



# Séminaire Laurent Schwartz

## EDP et applications

Année 2022-2023

Exposé n° II (18 octobre 2022)

Claude Le Bris

DEFECTS IN HOMOGENIZATION THEORY

<https://doi.org/10.5802/slsedp.157>

© Les auteurs, 2022-2023.



Cet article est mis à disposition selon les termes de la licence  
LICENCE INTERNATIONALE D'ATTRIBUTION CREATIVE COMMONS BY 4.0.  
<https://creativecommons.org/licenses/by/4.0/>

Institut des hautes études scientifiques  
Le Bois-Marie • Route de Chartres  
F-91440 BURES-SUR-YVETTE  
<http://www.ihes.fr/>

Centre de mathématiques Laurent Schwartz  
CMLS, École polytechnique, CNRS, Université  
Paris-Saclay  
F-91128 PALAISEAU CEDEX  
<http://www.math.polytechnique.fr/>



Publication membre du  
Centre Mersenne pour l'édition scientifique ouverte  
[www.centre-mersenne.org](http://www.centre-mersenne.org)  
e-ISSN : 2266-0607

# Defects in homogenization theory

Claude Le Bris<sup>\*†</sup>

## Abstract

We review a series of works that address homogenization for partial differential equations with highly oscillatory coefficients. A prototypical setting is that of periodic coefficients that are locally, or more globally perturbed. We investigate the homogenization limits obtained, first for linear elliptic equations, both in conservative and non conservative forms, and next for nonlinear equations such as Hamilton-Jacobi type equations.

## 1 Introduction

Consider the simple (yet ubiquitous) equation

$$-\operatorname{div} (a(x/\varepsilon) \nabla u_\varepsilon(x)) = f(x), \quad (1)$$

posed on a domain  $\mathcal{D}$  of the ambient space  $\mathbb{R}^d$ , and supplied with, say, homogeneous Dirichlet boundary condition on  $\partial\mathcal{D}$ . The coefficient within the divergence operator is a rescaled function  $a$ , highly oscillatory at the presumably small scale  $\varepsilon$ . It is supposed to be bounded and bounded away from zero, so that the equation is well-posed in  $H_0^1(\mathcal{D})$ , for, say,  $f \in L^2(\mathcal{D})$ .

We intend to study the homogenization limit of this equation. Our important assumption, for this purpose, is that the coefficient  $a$  is not necessarily periodic. It does not belong either to any of the classes of functions commonly considered in the literature of homogenization theory, such as quasi-periodic, almost-periodic or stationary ergodic functions. In the sequel, the function  $a$  will typically be a *perturbation* of a periodic function  $a_{per}$ , in a sense that will be made precise. Because of the relevance of this issue in materials modeling, we call such a perturbation a *defect*.

Of course, under our above assumptions on the coefficient within (1), the general theory of homogenization (see [49, 47]) applies, and we know there exists

---

<sup>\*</sup>Ecole des Ponts & Inria, 6 & 8 Avenue Blaise Pascal, Cité Descartes, Champs sur Marne, 77455 Marne La Vallée Cedex 2, France, [claude.le-bris@enpc.fr](mailto:claude.le-bris@enpc.fr).

<sup>†</sup>The research of the author is partly supported by ONR and EOARD, currently under the Grants ONR N00014-20-1-2691 and EOARD FA8655-20-1-7043. The writing of this manuscript has been completed during a visit at the Math+ Institute in Berlin. Partial financial support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy —The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689) —is gratefully acknowledged.

an homogenized limit, of the form

$$-\operatorname{div} (A^* \nabla u^*(x)) = f(x), \tag{2}$$

up to an extraction in  $\varepsilon$ . This compactness result, however, does not say anything, in whole generality, on the homogenized coefficient  $A^*$ . It does not say anything either on the types of convergence of  $u_\varepsilon$  to  $u^*$ , and *a fortiori* on the *rates* of this convergence, in terms of  $\varepsilon$ , in suitable functional spaces.

For the particular class of perturbations we consider (which makes our coefficient less general than for the classical homogenization theory we have just recalled), our purpose is to put homogenization theory for (1) on an equal footing with periodic homogenization theory for (1). More precisely, we expect to establish convergence of the *whole* sequence of solutions  $u_\varepsilon$  to the solution  $u^*$  of (2) and we expect the effective coefficient  $A^*$  to be *explicitly* expressed in terms of the data. We also expect to define a corrector function and to prove that this allows convergence of  $u_\varepsilon$  to hold strongly in  $H^1(\mathcal{D})$  and in other suitable Schauder and Sobolev spaces, once  $u^*$  is appropriately corrected by a term accounting for the fine scale oscillations. We finally expect to determine the rate of the convergences in the various functional spaces.

The particular elliptic linear, divergence form equation (1) is chosen for simplicity. Other, more sophisticated equations, such as linear elliptic equations that are not in divergence forms, or also nonlinear equations such as Hamilton-Jacobi type equations will also be discussed.

Let us at once make it clear that, *even* if (1) is linear, the question we examine has an intrinsic *nonlinear* nature. We indeed focus on the *nonlinear* (and in most cases also *nonlocal*) character of the application that maps our input parameter  $a$  to our output solution  $u$ . This can already be illustrated upon forcefully deleting the differential operators in (1): the application then reads as  $a \mapsto u = f/a$ . Put differently, we are investigating how a variation in  $a$  affects  $u$ . In this specific instance, the variation is a rescaling and a perturbation. We note in passing that the theory we present heavily relies upon the fact that the oscillatory coefficient  $a_\varepsilon$  is a *fixed rescaled* function  $a$ . The case of a general coefficient  $a_\varepsilon$  depending differently on the small scale parameter  $\varepsilon$  is irrelevant.

Our motivation for considering such a line of work is twofold.

The first incentive is modeling in *materials science* (see [38] and note that one could presumably apply the same observations to modeling in more general physical media). It is indeed our considered opinion that theoretical and computational materials science has witnessed a major evolution in the past decades. The major two novel features in this discipline are (a) an increasing importance of *multi-scale* phenomena (with the inclusion of the micro-scale in macro-scale simulations, both sequentially and concurrently) and (b) the consideration of materials with microstructures that are not necessarily periodic but contain possibly random, features —defects in structures, dislocations in lattices—breaking the idealized picture of an otherwise periodic model. Examples abound in many

fields of the engineering sciences and life sciences that testify of this evolution: composite materials for the aerospace industry, metallic alloys at use in nuclear engineering, etc.

Our second incentive is purely mathematical in nature. It specifically concerns *PDE theory*, and, to some extent, has little to do with homogenization theory. The rescaling  $x \rightarrow x/\varepsilon$  of the coefficient  $a$  in the equation (1) and its possible perturbation from a periodic function  $a_{per}$  to a more general function  $a$  is nothing but a possible practical means to understand the dependency of the solution upon the parameters of the equation. In addition, it is also likely to *create* an equation (the limit equation such as (2)) from a class of equations (the equations (1) for the family of parameters  $\varepsilon$ ), possibly extending the former class (a so-called “closure” problem—we shall see such a problem in Section 3) and explore which structures and regularities are carried over from one scale to another scale. It also allows (but the latter question will not be examined in the present contribution) to consider from this specific perspective *inverse problems*, where information on  $a$  is sought, based upon the observation of  $u_\varepsilon$  for various parameters  $\varepsilon$  and right-hand sides  $f$ .

Given the above twofold motivation, we now present a set of works where the common denominators are as follows. We explore the boundaries of homogenization theory by considering coefficients beyond the idealistic setting of periodic coefficients. When possible, we try and avoid fully general random coefficients, which (a) are difficult theoretically and (b) given our practical considerations, are often prohibitively expensive to address practically. We specifically consider coefficients that are *perturbations* of periodic materials.

Our Section 2 exclusively concerns itself with *linear* equations. The results we overview there, although necessarily partial, cover a large part of the issues mentioned above. Our Section 3 next presents some of the first steps of a similar mathematical endeavor put in action on *some* nonlinear equations, here Hamilton-Jacobi type equations. Our final Section 4 lists a few topics that have been left aside in our presentation, together with some pending issues in various directions of research. We also mention both our successes and the limitations of our results and techniques.

Most of the results overviewed in this article have been obtained in collaboration with the following colleagues: Yves Achdou (Université Paris-Cité), Xavier Blanc (Université Paris-Cité), Pierre Cardaliaguet (Université Paris-Dauphine), Pierre-Louis Lions (Collège de France), Panagiotis Souganidis (University of Chicago).

## 2 Linear elliptic equations

### 2.1 Our expectations

We first need to briefly recall the basics of periodic homogenization theory, before we extend the theory to the case of *perturbed* periodic coefficients. When

the possibly matrix-valued coefficient  $a$  in (1) is, say,  $\mathbb{Z}^d$ -periodic, then, as  $\varepsilon \rightarrow 0$ ,  $u_\varepsilon$  converges to  $u^*$  solution to (2). The homogenized coefficient  $A^*$  is then given, for  $1 \leq i, j \leq d$ , by

$$[A^*]_{ij} = \int_Q (e_i + \nabla w_{e_i, per}(y))^T A_{per}(y) e_j dy, \quad (3)$$

where  $Q = [0, 1]^d$  is the unit cube, and where, for any  $p \in \mathbb{R}^d$ , the function  $w_{p, per}$ , called the *corrector function*, also assumed a  $\mathbb{Z}^d$ -periodic function, solves the *corrector problem*. The latter problem reads

$$-\operatorname{div} [a_{per}(y) (p + \nabla w_{p, per})] = 0 \quad \text{in } Q. \quad (4)$$

The success of periodic homogenization theory and its impact on practical problems is then justified by the fact that, solving the  $d$  equations (4) corresponding to each  $p = e_i$ ,  $1 \leq i \leq d$ , on the bounded domain  $Q$  (a task that is considered straightforward by the metric of today's computational technology) allows to indeed determine the homogenized coefficient  $A^*$  thus the homogenized limit  $u^*$ . The “explicitness” of the expression (3) and the equation (4) is what we target in our more general setting of perturbed periodic coefficients.

In addition, the periodic theory also allows to accurately approximate  $u_\varepsilon$  in the regime of small parameters  $\varepsilon$ , using the so-called *two-scale expansion*

$$u_{\varepsilon,1}(x) = u^*(x) + \varepsilon \sum_{i=1}^d \frac{\partial u^*}{\partial x_i}(x) w_{e_i, per}\left(\frac{x}{\varepsilon}\right). \quad (5)$$

Rates of convergence for the difference

$$R_\varepsilon = u_\varepsilon - u_{\varepsilon,1} \quad (6)$$

in various functional spaces (starting from  $H^1(\mathcal{D})$ ) are also available. We again similarly intend to exhibit such rates of convergence in our perturbed setting.

In any event, there exists an enormous literature on homogenization theory and we only cite here some of the most famous references in the domain, with no claim whatsoever about exhaustiveness: [13, 49, 47, 4, 45], etc.

A substantial extension of the above periodic theory was accomplished in the ergodic stationary setting. The homogenized limit of the same equation as (1) but with a stationary ergodic coefficient  $a(x/\varepsilon, \omega)$  also reads as (1) where this time

$$[A^*]_{ij} = \mathbb{E} \left( \int_Q (e_i + \nabla w_{e_i, sto}(y, \cdot))^T A(y, \cdot) e_j dy \right), \quad (7)$$

and  $w_{p, sto}(y, \omega)$  is now the solution to the equation

$$-\operatorname{div} [A(y, \omega) (p + \nabla w_{p, sto}(y, \omega))] = 0 \quad (8)$$

in the *whole* space  $\mathbb{R}^d$ , with  $\nabla w_{p, sto}$  stationary and  $\mathbb{E} \left( \int_Q \nabla w_{p, sto}(y, \cdot) dy \right) = 0$ .

The list of contributors to random homogenization theory is also considerable. We only cite here the classical works [44, 25, 24]. Some recent significant developments appeared in [31, 30, 7, 6]. The random nonlinear setting was also considered and one of pioneering works in this direction is [26].

Formulae (7)-(8) show that, as in the periodic setting, the homogenized coefficient may thus be *explicitly* expressed. If (3) and (7) really look alike, the striking difference between (4) and (8), however, is that the latter equation is posed on the *unbounded* domain  $\mathbb{R}^d$ . Several mathematical difficulties regarding the well-posedness of the problem originate from this difference. This is a feature we will also find in our *perturbed* periodic setting. Likewise, huge computational difficulties also arise, but these are not our focus here. On the other hand, rates of convergence are also a question significantly more difficult in the random setting than in the periodic setting. In our perturbed setting, we will also observe some flavor of this.

## 2.2 Intuitive description of our line of research

The brutal technique consisting in eliminating the differential operators in (1) already used in Section 1, suggests that, in order for the entire sequence of “solutions”  $u_\varepsilon = f/a(\cdot/\varepsilon)$  to weakly converge to a limit that can be explicitly identified, it is sufficient for the function on the right hand side to admit an average.

Our strategy has therefore been to look for classes of functions for this to happen, with the hope (indeed often fulfilled) that the same classes will be suitable for coefficients  $a$  in (1) and for the expected homogenization theory.

It is well known that periodic, quasiperiodic, almost periodic, stationary ergodic, functions are all such admissible classes. On the other hand, functions modeling a periodic background perturbed by a so-called local defect, such as functions that read as  $a_{per} + C_0^\infty$ , for  $a_{per}$  a periodic function and  $C_0^\infty$  denoting the space of smooth compactly supported functions, are also convenient. Further, some specific functions that are more global perturbations of a periodic functions may also be used. For instance, we may be willing to consider functions such as  $\sum_{k \in \mathbb{Z}^d} \psi(x - k - Z_k)$  where  $Z_k$  denotes a small displacement of the original periodic position  $k$ .

In any event, we also learn from the consideration of  $u_\varepsilon = f/a(\cdot/\varepsilon)$  that we will have to consider inverses and products of such functions. Ideally, this is a notion of *algebras* and not vector spaces that is indeed relevant. We will return to this in Section 4.

Once a suitable class of coefficients  $a$  is anticipated, the crucial task is to establish the well-posedness of the corrector equation in that class. This is perfectly understable in particular since our setting above is linear. Homogenization theory almost reduces then to the corrector equation. The purpose is to solve an equation analogous to (4) and next build the corresponding two-scale expansion in the vein of (5).

### 2.3 Existence of a suitable corrector function

In the case of our simple elliptic, linear, divergence form equation (1), the corrector equation we have to solve is the analogous equation to (4), namely

$$-\operatorname{div} (a(y) (p + \nabla w(y))) = 0, \quad (9)$$

where the coefficient is  $a$  and not the periodic coefficient  $a_{per}$ . The equation is this time posed on the whole ambient space  $\mathbb{R}^d$ , just as the corrector equation (8) for the ergodic stationary case is. It is supplied with a boundary condition at infinity that should express the strict sublinearity  $\frac{w(y)}{1 + |y|} \rightarrow 0$  (this property ensures that the rightmost term in (5) is indeed a correction to the leading term). In the absence of any structure, we are unaware of any approach that allows to establish existence for equation (9).

One may indeed realize that, in the ergodic stationary case, as well as in all the related (periodic, quasiperiodic, almost periodic) settings, the proof of existence relies upon a reinterpretation of the equation (9) that explicitly uses the structure of the coefficient. Put differently, the equation is lifted from an equation on the space  $\mathbb{R}^d$  to an equation solved on the torus, or, say, on the abstract probability space. In all such settings, some type of “compactness”, originally absent from the equation posed in  $\mathbb{R}^d$  is reinstated. A possible alternative perspective on this difficulty is to express that, in all the above settings, it is possible to pass from an estimate on large balls to a local estimate. The most natural estimate on approximated solutions are obtained on average, and, precisely because of the structure imposed, they translate into local estimates that in turn allow to pass to the limit, at least in the sense of distributions, in the sequence of regularizations.

In our own setting, we are going to also impose a structure on the coefficient  $a$ . We consider

$$a = a_{per} + \tilde{a}, \quad (10)$$

where  $a_{per}$  denotes the unperturbed, periodic background, and  $\tilde{a}$  denotes the perturbation, which belongs to a Lebesgue space, that is

$$\tilde{a} \in L^r(\mathbb{R}^d), \quad \text{for some } 1 \leq r < +\infty. \quad (11)$$

Since we are also going to assume, in most cases, that  $\tilde{a}$  is (uniformly) Hölder continuous, this global integrability implies that  $\tilde{a}$  vanishes at infinity. The defect we consider is therefore, in that sense, *local*. Evidently, our setting is an over-simplification of the much more practically relevant condition  $\tilde{a} \xrightarrow{|x| \rightarrow \infty} 0$  (a condition that is reminiscent of the space  $a_{per} + C_0^\infty$  we were mentioning in the previous section). We are indeed unable to proceed in the full mathematical generality of the latter condition.

Given (10)-(11), it is anticipated that the homogenized equation obtained is identical to that for the periodic coefficient  $a_{per}$ . Intuitively, the reason is,

the coefficient  $\tilde{a}$  does not contribute to averages over large balls. The detailed mathematical study indeed confirms that the homogenized limit is the periodic one.

The key task is, on the other hand, to solve the corrector equation (9). It is readily seen, introducing  $\tilde{w}_p = w_p - w_{p,per}$  where  $w_{p,per}$  is the solution to (4), that the suitable functional class where to look for  $\tilde{w}_p$  is such that  $\nabla \tilde{w}_p \in L^r(\mathbb{R}^d)$ .

The work [19] contains a first theoretical study in the case  $r = 2$ , along with some computational illustrations. On the one hand, the proof of existence of the corrector function, solution to (4) for (10) and (11) with  $r = 2$ , is performed on the basis of arguments only relevant in this Hilbertian setting. In addition, it is observed there, numerically, that a two-scale expansion of the type (5) employing the periodic corrector  $w_{p,per}$  does not provide an accurate approximation at the vicinity of the defects, that is the region where  $\tilde{a}$  is large. On the other hand, the same expansion with  $w_p$  instead of  $w_{p,per}$  reinstates everywhere in the domain the quality of approximation observed in the absence of defect.

The general case of a defect  $\tilde{a} \in L^r(\mathbb{R}^d)$ , for  $r$  not necessarily equal to 2, was next addressed in the work [20]. We prove there the

**Theorem** [ *$L^r$ -perturbation of a periodic coefficient, [20]*] : *Assume periodicity of the background coefficient  $a_{per}$  and (coercivity, boundedness and) Hölder regularity of both  $a_{per}$  and  $a$ . Then, the corrector problem has a unique solution  $w_p$ , up to the addition of a constant. Moreover,  $w_p = w_{p,per} + \tilde{w}_p$ , where  $w_{p,per}$  is the periodic corrector and*

- *if  $1 \leq r < d$ , then,  $\lim_{|x| \rightarrow +\infty} \tilde{w}_p(x) = 0$  ;*
- *if  $2 \leq r$ , then  $\nabla \tilde{w}_p \in L^r(\mathbb{R}^d)$ .*

The proof of this theorem performed in [20] uses estimates of the Green function on dyadic rings. The corrector equation is written under the form

$$-\operatorname{div}(a_{per} \nabla \tilde{w}_p) = \operatorname{div}(\tilde{a} \nabla \tilde{w}_p) + \operatorname{div}(\tilde{a}(p + \nabla w_{p,per})). \quad (12)$$

This isolates in the left-hand side the operator with periodic coefficients for which the fundamental results established by M. Avellaneda and F.-H. Lin, in [8, 9, 10] are then used.

A more general and versatile proof of the same result was then presented in [21]. Since the corrector equation also reads as

$$-\operatorname{div}(a \nabla \tilde{w}_p) = \operatorname{div}(\tilde{a}(p + \nabla w_{p,per})), \quad (13)$$

it is immediate to see that proving the existence and uniqueness (up to the addition of a constant) of the corrector function  $\tilde{w}_p$  amounts to establishing the following (Calderón-Zygmund theory type) estimate

$$-\operatorname{div}(a \nabla u) = \operatorname{div}(f) \quad \Rightarrow \quad \|\nabla u\|_{L^q(\mathbb{R}^d)} \leq C_q \|f\|_{L^q(\mathbb{R}^d)} \quad (14)$$



for the coefficient  $a = a_{per} + \tilde{a}$  and  $\tilde{a} \in L^r(\mathbb{R}^d)$ . The result is then readily applied to  $f = \tilde{a}(p + \nabla w_{p,per})$  and  $q = r$ .

We indeed show that such an estimate (14) holds true using the (locally compact version of the) concentration-compactness principle [41] to reduce the problem to the periodic result of [10]. A quick outline of the proof goes as follows. Contradict (14) assuming the existence of two sequences  $\nabla u_n \in L^q(\mathbb{R}^d)$  such that  $\|\nabla u_n\|_{L^q(\mathbb{R}^d)} = 1$  and  $f_n \in L^r(\mathbb{R}^d)$  such that  $\|f_n\|_{L^q(\mathbb{R}^d)} \rightarrow 0$  as  $n \rightarrow +\infty$ , while  $-\operatorname{div}(a \nabla u_n) = \operatorname{div}(f_n)$  for all  $n \in \mathbb{N}$ . If all the mass of  $\nabla u_n$  escapes at infinity, then the product  $\tilde{a} \nabla u_n$  vanishes at infinity (since  $\tilde{a}$  itself vanishes there in some loose sense). The product term  $a \nabla u_n$  on the left-hand side of the equation thus behaves like  $a_{per} \nabla u_n$ . This then contradicts the estimate analogous to (14) for the periodic operator, which we know is true by the results of [10]. On the other hand, if *some* mass of  $\nabla u_n$  remains at finite distance from the origin, then we now essentially contradict the estimate on a bounded domain. But that estimate is again true using standard arguments. Thus the result.

The flexibility of the above strategy of proof allows it to carry over to other elliptic linear equations than (1), namely equations that are not in divergence form, or advection-diffusion type equations. The details may be found in [21, 22].

## 2.4 Rates of convergence

Once the existence of a suitable corrector function  $w_p$  is established, we may use this function to construct a two-scale approximation (5) of the solution  $u_\varepsilon$  for sufficiently small  $\varepsilon$  and study the rate of convergence of the remainder  $R_\varepsilon$  defined in (6). Our main result in this direction is the following.

**Theorem [14, 15]:** *Assume  $d \geq 3$ ,  $r \neq d$  and  $\tilde{a} \in L^r(\mathbb{R}^d)$ . Take  $a = a^{per} + \tilde{a}$  with the usual properties of ellipticity and Hölder regularity. Consider a right-hand side  $f \in L^2(\Omega)$ , a strict subdomain  $\Omega_1 \subset\subset \Omega$  and the residual*

$$R_\varepsilon = u^\varepsilon - u^* - \varepsilon \sum_{i=1}^d \partial_i u^*(\cdot) w_i(\cdot/\varepsilon).$$

Then

$$\|\nabla R_\varepsilon\|_{L^2(\Omega_1)} \leq C \varepsilon^{\min(1, d/r)} \|f\|_{L^2(\Omega)}.$$

When in addition  $f \in L^q(\Omega)$  for  $q \geq 2$ , we have

$$\|\nabla R_\varepsilon\|_{L^q(\Omega_1)} \leq C \varepsilon^{\min(1, d/r)} \|f\|_{L^q(\Omega)}$$

If  $f$  is Hölder continuous, then

$$\|\nabla R_\varepsilon\|_{L^\infty(\Omega_1)} \leq C \varepsilon^{\min(1, d/r)} (1 + |\ln \varepsilon^{-1}|) \|f\|_{C^{0,\beta}(\Omega)}.$$

The proof follows the same pattern as those by M. Avellaneda and F.-H. Lin in [8, 9, 10] and by C. Kenig and coll. in [37] in the periodic case. It combines the following five main ingredients:

1. the differential operator  $L_\varepsilon$  in (1) converges to the constant coefficient homogenized operator  $L^*$  in (2), thus the properties of the latter operator also hold for the former operators when  $\varepsilon$  is small
2. a general estimate of the Green function  $\mathcal{G}_\varepsilon(x, y)$  of the operator  $L_\varepsilon$  established by M. Grueter and K.-O. Widman in [34] and for which only ellipticity of the operator is needed
3. an estimate of the derivatives  $\partial_x \mathcal{G}_\varepsilon(x, y)$  and  $\partial_x \partial_y \mathcal{G}_\varepsilon(x, y)$ , for which, this time, the specific structure of the coefficient is needed
4. an estimate of the rate of convergence of  $R_\varepsilon$  for a regular right-hand side
5. an argument by duality for the convergence of  $\mathcal{G}_\varepsilon(x, y) - \mathcal{G}^*(x, y)$  (where  $\mathcal{G}^*$  is the Green function associated to  $L^*$ ).

### 3 Some nonlinear equations

#### 3.1 A striking difference

One of our first observations when studying coefficients of the form (10)-(11) was that the homogenized equation obtained would then be equal to that obtained in the absence of perturbation. Our intuition was then driven by formal arguments about averages of functions over large balls. The result is also intuitive because, in a diffusion equation such as (1), there are all reasons to think that local microscopic defects do not percolate at the macroscale. They only matter when zooming in microscopically. The above two assertions are mathematically translated into the fact that the homogenized equation remains unperturbed while the corrector equation is different.

Figuratively speaking, we may express this upon claiming that an elliptic equation is very forgiving. But not all equations are...

In order to emphasize the difference of behavior between different categories of equations, let us consider the following one-dimensional, simple, first-order Hamilton-Jacobi equation

$$u_\varepsilon + |(u_\varepsilon)'| = \tilde{V}(x/\varepsilon) \quad \text{in } \mathbb{R}. \quad (15)$$

This equation should be understood as the original unperturbed equation  $u_\varepsilon + |(u_\varepsilon)'| = V_{per}(x/\varepsilon)$  specifically considered for a null periodic potential  $V_{per} = 0$  and that is subsequently perturbed by the potential  $\tilde{V}$ . In this new language, the potentials  $V_{per}$  and  $\tilde{V}$  respectively play the role of our coefficients  $a_{per}$  and  $\tilde{a}$  of the previous section.

In the absence of perturbation, the equation admits the only trivial solution  $u_\varepsilon = 0$  and therefore homogenizes in the same equation  $u + |u'| = 0$ . With a perturbation  $\tilde{V}$ , interesting phenomena appear.

If, for instance (and for simplicity),  $\tilde{V}$  is a nonpositive, compactly supported potential such that  $\tilde{V}(0) = \inf_{\mathbb{R}} \tilde{V} < 0$ , then it may be easily shown (in fact explicitly exhibiting  $u_\varepsilon$  analytically) that  $u_\varepsilon$  converges uniformly to  $\bar{u} = \tilde{V}(0)e^{-|x|}$ , solution to

$$\begin{cases} \bar{u}(x) + |(\bar{u})'(x)| = 0 & \forall x \neq 0, \\ \bar{u}(0) = \tilde{V}(0). \end{cases} \quad (16)$$

Obviously, the limiting equation as  $\varepsilon \rightarrow 0$  is thus different from the trivial equation  $u + |u'| = 0$ . But, more importantly and interestingly enough, (16) is *not* a differential equation on the real line, but only two separate equations of the half-lines, combined to one another using a Dirichlet type condition at the origin. Put differently, the defect  $\tilde{V}$  macroscopically (and tremendously) affects the homogenized limit.

Even more interestingly, it might be the case, with a different “alignment of planets” and in the same equation, that the defect does not at all affect the homogenized equation. It suffices to now consider a *nonnegative* perturbation  $\tilde{V}$  (still smooth and compactly supported, for simplicity). In that case, and in sharp contrast with the former situation, an argument equally simple as the previous one shows that the solution  $u_\varepsilon$  then converges to  $u = 0$ , the solution to  $u + |u'| = 0$ . The defect does not show up in the homogenized equation.

In a nutshell, defects in elliptic equations are somewhat harmless (they only matter after the dominant order) and are all about averages. Defects in hyperbolic equations are more treacherous, and, specifically for problems that take root in control theory (and the above first order Hamilton-Jacobi equation is one such problem), are all about infimums.

In the sequel of this section, we only consider the homogenized equation. In one particular case of the setting we consider, this homogenized equation itself is modified. Should it not be the case, that is when the homogenized equation remains identical to that of the periodic case and the perturbation only interferes at the next order (an option closer to that of the linear case we have studied in Section 2), the result we establish has to be complemented by some other results regarding the corrector equation specifically. Such results have been obtained by P.-L. Lions and P. Souganidis in [40, 42].

### 3.2 A result for some first order Hamilton-Jacobi equations

In the work [1], we have considered the following general class of first order Hamilton-Jacobi equations

$$u_\varepsilon + H(x/\varepsilon, Du_\varepsilon) = 0 \quad \text{in } \mathbb{R}^d, \quad (17)$$

where the Hamiltonian  $H$ , which has all the usual nice properties in terms of regularity, convexity and coerciveness, is the perturbation of a periodic Hamiltonian  $H_{per}$  by a local defect. For simplicity, one may think of  $H(y, p) = H_{per}(y, p) - \tilde{V}(y)$ .

We have then established the following result.

**Theorem [1]**

As  $\epsilon \rightarrow 0$ , the solution  $u_\epsilon$  converges locally uniformly to the unique bounded, uniformly continuous function  $u$  defined by:

- $u$  is a viscosity solution of

$$u + \overline{H}_{per}(Du) = 0 \quad \text{in } \mathbb{R}^d \setminus \{0\},$$

with  $\overline{H}_{per}$  defined by periodic homogenization.

- 

$$u(0) \leq -E, \tag{18}$$

where  $E$  is the ergodic constant, or effective Dirichlet datum.

If  $\phi \in C^1(\mathbb{R}^d)$  is such that  $u - \phi$  has a local maximum at the origin, then

$$u(0) + \overline{H}_{per}(D\phi(0)) \leq 0. \tag{19}$$

- If  $\phi \in C^1(\mathbb{R}^d)$  is such that  $u - \phi$  has a local minimum at the origin, then

$$u(0) + \max(E, \overline{H}_{per}(D\phi(0))) \geq 0. \tag{20}$$

The ergodic constant  $E$  appearing in the statement of the above theorem is defined in the course of the proof of this theorem. Somewhat more precisely, its definition proceeds as follows. We first consider the approximate/truncated corrector problem

$$\lambda w^{\lambda,R} + H(y, Dw^{\lambda,R}) = 0 \quad \text{in } B(0, R),$$

with suitable, so called “state-constrained” boundary conditions on  $\partial B(0, R)$  (see [27, 46]). As  $\lambda \rightarrow 0$ , the difference  $w^{\lambda,R} - w^{\lambda,R}(0)$  can then be shown to converge to some function  $w^R$ , viscosity solution of

$$\begin{aligned} H(y, Dw^R) &\leq E^R && \text{in } B(0, R), \\ H(y, Dw^R) &\geq E^R && \text{in } \overline{B(0, R)}. \end{aligned}$$

Then the ergodic constant is defined as  $E = \lim_{R \rightarrow \infty} E^R$ , the sequence  $E^R$  being proven monotonic. Once the ergodic constant is defined, the proof of the homogenization limit makes use of the classical perturbed test functions method [29] adjusted to the case at hand. It also uses several techniques from the control theoretic interpretation of the problem, see [11] for a general exposition on the subject.

In view of the condition (20), it is clear that the defect affects, or not, the homogenized equation itself, depending upon whether

$$E > \min_{p \in \mathbb{R}^d} \overline{H}_{per}(p) \quad \text{or not.}$$

This condition in turn depends, say in a simple setting such as that we introduced in Section 3.1, on the “sign” of defect. We of course easily recognize the specific results of Section 3.1 in the general statements of the above Theorem. In particular then,  $\overline{H}_{per}(p) = |p|$  and  $E = -\tilde{V}(0)$ .

The statement of the above Theorem illustrates the relation of the problem considered with some previous works on Hamilton-Jacobi equations on heterogeneous structures (networks, stratified media, ...), such as the works by Y. Achdou and N. Tchou [3, 2], G. Barles [12], N. Forcadel, C. Imbert [35], all together with their respective collaborators.

## 4 Some topics left aside and some questions for future research

A general recollection of the works performed and of most of the issues related to those overviewed in this article may be found in the textbooks [16, 17]. Nevertheless, we would like to mention in this final section a few issues that have been omitted in the previous three sections.

The case of a localized defect, vanishing at infinity in some loose sense such as (10)-(11) and inserted in an elliptic equation, is one among many that may be considered, even in the setting of linear equations of Section 2 only. In [20], were also considered some prototypical interface problems where two different, incommensurable periodic structures are separated by a flat interface, that is

$$a^{per}(x) = a_{per,1,2}(x) = \begin{cases} a_{per,1}(x) & \text{when } x_1 \leq 0, \\ a_{per,2}(x) & \text{when } x_1 > 0. \end{cases}$$

This setting is the mathematical formalization of the physically relevant problem of *twin-boundaries*. After [20], it was more thoroughly explored by M. Josien and C. Raithel in [36].

The defects may also affect the geometry of the domain itself, as is the case for domains with nonperiodic arrays of perforations, a case studied in the works [23, 48] by X. Blanc and S. Wolf.

Another option is to study periodic coefficients that are perturbed by defects that are not vanishing at infinity but that are only “rare” at infinity. This is the case of the work [32] by R. Goudey.

In the context of the Hamilton-Jacobi equations approached in Section 3, it is worth mentioning that other geometries (defects on interfaces, etc) and extensions to *viscous* Hamilton-Jacobi equations are yet to be considered.

In both the linear and the nonlinear settings, some randomized variants of the problems with defects may also be studied. In short, the defects are then supposed to appear with a certain probability and the homogenized problems obtained are then identified. Such a setting may be seen as a compromise between a somehow idealistic scenario of a deterministic set of defects and a

prohibitively computationally expensive and theoretically demanding general random setting. Examples of research efforts in this direction are [5, 39] in the linear elliptic case and [28, 1] in the Hamilton-Jacobi case.

But more generally speaking, we would like to conclude this review upon mentioning that all the settings considered are particular examples or variants of a general theory that we were originally aiming at developing for homogenization problems.

The underlying formalism for the theory was originally introduced in [18] in a slightly different (but intrinsically related) context, that of thermodynamic limit problems. It all starts from the consideration of a suitable set of points  $\{X_k\}_{\mathbb{Z}^d}$ , distributed over the ambient space  $\mathbb{R}^d$ , that are not necessarily arranged in a periodic array, but that are sufficiently well organized geometrically. Prototypical functions are constructed using translations along this set of points, that is functions of the form  $\sum_{k \in \mathbb{Z}^d} \psi(x - X_k)$  for  $\psi \in C_0^\infty(\mathbb{R}^d)$ . If some adequate geometric conditions such as

$$\begin{cases} \#\{X_k \in B\} & \propto \text{volume}(B) \\ \#\{X_k - X_{k'} \approx L\} & \text{controlled, for all } L \\ \#\{(X_k, X_{k'}, X_{k''})\} & \dots \end{cases} \quad (21)$$

ruling the correlations of these points are imposed, then it is possible to then construct some *algebras*  $\mathcal{A}$  of functions that have interesting averaging properties. Omitting some technicalities, the question of developing an homogenization theory for (say) equations of the form (1) with coefficients  $a$  in such an algebra  $\mathcal{A}$ , then reduces to establishing the existence of a solution  $w_p$  to the corrector equation (9) that satisfies  $\nabla w_p \in \mathcal{A}$  and has zero average in this algebra. A related line of thought is presented in [43] and other works by the same author and his collaborators, where some algebras for homogenization theory are also constructed. The corrector equation (9) is however then solved in a sense different from the sense of distribution. We therefore cannot use a similar construction in our own endeavor. In the absence of a general strategy for solving this question, we have only been able to consider some specific *instances* of this general problem. The most recent example in this line of research is the work [33] where homogenization of the Schrödinger equation  $-\Delta u_\varepsilon + \varepsilon^{-\alpha} V(\cdot/\varepsilon) u_\varepsilon = f$  is considered for a general class of highly oscillatory potentials  $V$  constructed using a set of points  $X_k$  as above.

## References

- [1] Yves Achdou and Claude Le Bris, *Homogenization of some periodic Hamilton-Jacobi equations with defects*, 2022, [arXiv:2211.16157](#).
- [2] Yves Achdou, Salomé Oudet, and Nicoletta Tchou, *Effective transmission conditions for Hamilton-Jacobi equations defined on two domains separated*

- by an oscillatory interface, *J. Math. Pures Appl.* (9) **106** (2016), no. 6, 1091–1121.
- [3] Yves Achdou and Nicoletta Tchou, *Hamilton-Jacobi equations on networks as limits of singularly perturbed problems in optimal control: dimension reduction*, *Comm. Partial Differential Equations* **40** (2015), no. 4, 652–693.
- [4] Grégoire Allaire, *Shape optimization by the homogenization method*, vol. 146, Springer, New York, NY, 2002.
- [5] Arnaud Anantharaman and Claude Le Bris, *A numerical approach related to defect-type theories for some weakly random problems in homogenization*, *Multiscale Model. Simul.* **9** (2011), no. 2, 513–544.
- [6] Scott Armstrong, Tuomo Kuusi, and Jean-Christophe Mourrat, *Quantitative stochastic homogenization and large-scale regularity*, vol. 352, Springer, Cham, 2019.
- [7] Scott N. Armstrong and Charles K. Smart, *Quantitative stochastic homogenization of elliptic equations in nondivergence form*, *Arch. Ration. Mech. Anal.* **214** (2014), no. 3, 867–911.
- [8] Marco Avellaneda and Fang-Hua Lin, *Compactness methods in the theory of homogenization*, *Commun. Pure Appl. Math.* **40** (1987), no. 6, 803–847.
- [9] ———, *Compactness methods in the theory of homogenization. II: Equations in non-divergence form*, *Commun. Pure Appl. Math.* **42** (1989), no. 2, 139–172.
- [10] ———,  *$L^p$  bounds on singular integrals in homogenization*, *Commun. Pure Appl. Math.* **44** (1991), no. 8-9, 897–910.
- [11] Martino Bardi and Italo Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, *Systems & Control: Foundations & Applications*, Birkhäuser Boston Inc., Boston, MA, 1997, With appendices by Maurizio Falcone and Pierpaolo Soravia.
- [12] Guy Barles and Emmanuel Chasseigne, *An illustrated guide of the modern approaches of Hamilton-Jacobi equations and control problems with discontinuities*, 2018, [arXiv:1812.09197](https://arxiv.org/abs/1812.09197).
- [13] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou, *Asymptotic analysis for periodic structures. Reprint of the 1978 original with corrections and bibliographical additions*, Providence, RI: AMS Chelsea Publishing, 2011.
- [14] Xavier Blanc, Marc Josien, and Claude Le Bris, *Local precised approximation in multiscale problems with local defects*, *C. R., Math., Acad. Sci. Paris* **357** (2019), no. 2, 167–174.

- [15] ———, *Precised approximations in elliptic homogenization beyond the periodic setting*, *Asymptotic Analysis* **116** (2020), no. 2, 93–137.
- [16] Xavier Blanc and Claude Le Bris, *Homogénéisation en milieu périodique... ou non : une introduction*, *Mathématiques & Applications*, vol. 88, Springer, 2023.
- [17] ———, *Homogenization theory for multiscale problems: an introduction*, *MS&A, Modeling, Simulation and Applications*, vol. 21, Springer, 2023.
- [18] Xavier Blanc, Claude Le Bris, and Pierre-Louis Lions, *A definition of the ground state energy for systems composed of infinitely many particles*, *Commun. Partial Differ. Equations* **28** (2003), no. 1-2, 439–475.
- [19] ———, *A possible homogenization approach for the numerical simulation of periodic microstructures with defects*, *Milan J. Math.* **80** (2012), no. 2, 351–367.
- [20] ———, *Local profiles for elliptic problems at different scales: defects in, and interfaces between periodic structures*, *Commun. Partial Differ. Equations* **40** (2015), no. 12, 2173–2236.
- [21] ———, *On correctors for linear elliptic homogenization in the presence of local defects*, *Commun. Partial Differ. Equations* **43** (2018), no. 6, 965–997.
- [22] ———, *On correctors for linear elliptic homogenization in the presence of local defects: the case of advection-diffusion*, *J. Math. Pures Appl. (9)* **124** (2019), 106–122.
- [23] Xavier Blanc and Sylvain Wolf, *Homogenization of the Poisson equation in a non-periodically perforated domain*, *Asymptotic Anal.* **126** (2022), no. 1-2, 129–155.
- [24] Alain Bourgeat and Andrey Piatnitski, *Estimates in probability of the residual between the random and the homogenized solutions of one-dimensional second-order operator*, *Asymptotic Anal.* **21** (1999), no. 3-4, 303–315.
- [25] ———, *Approximations of effective coefficients in stochastic homogenization*, *Ann. Inst. H. Poincaré Probab. Statist.* **40** (2004), no. 2, 153–165.
- [26] Luis A. Caffarelli, Panagiotis E. Souganidis, and L. Wang, *Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media*, *Comm. Pure Appl. Math.* **58** (2005), no. 3, 319–361.
- [27] Italo Capuzzo-Dolcetta and Pierre-Louis Lions, *Hamilton-Jacobi equations with state constraints*, *Trans. Amer. Math. Soc.* **318** (1990), no. 2, 643–683.
- [28] Pierre Cardaliaguet, Claude Le Bris, and Panagiotis E. Souganidis, *Perturbation problems in homogenization of Hamilton-Jacobi equations*, *J. Math. Pures Appl. (9)* **117** (2018), 221–262.



- [29] Lawrence Craig Evans, *The perturbed test function method for viscosity solutions of nonlinear PDE*, Proc. Roy. Soc. Edinburgh Sect. A **111** (1989), no. 3-4, 359–375.
- [30] Antoine Gloria, Stefan Neukamm, and Felix Otto, *A regularity theory for random elliptic operators*, Milan J. Math. **88** (2020), no. 1, 99–170.
- [31] Antoine Gloria and Felix Otto, *Quantitative results on the corrector equation in stochastic homogenization*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 11, 3489–3548.
- [32] Rémi Goudey, *A periodic homogenization problem with defects rare at infinity*, Netw. Heterog. Media **17** (2022), no. 4, 547–592.
- [33] Rémi Goudey and Claude Le Bris, *Linear elliptic homogenization for a class of highly oscillating non-periodic potentials*, 2022, [arXiv:2205.15600](https://arxiv.org/abs/2205.15600).
- [34] Michael Grueter and Kjell-Ove Widman, *The Green function for uniformly elliptic equations*, Manuscr. Math. **37** (1982), 303–342 (English).
- [35] Cyril Imbert and Régis Monneau, *Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 2, 357–448.
- [36] Marc Josien and Claudia Raithel, *Quantitative homogenization for the case of an interface between two heterogeneous media*, SIAM J. Math. Anal. **53** (2021), no. 1, 813–854.
- [37] Carlos Kenig, Fanghua Lin, and Zhongwei Shen, *Periodic homogenization of Green and Neumann functions*, Commun. Pure Appl. Math. **67** (2014), no. 8, 1219–1262.
- [38] Claude Le Bris, *Mathematics for the modeling of defects in materials*, Notices Amer. Math. Soc. **67** (2020), no. 6, 788–796.
- [39] Claude Le Bris and Florian Thomines, *A reduced basis approach for some weakly stochastic multiscale problems*, Chin. Ann. Math. Ser. B **33** (2012), no. 5, 657–672.
- [40] Pierre-Louis Lions, *Équations elliptiques ou paraboliques, et homogénéisation précisée [Elliptic or parabolic equations and precised homogenization]*, Lectures at Collège de France 2013/14, <https://www.college-de-france.fr/site/pierre-louis-lions>.
- [41] Pierre-Louis Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I & II*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **1** (1984), 109–145 & 223–283 (English).
- [42] Pierre-Louis Lions and Panagiotis Souganidis, in Lectures at Collège de France [40].

- [43] Gabriel Nguetseng, *Homogenization structures and applications. I*, Z. Anal. Anwend. **22** (2003), no. 1, 73–107.
- [44] George C. Papanicolaou and S. R. Srinivasa Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, Random fields. Rigorous results in statistical mechanics and quantum field theory, Esztergom 1979, Colloq. Math. Soc. Janos Bolyai 27, 835-873, 1981.
- [45] Zhongwei Shen, *Periodic homogenization of elliptic systems*, Operator Theory: Advances and Applications, vol. 269, Birkhäuser/Springer, Cham, 2018.
- [46] Halil Mete Soner, *Optimal control with state-space constraint. I*, SIAM J. Control Optim. **24** (1986), no. 3, 552–561.
- [47] Luc Tartar, *The general theory of homogenization. A personalized introduction*, Lect. Notes Unione Mat. Ital., vol. 7, Springer, Berlin, 2009.
- [48] Sylvain Wolf, *Homogenization of the  $p$ -Laplace equation in a periodic setting with a local defect*, 2022, [arXiv:2206.03071](https://arxiv.org/abs/2206.03071).
- [49] Vasilii V. Zhikov, Sergei M. Kozlov, and Olga A. Olejnik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994.