Séminaire Laurent Schwartz

EDP et applications

Année 2022-2023

Exposé n° II (18 octobre 2022)

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https://doi.org/10.5802/slsedp.157

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Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2266-0607

Séminaire Laurent-Schwartz — EDP et applications
Institut des hautes études scientifiques, 2022-2023
Exposé n° II, 1-17

Defects in homogenization theory

Claude Le Bris*†

Abstract

We review a series of works that address homogenization for partial differential equations with highly oscillatory coefficients. A prototypical setting is that of periodic coefficients that are locally, or more globally perturbed. We investigate the homogenization limits obtained, first for linear elliptic equations, both in conservative and non conservative forms, and next for nonlinear equations such as Hamilton-Jacobi type equations.

1 Introduction

Consider the simple (yet ubiquitous) equation

$$-\operatorname{div}\ \left(a(x/\varepsilon)\,\nabla u_\varepsilon(x)\right) \;=\; f(x), \tag{1}$$

posed on a domain $\mathcal{D}$ of the ambient space $\mathbb{R}^d$, and supplied with, say, homogeneous Dirichlet boundary condition on $\partial\mathcal{D}$. The coefficient within the divergence operator is a rescaled function a, highly oscillatory at the presumably small scale ε. It is supposed to be bounded and bounded away from zero, so that the equation is well-posed in $H^1_0(\mathcal{D})$, for, say, $f \in L^2(\mathcal{D})$.

We intend to study the homogenization limit of this equation. Our important assumption, for this purpose, is that the coefficient a is not necessarily periodic. It does not belong either to any of the classes of functions commonly considered in the literature of homogenization theory, such as quasi-periodic, almost-periodic or stationary ergodic functions. In the sequel, the function a will typically be a *perturbation* of a periodic function a_{per}, in a sense that will be made precise. Because of the relevance of this issue in materials modeling, we call such a perturbation a *defect*.

Of course, under our above assumptions on the coefficient within (1), the general theory of homogenization (see [49, 47]) applies, and we know there exists

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†The research of the author is partly supported by ONR and EOARD, currently under the Grants ONR N00014-20-1-2691 and EOARD FA8655-20-1-7043. The writing of this manuscript has been completed during a visit at the Math+ Institute in Berlin. Partial financial support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy —The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689) —is gratefully acknowledged.

an homogenized limit, of the form

$$- \operatorname{div} \left(A^* \nabla u^*(x)\right) = f(x), \tag{2}$$

up to an extraction in ε. This compactness result, however, does not say anything, in whole generality, on the homogenized coefficient A^*. It does not say anything either on the types of convergence of u_ε to u^*, and *a fortiori* on the *rates* of this convergence, in terms of ε, in suitable functional spaces.

For the particular class of perturbations we consider (which makes our coefficient less general than for the classical homogenization theory we have just recalled), our purpose is to put homogenization theory for (1) on an equal footing with periodic homogenization theory for (1). More precisely, we expect to establish convergence of the *whole* sequence of solutions u_ε to the solution u^* of (2) and we expect the effective coefficient A^* to be *explicitly* expressed in terms of the data. We also expect to define a corrector function and to prove that this allows convergence of u_ε to hold strongly in $H^1(\mathcal{D})$ and in other suitable Schauder and Sobolev spaces, once u^* is appropriately corrected by a term accounting for the fine scale oscillations. We finally expect to determine the rate of the convergences in the various functional spaces.

The particular elliptic linear, divergence form equation (1) is chosen for simplicity. Other, more sophisticated equations, such as linear elliptic equations that are not in divergence forms, or also nonlinear equations such as Hamilton-Jacobi type equations will also be discussed.

Let us at once make it clear that, *even* if (1) is linear, the question we examine has an intrinsic *nonlinear* nature. We indeed focus on the *nonlinear* (and in most cases also *nonlocal*) character of the application that maps our input parameter a to our output solution u. This can already be illustrated upon forcefully deleting the differential operators in (1): the application then reads as $a \longmapsto u = f/a$. Put differently, we are investigating how a variation in a affects u. In this specific instance, the variation is a rescaling and a perturbation. We note in passing that the theory we present heavily relies upon the fact that the oscillatory coefficient a_ε is a *fixed rescaled* function a. The case of a general coefficient a_ε depending differently on the small scale parameter ε is irrelevant.

Our motivation for considering such a line of work is twofold.

The first incentive is modeling in *materials science* (see [38] and note that one could presumably apply the same observations to modeling in more general physical media). It is indeed our considered opinion that theoretical and computational materials science has witnessed a major evolution in the past decades. The major two novel features in this discipline are (a) an increasing importance of *multi-scale* phenomena (with the inclusion of the micro-scale in macro-scale simulations, both sequentially and concurrently) and (b) the consideration of materials with microstructures that are not necessarily periodic but contain possibly random, features —defects in structures, dislocations in lattices—breaking the idealized picture of an otherwise periodic model. Examples abound in many fields of the engineering sciences and life sciences that testify of this evolution: composite materials for the aerospace industry, metallic alloys at use in nuclear engineering, etc.

Our second incentive is purely mathematical in nature. It specifically concerns *PDE theory*, and, to some extent, has little to do with homogenization theory. The rescaling $x \to x/\varepsilon$ of the coefficient a in the equation (1) and its possible perturbation from a periodic function a_{per} to a more general function a is nothing but a possible practical means to understand the dependency of the solution upon the parameters of the equation. In addition, it is also likely to *create* an equation (the limit equation such as (2)) from a class of equations (the equations (1) for the family of parameters ε), possibly extending the former class (a so-called "closure" problem —we shall see such a problem in Section 3) and explore which structures and regularities are carried over from one scale to another scale. It also allows (but the latter question will not be examined in the present contribution) to consider from this specific perspective *inverse problems*, where information on a is sought, based upon the observation of u_ε for various parameters ε and right-hand sides f.

Given the above twofold motivation, we now present a set of works where the common denominators are as follows. We explore the boundaries of homogenization theory by considering coefficients beyond the idealistic setting of periodic coefficients. When possible, we try and avoid fully general random coefficients, which (a) are difficult theoretically and (b) given our practical considerations, are often prohibitively expensive to address practically. We specifically consider coefficients that are *perturbations* of periodic materials.

Our Section 2 exclusively concerns itself with *linear* equations. The results we overview there, although necessarily partial, cover a large part of the issues mentioned above. Our Section 3 next presents some of the first steps of a similar mathematical endeavor put in action on *some* nonlinear equations, here Hamilton-Jacobi type equations. Our final Section 4 lists a few topics that have been left aside in our presentation, together with some pending issues in various directions of research. We also mention both our successes and the limitations of our results and techniques.

Most of the results overviewed in this article have been obtained in collaboration with the following colleagues: Yves Achdou (Université Paris-Cité), Xavier Blanc (Université Paris-Cité), Pierre Cardaliaguet (Université Paris-Dauphine), Pierre-Louis Lions (Collège de France), Panagiotis Souganidis (University of Chicago).

2 Linear elliptic equations

2.1 Our expectations

We first need to briefly recall the basics of periodic homogenization theory, before we extend the theory to the case of *perturbed* periodic coefficients. When the possibly matrix-valued coefficient a in (1) is, say, $\mathbb{Z}^d$-periodic, then, as $\varepsilon \to 0$, u_ε converges to u^* solution to (2). The homogenized coefficient A^* is then given, for $1 \leq i, j \leq d$, by

$$[A^*]_{ij} = \int_Q (e_i + \nabla w_{e_i,per}(y))^T A_{\rm per}(y)\, e_j \, dy, \tag{3}$$

where $Q = [0,1]^d$ is the unit cube, and where, for any $p \in \mathbb{R}^d$, the function $w_{p,per}$, called the *corrector function*, also assumed a $\mathbb{Z}^d$-periodic function, solves the *corrector problem*. The latter problem reads

$$-\mathrm{div}\left[a_{\rm per}(y)\,(p + \nabla w_{p,per})\right] = 0 \quad \text{in} \quad Q. \tag{4}$$

The success of periodic homogenization theory and its impact on practical problems is then justified by the fact that, solving the d equations (4) corresponding to each $p = e_i$, $1 \leq i \leq d$, on the bounded domain Q (a task that is considered straightforward by the metric of today's computational technology) allows to indeed determine the homogenized coefficient A^* thus the homogenized limit u^*. The "explicitness" of the expression (3) and the equation (4) is what we target in our more general setting of perturbed periodic coefficients.

In addition, the periodic theory also allows to accurately approximate u_ε in the regime of small parameters ε, using the so-called *two-scale expansion*

$$u_{\varepsilon,1}(x) \;=\; u^*(x) \;+\; \varepsilon \sum_{i=1}^d \frac{\partial u^*}{\partial x_i}(x)\, w_{e_i,per}\left(\frac{x}{\varepsilon}\right). \tag{5}$$

Rates of convergence for the difference

$$R_\varepsilon \;=\; u_\varepsilon - u_{\varepsilon,1} \tag{6}$$

in various functional spaces (starting from $H^1(\mathcal{D})$) are also available. We again similarly intend to exhibit such rates of convergence in our perturbed setting.

In any event, there exists an enormous literature on homogenization theory and we only cite here some of the most famous references in the domain, with no claim whatsoever about exhaustiveness: [13, 49, 47, 4, 45], etc.

A substantial extension of the above periodic theory was accomplished in the ergodic stationary setting. The homogenized limit of the same equation as (1) but with a stationary ergodic coefficient $a\,(x/\varepsilon, \omega)$ also reads as (1) where this time

$$[A^*]_{ij} = \mathbb{E}\left(\int_Q (e_i + \nabla w_{e_i,sto}(y,\cdot))^T A\,(y,\cdot)\, e_j\, dy\right), \tag{7}$$

and $w_{p,sto}(y,\omega)$ is now the solution to the equation

$$-\mathrm{div}\left[A\,(y,\omega)\,(p + \nabla w_{p,sto}(y,\omega))\right] = 0 \tag{8}$$

in the *whole* space $\mathbb{R}^d$, with $\nabla w_{p,sto}$ stationary and $\mathbb{E}\left(\int_Q \nabla w_{p,sto}(y,\cdot)\, dy\right) = 0$.

The list of contributors to random homogenization theory is also considerable. We only cite here the classical works [44, 25, 24]. Some recent significant developments appeared in [31, 30, 7, 6]. The random nonlinear setting was also considered and one of pioneering works in this direction is [26].

Formulae (7)-(8) show that, as in the periodic setting, the homogenized coefficient may thus be *explicitly* expressed. If (3) and (7) really look alike, the striking difference between (4) and (8), however, is that the latter equation is posed on the *unbounded* domain $\mathbb{R}^d$. Several mathematical difficulties regarding the well-posedness of the problem originate from this difference. This is a feature we will also find in our *perturbed* periodic setting. Likewise, huge computational difficulties also arise, but these are not our focus here. On the other hand, rates of convergence are also a question significantly more difficult in the random setting than in the periodic setting. In our perturbed setting, we will also observe some flavor of this.

2.2 Intuitive description of our line of research

The brutal technique consisting in eliminating the differential operators in (1) already used in Section 1, suggests that, in order for the entire sequence of "solutions" $u_\varepsilon = f/a\,(./\varepsilon)$ to weakly converge to a limit that can be explicitly identified, it is sufficient for the function on the right hand side to admit an average.

Our strategy has therefore been to look for classes of functions for this to happen, with the hope (indeed often fulfilled) that the same classes will be suitable for coefficients a in (1) and for the expected homogenization theory.

It is well known that periodic, quasiperiodic, almost periodic, stationary ergodic, functions are all such admissible classes. On the other hand, functions modeling a periodic background perturbed by a so-called local defect, such as functions that read as $a_{per} + C_0^\infty$, for a_{per} a periodic function and C_0^∞ denoting the space of smooth compactly supported functions, are also convenient. Further, some specific functions that are more global perturbations of a periodic functions may also be used. For instance, we may be willing to consider functions such as $\sum_{k\in\mathbb{Z}^d}\psi(x-k-Z_k)$ where Z_k denotes a small displacement of the original periodic position k.

In any event, we also learn from the consideration of $u_\varepsilon = f/a\,(./\varepsilon)$ that we will have to consider inverses and products of such functions. Ideally, this is a notion of *algebras* and not vector spaces that is indeed relevant. We will return to this in Section 4.

Once a suitable class of coefficients a is anticipated, the crucial task is to establish the well-posedness of the corrector equation in that class. This is perfectly understable in particular since our setting above is linear. Homogenization theory almost reduces then to the corrector equation. The purpose is to solve an equation analogous to (4) and next build the corresponding two-scale expansion in the vein of (5).

2.3 Existence of a suitable corrector function

In the case of our simple elliptic, linear, divergence form equation (1), the corrector equation we have to solve is the analogous equation to (4), namely

$$-\operatorname{div}\ (a(y)\,(p+\nabla w(y)))\ =\ 0, \tag{9}$$

where the coefficient is a and not the periodic coefficient a_{per}. The equation is this time posed on the whole ambient space $\mathbb{R}^d$, just as the corrector equation (8) for the ergodic stationary case is. It is supplied with a boundary condition at infinity that should express the strict sublinearity $\dfrac{w(y)}{1+|y|} \to 0$ (this property ensures that the rightmost term in (5) is indeed a correction to the leading term). In the absence of any structure, we are unaware of any approach that allows to establish existence for equation (9).

One may indeed realize that, in the ergodic stationary case, as well as in all the related (periodic, quasiperiodic, almost periodic) settings, the proof of existence relies upon a reinterpretation of the equation (9) that explicitly uses the structure of the coefficient. Put differently, the equation is lifted from an equation on the space $\mathbb{R}^d$ to an equation solved on the torus, or, say, on the abstract probability space. In all such settings, some type of "compactness", originally absent from the equation posed in $\mathbb{R}^d$ is reinstated. A possible alternative perspective on this difficulty is to express that, in all the above settings, it is possible to pass from an estimate on large balls to a local estimate. The most natural estimate on approximated solutions are obtained on average, and, precisely because of the structure imposed, they translate into local estimates that in turn allow to pass to the limit, at least in the sense of distributions, in the sequence of regularizations.

In our own setting, we are going to also impose a structure on the coefficient a. We consider

$$a = a_{per} + \widetilde{a}, \tag{10}$$

where a_{per} denotes the unperturbed, periodic background, and $\widetilde{a}$ denotes the perturbation, which belongs to a Lebesgue space, that is

$$\widetilde{a} \in L^r(\mathbb{R}^d), \quad \text{for some} \quad 1 \le r < +\infty. \tag{11}$$

Since we are also going to assume, in most cases, that $\widetilde{a}$ is (uniformly) Hölder continuous, this global integrability implies that $\widetilde{a}$ vanishes at infinity. The defect we consider is therefore, in that sense, *local.* Evidently, our setting is an over-simplification of the much more practically relevant condition $\widetilde{a} \overset{|x|\to\infty}{\longrightarrow} 0$ (a condition that is reminiscent of the space $a_{per} + C_0^\infty$ we were mentioning in the previous section). We are indeed unable to proceed in the full mathematical generality of the latter condition.

Given (10)-(11), it is anticipated that the homogenized equation obtained is identical to that for the periodic coefficient a_{per}. Intuitively, the reason is, the coefficient $\widetilde{a}$ does not contribute to averages over large balls. The detailed mathematical study indeed confirms that the homogenized limit is the periodic one.

The key task is, on the other hand, to solve the corrector equation (9). It is readily seen, introducing $\widetilde{w}_p = w_p - w_{p,per}$ where $w_{p,per}$ is the solution to (4), that the suitable functional class where to look for $\widetilde{w}_p$ is such that $\nabla \widetilde{w}_p \in L^r(\mathbb{R}^d)$.

The work [19] contains a first theoretical study in the case $r = 2$, along with some computational illustrations. On the one hand, the proof of existence of the corrector function, solution to (4) for (10) and (11) with $r = 2$, is performed on the basis of arguments only relevant in this Hilbertian setting. In addition, it is observed there, numerically, that a two-scale expansion of the type (5) employing the periodic corrector $w_{p,per}$ does not provide an accurate approximation at the vicinity of the defects, that is the region where $\widetilde{a}$ is large. On the other hand, the same expansion with w_p instead of $w_{p,per}$ reinstates everywhere in the domain the quality of approximation observed in the absence of defect.

The general case of a defect $\widetilde{a} \in L^r(\mathbb{R}^d)$, for r not necessarily equal to 2, was next addressed in the work [20]. We prove there the

Theorem [L^r-perturbation of a periodic coefficient, [20]] : *Assume periodicity of the background coefficient a_{per} and (coercivity, boundedness and) Hölder regularity of both a_{per} and a. Then, the corrector problem has a unique solution w_p, up to the addition of a constant. Moreover, $w_p = w_{p,per} + \widetilde{w}_p$, where $w_{p,per}$ is the periodic corrector and*

- *if* $1 \leq r < d$, *then,* $\lim_{|x| \to +\infty} \widetilde{w}_p(x) = 0$ *;*
- *if* $2 \leq r$, *then* $\nabla \widetilde{w}_p \in L^r(\mathbb{R}^d)$.

The proof of this theorem performed in [20] uses estimates of the Green function on dyadic rings. The corrector equation is written under the form

$$- \operatorname{div} (a_{per} \nabla \widetilde{w}_p) = \operatorname{div} (\widetilde{a} \nabla \widetilde{w}_p) + \operatorname{div} (\widetilde{a} (p + \nabla w_{p,per})) . \tag{12}$$

This isolates in the left-hand side the operator with periodic coefficients for which the fundamental results established by M. Avellaneda and F.-H. Lin, in [8, 9, 10] are then used.

A more general and versatile proof of the same result was then presented in [21]. Since the corrector equation also reads as

$$- \operatorname{div} (a \nabla \widetilde{w}_p) = \operatorname{div} (\widetilde{a} (p + \nabla w_{p,per})) , \tag{13}$$

it is immediate to see that proving the existence and uniqueness (up to the addition of a constant) of the corrector function $\widetilde{w}_p$ amounts to establishing the following (Calderón-Zygmund theory type) estimate

$$- \operatorname{div} (a \nabla u) = \operatorname{div} (f) \quad \Rightarrow \|\nabla u\|_{L^q(\mathbb{R}^d)} \leq C_q \, \|f\|_{L^q(\mathbb{R}^d)} \tag{14}$$

for the coefficient $a = a_{per} + \widetilde{a}$ and $\widetilde{a} \in L^r(\mathbb{R}^d)$. The result is then readily applied to $f = \widetilde{a}\,(p + \nabla w_{p,per})$ and $q = r$.

We indeed show that such an estimate (14) holds true using the (locally compact version of the) concentration-compactness principle [41] to reduce the problem to the periodic result of [10]. A quick outline of the proof goes as follows. Contradict (14) assuming the existence of two sequences $\nabla u_n \in L^q(\mathbb{R}^d)$ such that $\|\nabla u_n\|_{L^q(\mathbb{R}^d)} = 1$ and $f_n \in L^r(\mathbb{R}^d)$ such that $\|f_n\|_{L^q(\mathbb{R}^d)} \to 0$ as $n \to +\infty$, while $-\operatorname{div}(a\,\nabla u_n) = \operatorname{div}(f_n)$ for all $n \in \mathbb{N}$. If all the mass of ∇u_n escapes at infinity, then the product $\widetilde{a}\,\nabla u_n$ vanishes at infinity (since $\widetilde{a}$ itself vanishes there in some loose sense). The product term $a\,\nabla u_n$ on the left-hand side of the equation thus behaves like $a_{per}\,\nabla u_n$. This then contradicts the estimate analogous to (14) for the periodic operator, which we know is true by the results of [10]. On the other hand, if *some* mass of ∇u_n remains at finite distance from the origin, then we now essentially contradict the estimate on a bounded domain. But that estimate is again true using standard arguments. Thus the result.

The flexibility of the above strategy of proof allows it to carry over to other elliptic linear equations than (1), namely equations that are not in divergence form, or advection-diffusion type equations. The details may be found in [21, 22].

2.4 Rates of convergence

Once the existence of a suitable corrector function w_p is established, we may use this function to construct a two-scale approximation (5) of the solution u_ε for sufficiently small ε and study the rate of convergence of the remainder R_ε defined in (6). Our main result in this direction is the following.

Theorem [14, 15]: *Assume $d \geq 3$, $r \neq d$ and $\widetilde{a} \in L^r(\mathbb{R}^d)$. Take $a = a^{per} + \widetilde{a}$ with the usual properties of ellipticity and Hölder regularity. Consider a right-hand side $f \in L^2(\Omega)$, a strict subdomain $\Omega_1 \subset\subset \Omega$ and the residual*

$$R_\varepsilon = u^\varepsilon - u^* - \varepsilon \sum_{i=1}^{d} \partial_i u^*(\cdot)\, w_i(\cdot/\varepsilon).$$

Then

$$\|\nabla R_\varepsilon\|_{L^2(\Omega_1)} \leq C\varepsilon^{\min(1,d/r)}\, \|f\|_{L^2(\Omega)}.$$

When in addition $f \in L^q(\Omega)$ for $q \geq 2$, we have

$$\|\nabla R_\varepsilon\|_{L^q(\Omega_1)} \leq C\varepsilon^{\min(1,d/r)}\, \|f\|_{L^q(\Omega)}$$

If f is Hölder continuous, then

$$\|\nabla R_\varepsilon\|_{L^\infty(\Omega_1)} \leq C\varepsilon^{\min(1,d/r)}\, \left(1 + |\ln \varepsilon^{-1}|\right)\, \|f\|_{C^{0,\beta}(\Omega)}.$$

The proof follows the same pattern as those by M. Avellaneda and F.-H. Lin in [8, 9, 10] and by C. Kenig and coll. in [37] in the periodic case. It combines the following five main ingredients:

1. the differential operator L_ε in (1) converges to the constant coefficient homogenized operator L^* in (2), thus the properties of the latter operator also hold for the former operators when ε is small

2. a general estimate of the Green function $\mathcal{G}_\varepsilon(x, y)$ of the operator L_ε established by M. Grueter and K.-O. Widman in [34] and for which only ellipticity of the operator is needed

3. an estimate of the derivatives $\partial_x \mathcal{G}_\varepsilon(x, y)$ and $\partial_x \partial_y \mathcal{G}_\varepsilon(x, y)$, for which, this time, the specific structure of the coefficient is needed

4. an estimate of the rate of convergence of R_ε for a regular right-hand side

5. an argument by duality for the convergence of $\mathcal{G}_\varepsilon(x, y) - \mathcal{G}^*(x, y)$ (where $\mathcal{G}^*$ is the Green function associated to L^*).

3 Some nonlinear equations

3.1 A striking difference

One of our first observations when studying coefficients of the form (10)-(11) was that the homogenized equation obtained would then be equal to that obtained in the absence of perturbation. Our intuition was then driven by formal arguments about averages of functions over large balls. The result is also intuitive because, in a diffusion equation such as (1), there are all reasons to think that local microscopic defects do not percolate at the macroscale. They only matter when zooming in microscopically. The above two assertions are mathematically translated into the fact that the homogenized equation remains unperturbed while the corrector equation is different.

Figuratively speaking, we may express this upon claiming that an elliptic equation is very forgiving. But not all equations are...

In order to emphasize the difference of behavior between different categories of equations, let us consider the following one-dimensional, simple, first-order Hamilton-Jacobi equation

$$u_\varepsilon + |(u_\varepsilon)'| = \widetilde{V}(x/\varepsilon) \quad \text{in } \mathbb{R}. \tag{15}$$

This equation should be understood as the original unperturbed equation $u_\varepsilon + |(u_\varepsilon)'| = V_{per}(x/\varepsilon)$ specifically considered for a null periodic potential $V_{per} = 0$ and that is subsequently perturbed by the potential $\widetilde{V}$. In this new language, the potentials V_{per} and $\widetilde{V}$ respectively play the role of our coefficients a_{per} and $\widetilde{a}$ of the previous section.

In the absence of perturbation, the equation admits the only trivial solution $u_\varepsilon = 0$ and therefore homogenizes in the same equation $u + |u'| = 0$. With a perturbation $\widetilde{V}$, interesting phenomena appear.

If, for instance (and for simplicity), $\widetilde{V}$ is a nonpositive, compactly supported potential such that $\widetilde{V}(0) = \inf_{\mathbb{R}} \widetilde{V} < 0$, then it may be easily shown (in fact explicitly exhibiting u_ε analytically) that u_ε converges uniformly to $\bar{u} = \widetilde{V}(0)e^{-|x|}$, solution to

$$\begin{cases} \bar{u}(x) + |(\bar{u})'(x)| = 0 \qquad \forall x \neq 0, \\ \bar{u}(0) = \widetilde{V}(0). \end{cases} \tag{16}$$

Obviously, the limiting equation as $\varepsilon \to 0$ is thus different from the trivial equation $u + |u'| = 0$. But, more importantly and interestingly enough, (16) is *not* a differential equation on the real line, but only two separate equations of the half-lines, combined to one another using a Dirichlet type condition at the origin. Put differently, the defect $\widetilde{V}$ macroscopically (and tremendously) affects the homogenized limit.

Even more interestingly, it might be the case, with a different "alignment of planets" and in the same equation, that the defect does not at all affect the homogenized equation. It suffices to now consider a *nonnegative* perturbation $\widetilde{V}$ (still smooth and compactly supported, for simplicity). In that case, and in sharp contrast with the former situation, an argument equally simple as the previous one shows that the solution u_ε then converges to $u = 0$, the solution to $u + |u'| = 0$. The defect does not show up in the homogenized equation.

In a nutshell, defects in elliptic equations are somewhat harmless (they only matter after the dominant order) and are all about averages. Defects in hyperbolic equations are more treacherous, and, specifically for problems that take root in control theory (and the above first order Hamilton-Jacobi equation is one such problem), are all about infimums.

In the sequel of this section, we only consider the homogenized equation. In one particular case of the setting we consider, this homogenized equation itself is modified. Should it not be the case, that is when the homogenized equation remains identical to that of the periodic case and the perturbation only interferes at the next order (an option closer to that of the linear case we have studied in Section 2), the result we establish has to be complemented by some other results regarding the corrector equation specifically. Such results have been obtained by P.-L. Lions and P. Souganidis in [40, 42].

3.2 A result for some first order Hamilton-Jacobi equations

In the work [1], we have considered the following general class of first order Hamilton-Jacobi equations

$$u_\epsilon + H\left(x/\epsilon, Du_\epsilon\right) = 0 \quad \text{ in } \mathbb{R}^d, \tag{17}$$

where the Hamiltonian H, which has all the usual nice properties in terms of regularity, convexity and coerciveness, is the perturbation of a periodic Hamiltonian H_{per} by a local defect. For simplicity, one may think of $H(y,p) = H_{per}(y,p) - \widetilde{V}(y)$.

We have then established the following result.

Theorem [1]
As $\epsilon \to 0$, the solution u_ϵ converges locally uniformly to the unique bounded, uniformly continuous function u defined by:

- *u is a viscosity solution of*

$$u + \overline{H}_{per}(Du) = 0 \quad \textit{in } \mathbb{R}^d \setminus \{0\},$$

  with $\overline{H}_{per}$ defined by periodic homogenization.

- $$u(0) \leq -E, \tag{18}$$

  where E is the ergodic constant, or effective Dirichlet datum.
  If $\phi \in C^1(\mathbb{R}^d)$ is such that $u - \phi$ has a local maximum at the origin, then

$$u(0) + \overline{H}_{per}(D\phi(0)) \leq 0. \tag{19}$$

- *If $\phi \in C^1(\mathbb{R}^d)$ is such that $u - \phi$ has a local minimum at the origin, then*

$$u(0) + \max\left(E, \overline{H}_{per}(D\phi(0))\right) \geq 0. \tag{20}$$

The ergodic constant E appearing in the statement of the above theorem is defined in the course of the proof of this theorem. Somewhat more precisely, its definition proceeds as follows. We first consider the approximate/truncated corrector problem

$$\lambda w^{\lambda,R} + H(y, Dw^{\lambda,R}) = 0 \quad \text{ in } B(0,R),$$

with suitable, so called "state-constrained" boundary conditions on $\partial B(0,R)$ (see [27, 46]). As $\lambda \to 0$, the difference $w^{\lambda,R} - w^{\lambda,R}(0)$ can then be shown to converge to some function w^R, viscosity solution of

$$\begin{array}{rcll} H(y, Dw^R) & \leq & E^R & \text{ in } B(0,R), \\ H(y, Dw^R) & \geq & E^R & \text{ in } \overline{B(0,R)}. \end{array}$$

Then the ergodic constant is defined as $E = \lim_{R\to\infty} E^R$, the sequence E^R being proven monotonic. Once the ergodic constant is defined, the proof of the homogenization limit makes use of the classical perturbed test functions method [29] adjusted to the case at hand. It also uses several techniques from the control theoretic interpretation of the problem, see [11] for a general exposition on the subject.

In view of the condition (20), it is clear that the defect affects, or not, the homogenized equation itself, depending upon whether

$$E > \min_{p \in \mathbb{R}^d} \overline{H}_{per}(p) \quad \text{ or not.}$$

This condition in turn depends, say in a simple setting such as that we introduced in Section 3.1, on the "sign" of defect. We of course easily recognize the specific results of Section 3.1 in the general statements of the above Theorem. In particular then, $\overline{H}_{per}(p) = |p|$ and $E = -\widetilde{V}(0)$.

The statement of the above Theorem illustrates the relation of the problem considered with some previous works on Hamilton-Jacobi equations on heterogeneous structures (networks, stratified media, ...), such as the works by Y. Achdou and N. Tchou [3, 2], G. Barles [12], N. Forcadel, C. Imbert [35], all together with their respective collaborators.

4 Some topics left aside and some questions for future research

A general recollection of the works performed and of most of the issues related to those overviewed in this article may be found in the textbooks [16, 17]. Nevertheless, we would like to mention in this final section a few issues that have been omitted in the previous three sections.

The case of a localized defect, vanishing at infinity in some loose sense such as (10)-(11) and inserted in an elliptic equation, is one among many that may be considered, even in the setting of linear equations of Section 2 only. In [20], were also considered some prototypical interface problems where two different, incommensurable periodic structures are separated by a flat interface, that is

$$a^{per}(x) = a_{per,1,2}(x) = \begin{cases} a_{per,1}(x) & \text{when} \quad x_1 \leq 0, \\ a_{per,2}(x) & \text{when} \quad x_1 > 0. \end{cases}$$

This setting is the mathematical formalization of the physically relevant problem of *twin-boundaries*. After [20], it was more thoroughly explored by M. Josien and C. Raithel in [36].

The defects may also affect the geometry of the domain itself, as is the case for domains with nonperiodic arrays of perforations, a case studied in the works [23, 48] by X. Blanc and S. Wolf.

Another option is to study periodic coefficients that are perturbed by defects that are not vanishing at infinity but that are only "rare" at infinity. This is the case of the work [32] by R. Goudey.

In the context of the Hamilton-Jacobi equations approached in Section 3, it is worth mentioning that other geometries (defects on interfaces, etc) and extensions to *viscous* Hamilton-Jacobi equations are yet to be considered.

In both the linear and the nonlinear settings, some randomized variants of the problems with defects may also be studied. In short, the defects are then supposed to appear with a certain probability and the homogenized problems obtained are then identified. Such a setting may be seen as a compromise between a somehow idealistic scenario of a deterministic set of defects and a prohibitively computationally expensive and theoretically demanding general random setting. Examples of research efforts in this direction are [5, 39] in the linear elliptic case and [28, 1] in the Hamilton-Jacobi case.

But more generally speaking, we would like to conclude this review upon mentioning that all the settings considered are particular examples or variants of a general theory that we were originally aiming at developing for homogenization problems.

The underlying formalism for the theory was originally introduced in [18] in a slightly different (but intrinsically related) context, that of thermodynamic limit problems. It all starts from the consideration of a suitable set of points $\{X_k\}_{\mathbb{Z}^d}$, distributed over the ambient space $\mathbb{R}^d$, that are not necessarily arranged in a periodic array, but that are sufficiently well organized geometrically. Prototypical functions are constructed using translations along this set of points, that is functions of the form $\sum_{k\in\mathbb{Z}^d} \psi(x - X_k)$ for $\psi \in C_0^\infty(\mathbb{R}^d)$. If some adequate geometric conditions such as

$$\begin{cases} \sharp\{X_k \in B\} \quad \propto \quad \text{volume}\,(B) \\ \sharp\{X_k - X_{k'} \approx L\} \quad \text{controlled, for all}\, L \\ \sharp\{(X_k, X_{k'}, X_{k''})\} \quad ... \end{cases} \tag{21}$$

ruling the correlations of these points are imposed, then it is possible to then construct some *algebras* $\mathcal{A}$ of functions that have interesting averaging properties. Omitting some technicalities, the question of developing an homogenization theory for (say) equations of the form (1) with coefficients a in such an algebra $\mathcal{A}$, then reduces to establishing the existence of a solution w_p to the corrector equation (9) that satisfies $\nabla w_p \in \mathcal{A}$ and has zero average in this algebra. A related line of thought is presented in [43] and other works by the same author and his collaborators, where some algebras for homogenization theory are also constructed. The corrector equation (9) is however then solved in a sense different from the sense of distribution. We therefore cannot use a similar construction in our own endeavor. In the absence of a general strategy for solving this question, we have only been able to consider some specific *instances* of this general problem. The most recent example in this line of research is the work [33] where homogenization of the Schrödinger equation $-\Delta u_\varepsilon + \varepsilon^{-\alpha} V(./\varepsilon)\, u_\varepsilon = f$ is considered for a general class of highly oscillatory potentials V constructed using a set of points X_k as above.

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