In particular, $g_\lambda \rightarrow g_0$ strongly in L_{loc}^∞ and $\partial g_\lambda \rightarrow \partial g_0$ weakly in L_{loc}^2 .

The Einstein-null dust system (4.1) is a discretized version of the massless Einstein-Vlasov system (1.4), where the density is of the form $f = F_0 \otimes \delta_{du_0}$, where δ_{du_0} is the Dirac measure on $T^*\mathcal{M}$ at du_0 . The Einstein-null dusts (plural) system is obtained from (1.4) by setting

$$f = \sum_{\mathbf{A} \in \mathcal{A}} F_{\mathbf{A}} \otimes \delta_{du_{\mathbf{A}}},$$

with \mathcal{A} a finite set and $u_{\mathbf{A}}$ solutions of the eikonal equations. As explained in Section 3.2, Huneau and Luk treat the case of arbitrary $|\mathcal{A}| \in \mathbb{N}$ in $U(1)$ symmetry and Luk and Rodnianski the case $|\mathcal{A}| = 2$ in double null foliation. Theorem 4.1 corresponds to $|\mathcal{A}| = 1$ and thus doesn't improve yet Luk and Rodnianski's result. However, the choice of the (generalised) wave gauge *a priori* allows the superposition of an arbitrary number of null dusts since it doesn't single out any null directions, as opposed to the double null foliation.

The generalised wave gauge differs from the wave gauge of Section 1.2 by allowing a RHS: ${}_g x^\alpha = 0$ becomes ${}_g x^\alpha = \mathcal{F}(g)$ with \mathcal{F} given. This type of gauge has been used in stability results to correct decay of the solutions, see for example [Hun18, HV18, Joh19]. In Theorem 4.1, the function \mathcal{F}_λ absorbs oscillations and satisfies $\mathcal{F}_\lambda = O(\lambda)$. Therefore, not only do the metrics g_λ oscillate but the gauge as well.

The strategy to propagate this generalised wave condition is similar to the one exposed in Section 1.2 and is based on the Bianchi identities. It also requires special initial data solving the constraint equations (1.6). The construction of high-frequency solutions to the constraint equations is performed in [Tou23], and is presented in the next section. In Sections 4.3, 4.4 and 4.5, we sketch the proof of Theorem 4.1.

4.2 High-frequency solutions to the constraint equations

In order to launch the evolution problem of Theorem 4.1, we need appropriate solutions of the constraint equations (1.6). As they are eventually the projection on $\{t = 0\}$ of the spacetime metric g_λ , these solutions $(\bar{g}_\lambda, K_\lambda)$ share its main properties. In particular, they are also given by high-frequency expansion. The main result of [Tou23] is the following.

Theorem 4.2. *Let $(\bar{g}_0, K_0, u_0, F_0)$ be an asymptotically flat smooth solution of the maximal null dust constraint equations*

$$\begin{cases} R(\bar{g}_0) - |K_0|_{\bar{g}_0}^2 = 2|\nabla u_0|_{\bar{g}_0}^2 F_0^2, \\ -\text{div}_{\bar{g}_0} K_0 = |\nabla u_0|_{\bar{g}_0} F_0^2 du_0, \\ \text{tr}_{\bar{g}_0} K_0 = 0. \end{cases} \quad (4.4)$$

There exists a family $(\bar{g}_\lambda, K_\lambda)_{\lambda \in (0,1]}$ of asymptotically flat solutions to (1.6) of the form

$$\bar{g}_\lambda = \bar{g}_0 + \lambda \cos\left(\frac{u_0}{\lambda}\right) \bar{F}^{(1)} + O(\lambda^2), \quad (4.5)$$

$$K_\lambda = K_0 + \frac{1}{2} \sin\left(\frac{u_0}{\lambda}\right) |\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)} + O(\lambda). \quad (4.6)$$

The system (1.6) is underdetermined, and in order to solve it we use the conformal method. Introduced by Lichnerowicz in [Lic44], this method transforms (1.6) into a determined system of equations composed of a vectorial equation and a scalar equation. The idea giving its name to the method is to seek solutions (\bar{g}, K) of the form

$$\bar{g} = \varphi^4 \gamma, \quad K = \varphi^{-2}(\sigma + L_\gamma W) + \frac{1}{3} \varphi^4 \gamma \tau,$$

with fixed parameters γ a Riemannian metric on \mathbb{R}^3 , τ a scalar function and σ a TT-tensor, i.e., a symmetric 2-tensor solving $\operatorname{div}_\gamma \sigma = 0$ and $\operatorname{tr}_\gamma \sigma = 0$. The unknowns (φ, W) then solve

$$\begin{cases} 8\Delta_\gamma \varphi = R(\gamma)\varphi + \frac{2}{3}\tau^2\varphi^5 - |\sigma + L_\gamma W|_\gamma^2 \varphi^{-7}, \\ \operatorname{div}_\gamma L_\gamma W = \frac{2}{3}\varphi^6 d\tau. \end{cases} \quad (4.7)$$

The main challenge in the proof Theorem 4.2 is to adapt the conformal method to the high-frequency context, i.e., choose the parameters so that the resulting solution $(\bar{g}_\lambda, K_\lambda)$ displays the behaviour (4.5)-(4.6). Not only are the solutions (φ, W) be defined by high-frequency ansatz, but also the parameters γ , τ and σ . In particular, we construct an oscillating TT-tensor, i.e., solve $\operatorname{div}_\gamma \sigma = 0$ and $\operatorname{tr}_\gamma \sigma = 0$, with γ itself oscillating.

4.3 Spacetime ansatz

The core of the proof Theorem 4.1 is to introduce a more complete high-frequency ansatz than (4.2), in particular with a remainder, and then to plug it into (1.2) to obtain a hierarchy of equations and algebraic conditions. This ansatz will capture the main feature of non-linear waves, i.e., the creation of harmonics. We consider the following ansatz:

$$g_\lambda = g_0 + \lambda \cos\left(\frac{u_0}{\lambda}\right) F^{(1)} + \lambda^2 \left(\sin\left(\frac{u_0}{\lambda}\right) \mathfrak{F}_\lambda + \cos\left(\frac{2u_0}{\lambda}\right) F^{(2)} + \mathfrak{h}_\lambda \right) + \lambda^3 g^{(3)}\left(\frac{u_0}{\lambda}\right), \quad (4.8)$$

where every tensors are symmetric 2-tensors, $F^{(1)}$ and $F^{(2)}$ are independent of λ , \mathfrak{h}_λ is the remainder of the ansatz and depends on λ , as well as the amplitudes \mathfrak{F}_λ ($g^{(3)}$ is oscillating but is irrelevant in the sequel). Let us plug the ansatz (4.8) into (1.2) expressed in generalised wave coordinates (see (1.5)), we want:

$$-g_\lambda(g_\lambda)_{\alpha\beta} + (g_\lambda)_{\rho(\alpha}\partial_{\beta)}H^\rho + P_{\alpha\beta}(g_\lambda)(\partial g_\lambda, \partial g_\lambda) = 0,$$

where $H^\rho = g_\lambda^{\mu\nu}\Gamma(g_\lambda)_{\mu\nu}^\rho$. We detail the three different types of terms in this equation.

- The wave operator $g_\lambda(g_\lambda)_{\alpha\beta}$ gives, at leading order, the transport operator

$$2g_0^{\alpha\beta}\partial_\alpha u_0 \mathbf{D}_\beta + (g_0 u_0),$$

applied to each amplitudes in (4.8) and the wave operator g_λ applied to \mathfrak{h}_λ . The quasi-linearity of $g_\lambda(g_\lambda)_{\alpha\beta}$ implies a loss of one derivative which is the main obstacle to well-posedness, see Section 4.4.

- The quadratic non-linearity is the main source of interactions between the different waves in (4.8). The lack of null condition produces backreaction, i.e., resonant interaction, and high-high interaction. The latter terms are problematic and we rely crucially on the weak polarized null condition $P_{\alpha\beta}(g_\lambda)(\partial g_\lambda, \partial g_\lambda)$ satisfies.
- The leading order terms in the gauge term $(g_\lambda)_{\rho(\alpha}\partial_{\beta)}H^\rho$ are given by $\partial_{(\alpha}u_0 \operatorname{Pol}_{\beta)}(T)$ where

$$\operatorname{Pol}_\beta(T) = g_0^{\mu\nu} \left(\partial_\mu u_0 T_{\beta\nu} - \frac{1}{2}\partial_\beta u_0 T_{\mu\nu} \right),$$

is the polarization tensor of T . Here T stands for either $F^{(1)}$, \mathfrak{F}_λ or $F^{(2)}$. The latters will solve transport equation, but we also ask them to satisfy polarization conditions of the form $\operatorname{Pol}(T) = \Omega$ to remove problematic terms from the PDEs.

In the last two sections, we will highlight two aspects of the proof: regain a missing derivative and controlling the high-high interaction. For the sake of clarity, we will think of these issues as independent, which is obviously a simplification since every aspects of the construction are intertwined. We deliberately leave aside the following aspects and refer to [Tou22b] for further details.

- The generalised wave gauge: its purpose is to remove second order derivatives of \mathfrak{h} from the RHS of the wave equation it satisfies.
- The tensor $g^{(3)}$: its purpose is to control the high-high interaction terms in $R_{\mu\nu}^{(1)}$ with a good structure.

4.4 Quasi-linear challenge

The quasi-linearity of the wave operator in (1.2) is responsible for the coupling between \mathfrak{F}_λ and \mathfrak{h}_λ . Indeed, when applied to the first wave in (4.2) g_λ gives

$$g_\lambda \left(\lambda \cos \left(\frac{u_0}{\lambda} \right) F^{(1)} \right) = -\frac{1}{\lambda} \cos \left(\frac{u_0}{\lambda} \right) g_\lambda^{-1} (du_0, du_0) F^{(1)} + O(\lambda^0). \quad (4.9)$$

Using again (4.8) to expand the inverse metric g_λ^{-1} , we see that $g_\lambda g_\lambda$ contains a term of the form

$$-\lambda \cos \left(\frac{u_0}{\lambda} \right) \mathfrak{h}_\lambda^{-1} (du_0, du_0) F^{(1)},$$

appearing as a source term in the transport equation satisfied by \mathfrak{F}_λ . Moreover, when the two derivatives of the wave operator hit the amplitude \mathfrak{F}_λ , they produce $\lambda^2 \sin \left(\frac{u_0}{\lambda} \right) g_\lambda \mathfrak{F}_\lambda$ appearing as a source term in the wave equation for \mathfrak{h}_λ . Therefore, the quasi-linearity of (1.2) is coupling the remainder and one of the amplitude through the eikonal term in (4.9). If we neglect terms coming from $F^{(1)}$ or $F^{(2;2)}$, this coupling is of the form

$$L_0 \mathfrak{F}_\lambda = \mathfrak{h}_\lambda, \quad (4.10)$$

$$g_\lambda \mathfrak{h}_\lambda = \sin \left(\frac{u_0}{\lambda} \right) g_\lambda \mathfrak{F}_\lambda, \quad (4.11)$$

where $L_0 = -g_0^{\alpha\beta} \partial_\alpha u_0 \partial_\beta$. The transport-wave system (4.10)-(4.11) is *a priori* ill-posed since it loses one derivative: from (4.10), \mathfrak{F}_λ and \mathfrak{h}_λ live at the same level of regularity, while from standard energy estimates applied to (4.11) one sees that $\partial \mathfrak{h}_\lambda$ lives at the level of regularity of $g_\lambda \mathfrak{F}_\lambda$, i.e., $\partial^2 \mathfrak{F}_\lambda$. To recover the missing derivative, we need to show that $g_\lambda \mathfrak{F}_\lambda$ is actually better than any second order derivatives $\partial^2 \mathfrak{F}_\lambda$.

First idea is to compute from (4.10) the transport equation satisfied by $g_\lambda \mathfrak{F}_\lambda$, which requires a nice expression for $[L_0, g_\lambda]$. However, L_0 is defined with the background metric g_0 , and nothing can be said on $[L_0, g_\lambda]$. Nevertheless, g_λ is closed to g_0 , and from (4.8) we obtain the schematic expansion

$$g_\lambda \mathfrak{F}_\lambda = g_0 \mathfrak{F}_\lambda + \lambda \partial^2 \mathfrak{F}_\lambda. \quad (4.12)$$

The first term in (4.12) has enough structure to be treated via transport. Namely, we can now compute the transport equation it satisfies from (4.10):

$$L_0 (g_0 \mathfrak{F}_\lambda) = [L_0, g_0] \mathfrak{F}_\lambda + \sin \left(\frac{u_0}{\lambda} \right) g_0 \mathfrak{F}_\lambda + \lambda \left(\sin \left(\frac{u_0}{\lambda} \right) \partial^2 \mathfrak{F}_\lambda + \partial^2 \mathfrak{h}_\lambda \right). \quad (4.13)$$

where we used $g_0 \mathfrak{h}_\lambda = g_\lambda \mathfrak{h}_\lambda + \lambda \partial^2 \mathfrak{h}_\lambda$, (4.11) and (4.12) again. The commutator $[L_0, g_0]$ has now some structure and one can show

$$|[L_0, g_0] f| \leq |\partial L_0 f| + |g_0 f|. \quad (4.14)$$

Using (4.10) and energy estimates for (4.11) we apply (4.14) to \mathfrak{F}_λ and conclude that

$$|[L_0, g_0] \mathfrak{F}_\lambda| \leq |g_0 \mathfrak{F}_\lambda| + \lambda (|\partial^2 \mathfrak{F}_\lambda| + |\partial^2 \mathfrak{h}_\lambda|).$$

Combining this with (4.13), we obtain a control of $g_0 \mathfrak{F}_\lambda$ by $\lambda (|\partial^2 \mathfrak{F}_\lambda| + |\partial^2 \mathfrak{h}_\lambda|)$. Coming back to (4.12), this gives the following estimate

$$|g_\lambda \mathfrak{F}_\lambda| \leq \lambda (|\partial^2 \mathfrak{F}_\lambda| + |\partial^2 \mathfrak{h}_\lambda|). \quad (4.15)$$

We now benefit from the small constant λ to invert one derivative. This naive idea is made rigorous by the introduction of a Fourier projector $\Pi_\lambda = \mathcal{F}^{-1} (\chi_\lambda \mathcal{F}(\cdot))$ where \mathcal{F} is the Fourier transform on \mathbb{R}^3 and χ_λ is supported on $[0, \lambda^{-1}]$. Indeed, Bernstein estimates imply $\|\lambda \partial \Pi_\lambda(f)\|_{L^2} \leq \|f\|_{L^2}$. Therefore, instead of solving (4.10)-(4.11) we solve

$$L_0 \mathfrak{F}_\lambda = \Pi_\lambda (\mathfrak{h}_\lambda), \quad (4.16)$$

$$g_\lambda \mathfrak{h}_\lambda = \sin \left(\frac{u_0}{\lambda} \right) g_\lambda \mathfrak{F}_\lambda + \frac{1}{\lambda} (\text{Id} - \Pi_\lambda) (\mathfrak{h}_\lambda). \quad (4.17)$$

The seemingly singular new term in the wave equation is not problematic since Bernstein estimates also imply $\|\lambda^{-1}(\text{Id} - \Pi_\lambda)(\mathfrak{h}_\lambda)\|_{L^2} \leq \|\nabla \mathfrak{h}_\lambda\|_{L^2}$, which is consistent with energy estimates for the wave equation. With the same strategy as above but now applied to (4.16)-(4.17), we obtain

$$|g_\lambda \mathfrak{F}_\lambda| \leq |\partial \mathfrak{F}_\lambda| + |\partial \mathfrak{h}_\lambda|,$$

instead of (4.15). This estimate clearly shows that we recover the missing derivative.

Remark 4.1. This strategy is highly adapted to our problem, and the system (4.16)-(4.17) is not a regularized version of (4.10)-(4.11). Indeed, the high-frequency term $(\text{Id} - \Pi_\lambda)(\mathfrak{h}_\lambda)$ is moved from the transport to the wave equation at the cost of the loss of one λ power, since the transport equation corresponds to λ terms in (1.2) and the wave equation to λ^2 terms. Therefore, the metric g_λ still satisfies (1.2) at fixed λ .

4.5 Semi-linear challenge

We finally turn our attention to a more formal issue. First, note that from (4.8) we obtain $R_{\mu\nu}^{(-1)} = -\cos\left(\frac{u_0}{\lambda}\right) \partial_{(\mu} u_0 \text{Pol}_{\nu)}(F^{(1)})$. As in [CB69] we thus impose $\text{Pol}(F^{(1)}) = 0$, which is the equivalent of the TT-gauge from linearized gravity. Moreover we have schematically

$$R_{\mu\nu}^{(0)} = R_{\mu\nu}(g_0) + \sin\left(\frac{u_0}{\lambda}\right) L_0 F^{(1)} + \sin^2\left(\frac{u_0}{\lambda}\right) (\partial u_0 F^{(1)})^2 + \partial_{(\mu} u_0 \text{Pol}_{\nu)}(\partial_\theta^2 g^{(2)}), \quad (4.18)$$

where $\partial_\theta^2 g^{(2)}$ stands for second derivatives in the oscillating variable of any λ^2 wave in (4.8), and $(\partial u_0 F^{(1)})^2$ stands for whatever interaction terms comes out of the quadratic non-linearity in the Ricci tensor. The lack of null condition precisely translates as these terms not identically vanishing. However, they satisfy what we call the weak polarized null condition, i.e

$$(\partial u_0 F^{(1)})^2 = (F^{(1)})^2 \partial_\mu u_0 \partial_\nu u_0 + \text{terms proportional to } \text{Pol}(F^{(1)}),$$

where the expression $(F^{(1)})^2$ has a sign. Since we impose $\text{Pol}(F^{(1)}) = 0$, we obtain $(\partial u_0 F^{(1)})^2 = (F^{(1)})^2 \partial_\mu u_0 \partial_\nu u_0$. After expanding the \sin^2 , (4.18) then becomes schematically

$$\begin{aligned} R_{\mu\nu}^{(0)} &= R_{\mu\nu}(g_0) + (F^{(1)})^2 \partial_\mu u_0 \partial_\nu u_0 + \sin\left(\frac{u_0}{\lambda}\right) L_0 F^{(1)} \\ &\quad + \cos\left(\frac{2u_0}{\lambda}\right) (F^{(1)})^2 \partial_\mu u_0 \partial_\nu u_0 + \partial_{(\mu} u_0 \text{Pol}_{\nu)}(\partial_\theta^2 g^{(2)}). \end{aligned}$$

The non-oscillating part in $R_{\mu\nu}^{(0)}$ vanishes if $(F^{(1)})^2 \sim F_0^2$, since g_0 solves the Einstein-null dust system. The first harmonic in $R_{\mu\nu}^{(0)}$ vanishes if $F^{(1)}$ solves the linear transport equation $L_0 F^{(1)} = 0$. In order to obtain $R_{\mu\nu}^{(0)} = 0$, it remains to deal with the high-high interaction term corresponding to the second harmonic. Without the polarization term depending on $g^{(2)}$ (which we recall comes from the gauge term in the Ricci tensor), we would be forced to put $\cos\left(\frac{2u_0}{\lambda}\right) (F^{(1)})^2 \partial_\mu u_0 \partial_\nu u_0$ in the equation for $F^{(1)}$, thus obtaining a non-linear transport equation for the first profile. Because of the particular structure $\partial_\mu u_0 \partial_\nu u_0$ of these terms, we can benefit from the presence of $\partial_{(\mu} u_0 \text{Pol}_{\nu)}(\partial_\theta^2 g^{(2)})$ by imposing a polarization condition for $g^{(2)}$ of the form

$$\text{Pol}(\partial_\theta^2 g^{(2)}) = (F^{(1)})^2 du_0.$$

This shows how transparency follows from the particular structure of the quadratic non-linearity, its lack of null condition being compensated by the gauge terms in the Ricci tensor. In conclusion, let us mention that these polarization conditions are algebraic conditions for the waves in (4.8) which already satisfy transport equations along the rays (we presented above the one for $F^{(1)}$, more complicated ones are required for \mathfrak{F}_λ and $F^{(2,2)}$). Therefore, we need to show the compatibility of the polarization conditions and the evolution. For $F^{(1)}$, we argue as in [CB69] and get a transport equation for $\text{Pol}(F^{(1)})$ from the one of $F^{(1)}$. For $g^{(2)}$, this strategy is not feasible and we treat the polarization conditions for $g^{(2)}$ as gauge terms, i.e., we solve (1.2) as they were zero and then propagate the fact that they are zero from the data with the Bianchi identities.

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