Séminaire Laurent Schwartz

EDP et applications

Année 2021-2022

Angeliki Menegaki and Clément Mouhot

A consistence-stability approach to hydrodynamic limit of interacting particle systems on lattices

Séminaire Laurent Schwartz — EDP et applications (2021-2022), Exposé n° VII, 15 p.

https://doi.org/10.5802/slsedp.154

Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
http://www.ihes.fr/

Centre de mathématiques Laurent Schwartz
CMLS, École polytechnique, CNRS, Université Paris-Saclay
F-91128 PALAISEAU CEDEX
http://www.math.polytechnique.fr/

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2266-0607

Séminaire Laurent-Schwartz — EDP et applications
Institut des hautes études scientifiques, 2021-2022
Exposé n° VII, 1-15

A CONSISTENCE-STABILITY APPROACH TO HYDRODYNAMIC LIMIT OF INTERACTING PARTICLE SYSTEMS ON LATTICES

ANGELIKI MENEGAKI AND CLÉMENT MOUHOT

Abstract. This is a review based on the presentation done at the seminar Laurent Schwartz in December 2021. It is announcing results in the forthcoming [MMM22]. This work presents a new simple quantitative method for proving the hydrodynamic limit of a class of interacting particle systems on lattices. We present here this method in a simplified setting, for the zero-range process and the Ginzburg-Landau process with Kawasaki dynamics, in the parabolic scaling and in dimension 1. The rate of convergence is quantitative and uniform in time. The proof relies on a consistence-stability approach in Wasserstein distance, and it avoids the use of both the so-called "block estimates".

Contents

1. The general method

We consider the hydrodynamic limit of interacting particle systems on a lattice. The problem is to show that under an appropriate scaling of time and space, the local particle densities of a stochastic lattice gas converge to the solution of a macroscopic partial differential equation. We first present our method abstractly and then sketch applications to three concrete models: the simple-exclusion process (SEP), the zero-range process (ZRP) and the Ginzburg Landau process with Kawasaki dynamics (GLK). The hydrodynamic limit is known at a qualitative level for all these models under both hyperbolic and parabolic scalings for the SEP and ZRP and under parabolic scaling for the GLK, see [GPV88, Yau91, Rez91, KL99]. However finding quantitative error estimates had remained an important opened question, as well as understanding the long-time behaviour of the hydrodynamic limit. First results towards quantitative error, in the particular case of the Ginzburg-Landau process with Kawasaki dynamics in dimension 1, were obtained in the two-parts work [DMOWa, DMOWb], which builds upon partial progresses in [GOVW09].

1.1. **Set up and notation.** We denote by X the state space at a given site (number of particles, spin, etc.), which will in this paper be N (ZRP) or R (GLK). Consider the discrete torus T^d_N and the corresponding phase space of particle configurations $\mathsf{X}_N := \mathsf{X}^{\mathsf{T}^d_N}$. Variables in T^d_N are called *microscopic* and denoted by x, y, z, whereas variables in the limit continuous torus T^d are called *macroscopic* and denoted by u;

finally particle *configurations* in X_N are denoted by η. The canonical embeddding $\mathsf{T}^d_N \to \mathsf{T}^d$, $x \mapsto x/N$ means the macroscopic distance between sites of the lattice is $1/N$. Given a particle configuration $\eta \in \mathsf{X}_N$, we define the *empirical measure*

$$\alpha^N_\eta := \sum_{x\in \mathsf{T}^d_N} \eta_x \delta_{x/N} \in \mathcal{M}_+(\mathsf{T}^d). \tag{1.1}$$

where η_x denotes the value of η at $x \in \mathsf{T}^d_N$, and $\mathcal{M}_+(\mathsf{T}^d)$ is the space of positive Radon measures on the torus, and $\sum$ denotes the "average sum", here $N^{-d}\sum_{x\in\mathsf{T}^d_N}$.

At the microscopic level, the interacting particle system evolves through a stochastic process and the time-dependent probability measure describing the law of η is denoted by $\mu^N_t \in P(\mathsf{X}_N)$. We consider a linear operator $\mathcal{L}_N : C_b(\mathsf{X}_N) \to C_b(\mathsf{X}_N)$ generating uniquely a Feller semigroup $e^{t\mathcal{L}_N}$ on $P(\mathsf{X}_N)$ (see [Lig85, Chapter 1]) so that given $\mu^N_0 \in P(\mathsf{X}_N)$ the solution $\mu^N_t = e^{t\mathcal{L}_N}\mu^N_0 \in P(\mathsf{X}_N)$ satisfies

$$\forall\, \Phi \in C_b(\mathsf{X}_N), \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle \Phi, \mu^N_t\rangle = \langle \mathcal{L}_N\Phi, \mu^N_t\rangle, \tag{1.2}$$

where $C_b(\mathsf{X}_N)$ denotes continuous bounded real-valued functions and $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $C_b(\mathsf{X}_N)$ and $P(\mathsf{X}_N)$.

At the macroscopic level, we consider a map $\mathcal{L}_\infty : \mathcal{M}_+(\mathsf{G}_\infty) \to \mathcal{M}_+(\mathsf{G}_\infty)$ (in general unbounded and nonlinear) and the evolution problem

$$\partial_t f_t = \mathcal{L}_\infty f_t, \quad f_{t=0} = f_0. \tag{1.3}$$

A measure $\mu^N \in P(\mathsf{X}_N)$ is called *invariant* for (1.2) if

$$\forall\, \Phi \in C_b(\mathsf{X}_N), \quad \left\langle \mu^N, \mathcal{L}_N\Phi\right\rangle = 0.$$

We also denote $\mathrm{Lip}(\mathsf{X}_N)$ the Lipschitz functions $\Phi : \mathsf{X}_N \to \mathsf{R}$ with respect to the (normalised) ℓ^1 norm: for every $\eta,\zeta \in \mathsf{X}_N$, $|\Phi(\eta)-\Phi(\zeta)| \le C_\Phi \sum_{x\in\mathsf{T}^d_N} |\eta_x - \zeta_x|$, and we denote the smallest such constant C_Φ by $[\Phi]_{\mathrm{Lip}(\mathsf{X}_N)} \in \mathsf{R}_+$.

1.2. **Abstract assumptions.** We make the following assumptions on (1.2)-(1.3):

(H0) Local equilibrium structure. There are $n_\lambda : \mathrm{Conv}(\mathsf{X}) \to \mathsf{R}_+$ depending on $\lambda \in \mathsf{R}$ (Conv denotes the convex hull) and $\sigma : \mathrm{Conv}(\mathsf{X}) \to \mathsf{R}$ so that: (i) $n_\lambda^{\otimes \mathsf{T}^d_N}$ is invariant on X_N for each λ, and (ii) for any $\rho \in \mathrm{Conv}(\mathsf{X})$, $\mathsf{E}_{n_{\sigma(\rho)}}[\eta_x] = \rho$. We then define, given a macroscopic profile f on T^d, the *local Gibbs measure*

$$\vartheta^N_f(\eta) := \nu^N_{\sigma(f(\frac{\cdot}{N}))}(\eta) \quad \text{where} \quad \nu^N_F(\eta) := \prod_{x\in\mathsf{G}_N} n_{F(x)}(\eta(x)).$$

The two maps $\eta \mapsto \alpha^N_\eta$ and $f \mapsto \vartheta^N_f$ allow comparisons between the microscopic and macroscopic scales, as summarized in Figure 1.

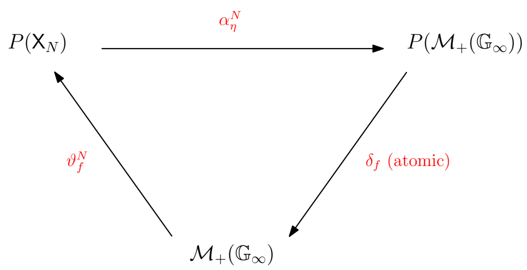

Figure 1. The functional setting.

(H1) Microscopic stability. The semigroup $e^{t\mathcal{L}_N}$ satisfies

(1.4) $$\forall\,\Phi \in \mathrm{Lip}(\mathsf{X}_N), \quad \left[e^{t\mathcal{L}_N}\Phi\right]_{\mathrm{Lip}(\mathsf{X}_N)} \le [\Phi]_{\mathrm{Lip}(\mathsf{X}_N)}.$$

(H2) Macroscopic stability. There is a Banach space $\mathfrak{B} \subset \mathcal{M}_+(\mathsf{G}_\infty)$ so that (1.3) is locally well-posed in $\mathfrak{B}$; given the maximal time of existence $T_m \in (0, +\infty]$ we denote for $t \in [0, T_m)$, $R(t) := \|f_t - f_\infty\|_{\mathfrak{B}}$ when (1.3) has a unique stationary solution $f_\infty \in \mathfrak{B}$ with mass $\int_{\mathsf{T}^d} f_\infty = \int_{\mathsf{T}^d} f_0$, otherwise we denote $R(t) := \|f_t\|_{\mathfrak{B}}$.

(H3) Consistency. There is a *consistency error* $\epsilon(N) \to 0$ as $N \to \infty$ so that for $T \in [0, T_m)$

$$\frac{1}{T}\int_0^T \int_0^t \left\langle \left(e^{(t-s)\mathcal{L}_N}\Phi\right), \left[\mathcal{L}_N^*\left(\frac{\mathrm{d}\vartheta^N_{f_s}}{\mathrm{d}\nu^N_\infty}\right) - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}\vartheta^N_{f_s}}{\mathrm{d}\nu^N_\infty}\right)\right]\mathrm{d}\nu^N_\infty \right\rangle \mathrm{d}s\,\mathrm{d}t$$
$$\le \epsilon(N)[\Phi]_{\mathrm{Lip}(\mathsf{X}_N)} \int_0^t R(s)\,\mathrm{d}s$$

for any $\Phi \in \mathrm{Lip}(\mathsf{X}_N)$, where ν^N_∞ is an equilibrium measure.

1.3. **The abstract strategy.**

Theorem 1.1. *Consider* (1.2)-(1.3) *with the assumptions* **(H0)–(H1)–(H2)–(H3)**. *Let* $\phi \in C^\infty(\mathsf{T}^d)$, $\mu^N_0 \in P_1(X_N)$ *for all* $N \ge 1$, $f_0 \in \mathcal{B}$. *Then*

(1.5)
$$\forall\, T \in [0, T_m), \quad \frac{1}{T}\int_0^T \left\|\mu^N_t - \vartheta^N_{f_t}\right\|_{\mathrm{Lip}^*} \mathrm{d}t \;\square\; \epsilon(N)\int_0^T R(s)\,\mathrm{d}s + \left\|\mu^N_0 - \vartheta^N_{f_0}\right\|_{\mathrm{Lip}^*}.$$

Remark 1. *Note that* $\|\mu^N_t - \vartheta^N_{f_t}\|_{\mathrm{Lip}^*} \to 0$ *as* $N \to \infty$ *implies that the empirical measure* (1.1) *sampled from the law* μ^N_t *satisfies*

(1.6) $$\forall\,\phi \in C_b(\mathsf{G}),\ \forall\,\epsilon > 0,\ \forall\, t \ge 0, \quad \lim_{N\to\infty} \mu^N_t\left(\left\{|\langle \alpha^N_\eta, \varphi\rangle - \langle f_t, \varphi\rangle| > \epsilon\right\}\right) = 0$$

with a rate of convergence (thus recovering quantitatively results from [GPV88]*):*

$$\mu^N_t\left(\left\{|\langle \alpha^N_\eta, \varphi\rangle - \langle f_t, \varphi\rangle| > \epsilon\right\}\right)$$
$$\le \mu^N_t\left(\left\{\langle \alpha^N_\eta, \varphi\rangle \ge \langle f_t, \varphi\rangle + \epsilon\right\}\right) + \mu^N_t\left(\left\{\langle \alpha^N_\eta, \varphi\rangle \le \langle f_t, \varphi\rangle - \epsilon\right\}\right)$$
$$\le \int_{\mathsf{X}_N}\left[F^+_\epsilon\left(\langle \phi, \alpha^N_\eta\rangle\right) - F^+_\epsilon\left(\langle \phi, f_t\rangle\right)\right]\mathrm{d}\mu^N_t + \int_{\mathsf{X}_N}\left[F^-_\epsilon\left(\langle \phi, \alpha^N_\eta\rangle\right) - F^-_\epsilon\left(\langle \phi, f_t\rangle\right)\right]\mathrm{d}\mu^N_t$$

where $F^\pm_\epsilon$ *are mollified version of the characteristic functions of respectively* $\{z \ge \langle \phi, f_t\rangle + \epsilon\}$ *and* $\{z \le \langle \phi, f_t\rangle - \epsilon\}$, *which yields*

$$\sup_{t\in[0,T]} \mu^N_t\left(\left\{|\langle \alpha^N_\eta, \varphi\rangle - \langle f_t, \varphi\rangle| > \epsilon\right\}\right) \;\square\; \epsilon^{-1}\|\mu^N_t - \vartheta^N_{f_t}\|_{\mathrm{Lip}^*} + \epsilon^{-2}N^{-d}.$$

2. Concrete applications

We apply the abstract result to two archetypical models, the zero-range process (ZRP), and the Ginzburg-Landau process with Kawasaki dynamics (GLK).

2.1. **The ZRP.** In this case, the state space at each site is $\mathsf{X} = \mathsf{N}$. Given the choice of a *transition function* $p \in P(\mathsf{T}^d_N)$ with $p(0) = 0$ and a *jump rate function* $g : \mathsf{N} \to \mathsf{R}_+$, the base generator $\widehat{\mathcal{L}}_N$ writes

(2.1) $$\forall\,\Phi \in C_b(\mathsf{X}_N),\ \forall\,\eta \in \mathsf{X}_N, \quad \widehat{\mathcal{L}}_N\Phi(\eta) := \sum_{x,y\in\mathsf{T}^d_N} p(y-x)g(\eta_x)\left[\Phi(\eta^{xy}) - \Phi(\eta)\right]$$

where η^{xy} is defined as before. The local equilibrium structure of **(H0)** is given by

$$n_\lambda(k) := \frac{\lambda^k}{g(k)!Z(\lambda)} \quad \text{with} \quad Z(\lambda) := \sum_{k=0}^{+\infty} \frac{\lambda^k}{g(k)!} \tag{2.2}$$

$$\sigma \;\text{ is defined implicitely by }\; \sigma(\rho)\frac{Z'(\sigma(\rho))}{Z(\sigma(\rho))} \equiv \rho \tag{2.3}$$

denoting $g(k)! := g(k)g(k-1)\cdots g(1)$. The pair (g,σ) thus constructed satisfies $\mathsf{E}_{n_{\sigma(\alpha)}}[g] = \sigma(\alpha)$. When $f \equiv \rho \in [0,+\infty)$ is constant, the local Gibbs measure $\vartheta^N_\rho = \nu^N_{\sigma(\rho)}$ is invariant with average number of particles ρ. The *mean transition rate* is defined by $\gamma := \sum_{x\in\mathsf{Z}^d} xp(x) \in \mathsf{R}^d$. When $\gamma \neq 0$, the first non-zero asymptotic dynamics as $N\to\infty$ is given by the hyperbolic scaling $\mathcal{L}_N := N\widehat{\mathcal{L}}_N$, and the corresponding expected limit equation is $\partial_t f = \gamma\cdot\nabla[\sigma(f)]$. When $\gamma = 0$, the first non-zero asymptotic dynamics as $N\to\infty$ is the given by the parabolic scaling $\mathcal{L}_N := N^2\widehat{\mathcal{L}}_N$, and the corresponding limit equation is formally

$$\partial_t f = \Delta_a[\sigma(f)] \quad \text{with} \quad \Delta_a := \sum_{i,j=1}^d a_{ij}\partial^2_{ij} \quad \text{and} \quad a_{ij} := \sum_{x\in\mathsf{Z}^d} p(x)x_i x_j. \tag{2.4}$$

We make the following assumptions on the jump rate function $g : \mathsf{N} \to [0,\infty)$.

(HZRP) The jump rate g satisfies $g(0) = 0$, $g(n) > 0$ for all $n > 0$, is non-decreasing, uniformly Lipschitz $\sup_{n\geq 0}|g(n+1) - g(n)| < +\infty$, and there are $n_0 > 0$ and $\beta > 0$ such that $g(n') - g(n) \geq \beta$ for any $n' \geq n + n_0$.

The main result on the ZRP is:

Theorem 2.1 (Hydrodynamic limit for the ZRP). *Consider $\widehat{\mathcal{L}}_N$ defined in* (2.1) *with g satisfying* **(HZRP)**. *Let $d = 1$, $f_0 \in C^3(\mathsf{T})$ with $f_0 \geq \delta > 0$, and $\mu^N_0 \in P_1(\mathsf{X}_N)$ for all $N \geq 1$. Assume $\gamma = 0$, define $\mu^N_t = e^{tN^2\widehat{\mathcal{L}}_N}$ and $f_t \in C([0,T), C^3(\mathsf{T}^d))$ solution to* (2.4)*, then the following convergence holds (with quantitative constants)*

$$\sup_{T\geq 0}\frac{1}{T}\int_0^T \left\|\mu^N_t - \vartheta^N_{f_t}\right\|_{\mathrm{Lip}^*} \,\mathrm{d}t \;\square\; N^{-1/8} + \left\|\mu^N_0 - \vartheta^N_{f_0}\right\|_{\mathrm{Lip}^*}. \tag{2.5}$$

2.2. **The GLK.** In this case, the state space at each site is $\mathsf{X} = \mathsf{R}$. Given the choice of a *single-site potential* $V \in C^2(\mathsf{R})$, the base generator $\widehat{\mathcal{L}}_N$ writes

$$\widehat{\mathcal{L}}_N\Phi(\eta) := \frac{1}{2}\sum_{x\sim y\in\mathsf{T}^d_N}\left(\frac{\partial}{\partial\eta_x} - \frac{\partial}{\partial\eta_y}\right)^2 - \frac{1}{2}\sum_{x\sim y\in\mathsf{T}^d_N}[V'(\eta_x) - V'(\eta_y)]\left(\frac{\partial}{\partial\eta_x} - \frac{\partial}{\partial\eta_y}\right) \tag{2.6}$$

where $x\sim y$ denotes neighbouring sites. The local equilibrium structure is given by

$$n_\lambda(r) := \frac{e^{\lambda r}}{Z(\lambda)} \quad \text{with} \quad Z(\lambda) := \int_{\mathsf{R}} e^{\lambda r - V(r)}\,\mathrm{d}r$$

$$\sigma \;\text{ is defined implicitely by }\; \frac{Z'(\sigma(\rho))}{Z(\sigma(\rho))} \equiv \rho.$$

When $f \equiv \rho \in \mathsf{R}$ is constant, the local Gibbs measure $\vartheta^N_\rho = \nu^N_{\sigma(\rho)}$ is invariant with average spin ρ. The hyperbolic scaling formally leads to zero and the parabolic scaling $\mathcal{L}_N := N^2\widehat{\mathcal{L}}_N$ formally leads to

$$\partial_t f = 2\Delta[\sigma(f)]. \tag{2.7}$$

We assume that the single-site potential satisfies

(HGLK) The potential V is C^2 and decomposes as $V(u) = V_0(u) + V_1(u)$ with $V_0''(u) \geq \kappa$ for all $u \in \mathsf{R}$ for some $\kappa > 0$ and $\|V_1\|_{W^{1,\infty}(\mathsf{R})} \lesssim 1$.

This assumption is similar with those in [GOVW09, DMOWa, Fat13]. One can take for example a double-well potential, provided it is uniformly convex at infinity.

Theorem 2.2 (Hydrodynamic limit for the GLK)**.** *Consider $\mathcal{L}_N$ defined in* (2.6) *with V satisfying* **(HGLK)**. *Let $d = 1$, $f_0 \in C^3(\mathsf{T}^d)$ and $\mu_0^N \in P_1(\mathsf{X}_N)$ for all $N \geq 1$. Define $\mu_t^N = e^{tN^2\widehat{\mathcal{L}}_N}$ and $f_t \in C([0,+\infty), C^3(\mathsf{T}^d))$ the global solution to* (2.7)*, then the following convergence holds (with quantitative constants)*

$$\sup_{T \geq 0} \frac{1}{T} \int_0^T \left\|\mu_t^N - \vartheta_{f_t}^N\right\|_{\mathrm{Lip}^*} \, \mathrm{d}t \lesssim N^{-1/8} + \left\|\mu_0^N - \vartheta_{f_0}^N\right\|_{\mathrm{Lip}^*}. \tag{2.8}$$

3. The abstract strategy

In this section we sketch the proof of Theorem 1.1. Let f_t be a solution to (1.3). Given $0 < \ell < N$, we denote by η^ℓ for the local ℓ-average $\eta_x^\ell := \sum_{|y-x| \leq \ell} \eta_y$.

Denote by $F_t^N := \mathrm{d}\mu_t^N / \mathrm{d}\nu_\infty^N$ and $G_t^N := \mathrm{d}\vartheta_{f_t}^N / \mathrm{d}\nu_\infty^N$ the densities with respect to ν_∞^N, and write

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(F_t^N - G_t^N\Big) = \mathcal{L}_N^*\Big(F_t^N - G_t^N\Big) + \left(\mathcal{L}_N^* G_t^N - \partial_t G_t^N\right)$$

so that Duhamel's formula yields

$$F_t^N - G_t^N = e^{t\mathcal{L}_N^*}\left(F_0^N - G_0^N\right) + \int_0^t e^{(t-s)\mathcal{L}_N^*}\left(\mathcal{L}_N^* G_s^N - \partial_s G_s^N\right) \mathrm{d}s.$$

Take $\Phi \in \mathrm{Lip}(\mathsf{X}_N)$ with $\|\Phi\|_{\mathrm{Lip}(\mathsf{X}_N)} \leq 1$ and integrate the above equation to get

$$\begin{aligned} &\int_{\mathsf{X}_N} \Phi\Big(F_t^N - G_t^N\Big) \, \mathrm{d}\nu_\infty^N \\ &= \underbrace{\int_{\mathsf{X}_N} \left(e^{t\mathcal{L}_N}\Phi\right)\Big(F_0^N - G_0^N\Big) \, \mathrm{d}\nu_\infty^N}_{I_1(t)} + \underbrace{\int_{\mathsf{X}_N} \int_0^t \Big(e^{(t-s)\mathcal{L}_N}\Phi\Big)\left(\mathcal{L}_N G_s^N - \partial_s G_s^N\right) \mathrm{d}\nu_\infty^N \, \mathrm{d}s}_{I_2(t)}. \end{aligned}$$

(H1) implies $I_1(t) \lesssim \|\mu_0^N - \vartheta_{f_0}^N\|_{\mathrm{Lip}^*}$ and **(H3)** implies $\frac{1}{T}\int_0^T I_2(t) \, \mathrm{d}t \leq \epsilon(N) \int_0^T R(s) \, \mathrm{d}s$, which implies the conclusion of Theorem 1.1.

4. Proof for the ZRP

In this section we prove Theorem 2.1) (hydrodynamical limit for the ZRP). Note for this model $\mathcal{L}_N = \mathcal{L}_N^*$ is symmetric with respect to equilibrium measures. Given $f_t \in C^3(\mathsf{T}^d)$ with $f > \delta$, $\delta > 0$, and $\rho := \int_{\mathsf{T}^d} f$, the density of the local Gibbs measure relatively to the invariant measure with mass ρ is:

$$G_t^N(\eta) := \frac{\mathrm{d}\vartheta_f^N(\eta)}{\mathrm{d}\vartheta_\rho^N(\eta)} = \prod_{x \in \mathsf{T}_N^d} \left(\frac{\sigma\left(f_t\left(x/N\right)\right)}{\sigma(\rho)}\right)^{\eta(x)} \left(\frac{Z(\sigma\left(f_t\left(x/N\right)\right))}{Z(\sigma(\rho))}\right)^{-1}. \tag{4.1}$$

where the function $\sigma(r)$ is defined by $\langle n_{\sigma(r)}, \eta(x)\rangle = r$ and the *partition function* $Z : [0, \lambda^*) \to \mathsf{R}$ is defined in (2.2), with $\lambda^* \in [0, +\infty]$ denoting the radius of convergence of the series.

It is proved in [KL99, Chapter 2, Section 3] that assumption **(HZRP)** on g implies that $\sigma = R^{-1} : [0,\infty) \to [0,\infty)$ is well-defined and strictly increasing, with

$$R(\lambda) = \lambda \partial_\lambda \log(Z(\lambda)) = \frac{1}{Z(\lambda)} \sum_{n \geq 0} \frac{n\lambda^n}{g(n)!}.$$

Then the building block n_ρ of the Gibbs measure satisfies $\langle n_{\sigma(\rho)}, g(\eta(x)) \rangle = \sigma(\rho)$. Moreover **(HZRP)** implies that the function σ is C^∞ with uniform bound on all derivatives on R_+, with Lipschitz constant less than g^*, see [KL99, Corollary 3.6], and with $\inf_{\lambda>0} \lambda^{-1}\sigma(\lambda) > 0$ (in particular $\sigma'(0) > 0$). Finally **(HZRP)** also implies the following comparison principle: if one starts from two ordered configurations $\eta \leq \zeta$ (at all points $x \in \mathsf{T}_N^d$) then the evolution preserves this inequality at later times: $\eta_t \leq \zeta_t$. This implies that if for any $f^N \in C_b(\mathsf{X}_N)$ so that $f^N(\eta) \leq f^N(\zeta)$ for all $\eta \leq \zeta$ one has $\langle \mu_0^{N,1}, f^N \rangle \leq \langle \mu_0^{N,2}, f^N \rangle$, then at later times $\mu_t^{N,1} \prec \mu_t^{N,2}$. It easy to deduce that the the kth moments ($k \in \mathsf{N}$)

$$M_k \left[\mu_t^N\right] := \left\langle \mu_t^N, \frac{1}{N^d} \sum_{x \in \mathsf{T}_N^d} \eta(x)^k \right\rangle$$

are uniformly bounded along time when $\mu_0^N \prec C\vartheta_\rho^N$ for some $C > 0$ and $\rho \in \mathsf{R}_+$.

4.1. **Microscopic Stability – hypothesis (H1).** We use again the "basic coupling" as in [Lig85, Rez91]. We define

$$\begin{aligned} \widetilde{\mathcal{L}}_N \Psi(\eta,\zeta) := &\sum_{x,y \in \mathsf{T}_N^d} p(y-x)\Big(g(\eta_x) \wedge g(\zeta_x)\Big)\Big[\Psi(\eta^{xy},\zeta^{xy}) - \Psi(\eta,\zeta)\Big] \\ &+ \sum_{x,y \in \mathsf{T}_N^d} p(y-x)\Big(g(\eta_x) - g(\eta_x) \wedge g(\zeta_x)\Big)\Big[\Psi(\eta^{xy},\zeta) - \Psi(\eta,\zeta)\Big] \\ &+ \sum_{x,y \in \mathsf{T}_N^d} p(y-x)\Big(g(\zeta_x) - g(\eta_x) \wedge g(\zeta_x)\Big)\Big[\Psi(\eta,\zeta^{xy}) - \Psi(\eta,\zeta)\Big]. \end{aligned} \tag{4.2}$$

for a two-variable test function $\Psi(\eta,\zeta)$. Then $\widetilde{\mathcal{L}}_N \Phi(\eta) = \widehat{\mathcal{L}}_N \Phi(\eta)$ and $\widetilde{\mathcal{L}}_N \Phi(\zeta) = \widehat{\mathcal{L}}_N \Phi(\zeta)$, and **(H1)** follows from the fact that $e^{t\widetilde{\mathcal{L}}_N}$ preserves sign and the inequality

$$\widetilde{\mathcal{L}}_N \Big(\sum_{z \in \mathsf{T}_N^d} |\eta_z - \zeta_z| \Big) \leq 0.$$

To prove the latter inequality, we compute

$$\begin{aligned} \widetilde{\mathcal{L}}_N \Big(\sum_{z \in \mathsf{T}_N^d} |\eta_z - \zeta_z| \Big) = &\sum_{x,y \in \mathsf{T}_N^d} p(y-x)\Big(g(\eta_x) - g(\eta_x) \wedge g(\zeta_x)\Big) \\ &\qquad \times \Big[|\eta_x^{xy} - \zeta_x| + |\eta_y^{xy} - \zeta_y| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y|\Big] \\ &+ \sum_{x,y \in \mathsf{T}_N^d} p(y-x)\Big(g(\zeta_x) - g(\eta_x) \wedge g(\zeta_x)\Big) \\ &\qquad \times \Big[|\eta_x - \zeta_x^{xy}| + |\eta_y - \zeta_y^{xy}| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y|\Big]. \end{aligned}$$

When $g(\eta_x) - g(\eta_x) \wedge g(\zeta_x) > 0$ necessarily $\eta_x - \zeta_y \geq 1$ and

$$\Big[|\eta_x^{xy} - \zeta_x| + |\eta_y^{xy} - \zeta_y| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y|\Big] \leq 0.$$

When $g(\zeta_x) - g(\eta_x) \wedge g(\zeta_x) > 0$ necessarily $\zeta_x - \eta_x \geq 1$ and

$$\Big[|\eta_x - \zeta_x^{xy}| + |\eta_y - \zeta_y^{xy}| - |\eta_x - \zeta_x| - |\eta_y - \zeta_y|\Big] \leq 0.$$

4.2. **Macroscopic stability – hypothesis (H2).** In the parabolic scaling the limit PDE is the nonlinear diffusion equation (2.4). We take $\mathcal{B} = C^3$ with its standard infinity Banach norm. The proof that this norm remains uniformly bounded in time is classical in dimension $d = 1$ (using the bounds on σ), and $f_t \in [\delta, 1-\delta]$ for all times by maximal principle. Moreover $f_t \to \rho$ exponentially fast as $t \to \infty$ in $\mathcal{B}$.

4.3. **Consistency estimate – hypothesis (H3).** Note that the operator is self-adjoint, $\mathcal{L}_N^* = \mathcal{L}_N$, with respect to the equilibrium measures. We assume $\gamma = 0$.

Proposition 4.1. *Given $d = 1$ and the solution $f_t \in C^3(\mathsf{T}^d)$ to (2.4) with $f \geq \delta$, $\delta > 0$, and $\rho := \int_{\mathsf{T}^d} f$, and G_t^N defined in (4.1), we have for every $\Phi \in \mathrm{Lip}(\mathsf{X}_N)$*

$$\frac{1}{T}\int_0^T I_t^N \,\mathrm{d}t := \frac{1}{T}\int_0^T \int_0^t \left\langle \left(e^{(t-s)\mathcal{L}_N}\Phi\right), \left[\mathcal{L}_N G_s^N - \frac{\mathrm{d}}{\mathrm{d}s} G_s^N\right] \mathrm{d}\nu_\infty^N \right\rangle \mathrm{d}s\,\mathrm{d}t = \mathcal{O}(N^{-1/8})$$

where the constant depends on the estimates in **(H2)**.

Proof. We start by computing

$$\mathcal{L}_N G_s^N - \frac{\mathrm{d}}{\mathrm{d}s} G_s^N = \sum_{x \in \mathsf{T}_N^d} A_x^N G_s^N$$

with (note that $f_t \to \rho$ exponentially fast)

$$\begin{aligned} A_x^N &:= N^2 \sum_{y \in \mathsf{T}_N^d} p(y-x) g(\eta_x) \left(\frac{\sigma(f_t(y/N))}{\sigma(f_t(x/N))} - 1 \right) - \eta_x \frac{\sigma'(f_t(x/N))}{\sigma(f_t(x/N))} \Delta_a[\sigma(f)](x/N) \\ &= \frac{g(\eta_x)}{\sigma(f_t(x/N))} \Delta_a[\sigma(f)](x/N) - \eta_x \frac{\sigma'(f_t(x/N))}{\sigma(f_t(x/N))} \Delta_a[\sigma(f)](x/N) + \mathcal{O}\left(e^{-Cs}/N\right) \end{aligned}$$

for some $C > 0$. Since (conservation of mass)

$$\int_{\mathsf{X}_N} \Big(\sum_{x \in \mathsf{T}_N^d} A_x^N G_s^N \Big) \mathrm{d}\nu_\infty^N = \int_{\mathsf{X}_N} \Big(\sum_{x \in \mathsf{T}_N^d} A_x^N \Big) \mathrm{d}\vartheta_{f_s}^N = 0,$$

we can replace $\Phi_{t-s} := e^{(t-s)\mathcal{L}_N}\Phi$ by

$$\widetilde{\Phi}_{t,s} := e^{(t-s)\mathcal{L}_N}\Phi - \mathbf{E}_{\vartheta_{f_s}^N}[e^{(t-s)\mathcal{L}_N}\Phi]$$

and use the Lipschitz bound on $e^{(t-s)\mathcal{L}_N}\Phi$ (microscopic stability) to get

$$I_t^N = \int_0^t \int_{\mathsf{X}_N} \widetilde{\Phi}_{t,s}(\eta) \Big(\sum_{x \in \mathsf{T}_N^d} \widetilde{A}_x^N \Big) \mathrm{d}\vartheta_{f_s}^N + \mathcal{O}(1/N)$$

with $\widetilde{A}_x^N$ defined by (note that it has zero average against $\mathrm{d}\vartheta_{f_s}^N$)

$$\widetilde{A}_x^N := \{g(\eta_x) - \sigma(f_t(x/N)) - \sigma'(f_t(x/N))[\eta_x - f_t(x/N)]\} \frac{\Delta_a[\sigma(f)](x/N)}{\sigma(f_t(x/N))}.$$

We then form sub-sum over non-overlapping cubes of size $\ell \in \{1, \ldots, N\}$ (this intermediate scale factor ℓ will be chosen later in terms of N). Let $\mathcal{R}_N^d \subset \mathsf{T}_N^d$ be a net of centers of non-overlapping cubes of the form $\mathcal{C}_x := \{y \in \mathsf{T}_N^d \ : \ \|x-y\|_\infty \le \ell\}$. Then

$$\begin{aligned} I_t^N &= \sum_{x\in\mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} \widetilde{\Phi}_{t,s}(\eta) \Big(\sum_{y\in\mathcal{C}_x} \widetilde{A}_y^N \Big) \, \mathrm{d}\vartheta_{f_s}^N + \mathcal{O}\left(1/N\right) \\ &= (2\ell+1)^d \sum_{x\in\mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} \widetilde{\Phi}_{t,s}(\eta) \widehat{A}_x^N \, \mathrm{d}\vartheta_{f_s}^N + \mathcal{O}\left(1/N\right) \end{aligned}$$

with the $\widehat{A}_x^N$ defined by

$$\widehat{A}_x^N := \left\{ \langle g(\eta)\rangle_{\mathcal{C}_x} - \sigma\left(f_t\left(x/N\right)\right) - \sigma'\left(f_t\left(x/N\right)\right) \left[\langle \eta \rangle_{\mathcal{C}_x} - f_t\left(x/N\right) \right] \right\} \frac{\Delta_a[\sigma(f)]\left(x/N\right)}{\sigma\left(f_t\left(x/N\right)\right)}$$

where $\langle F(\eta)\rangle_{\mathcal{C}_x}$, for $F = F(\eta_x)$, denotes taking the average over the cube $\mathcal{C}_x$. Note that the average of $\widehat{A}_x^N$ against $\mathrm{d}\vartheta_{f_s}^N$ is $\mathcal{O}(e^{-Cs}\ell/N)$. Then

$$\begin{aligned} &\sum_{x\in\mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} \widetilde{\Phi}_{t,s} \widehat{A}_x^N \, \mathrm{d}\vartheta_{f_s}^N \\ &= \sum_{x\in\mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} \left(\widetilde{\Phi}_{t,s} - \Pi_x^N \widetilde{\Phi}_{t,s} \right) \widehat{A}_x^N \, \mathrm{d}\vartheta_{f_s}^N + \sum_{x\in\mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} \Pi_x^N \widetilde{\Phi}_{t,s} \widehat{A}_x^N \, \mathrm{d}\vartheta_{f_s}^N \\ &= \sum_{x\in\mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} \left(\Phi_{t-s} - \Pi_x^N \Phi_{t-s} \right) \widehat{A}_x^N \, \mathrm{d}\vartheta_{f_s}^N \\ &\qquad + \sum_{x\in\mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} \left(\Pi_x^N \Phi_{t-s} - \mathbf{E}_{\vartheta_{f_s}^N}[\Pi_x^N \Phi_{t-s}] \right) \widehat{A}_x^N \, \mathrm{d}\vartheta_{f_s}^N =: J_t^1 + J_t^2 \end{aligned}$$

where Π_x^N projects on the local equilibrium with same mass in the cube $\mathcal{C}_x$ (and does not touch the other site):

$$\left\{ \begin{aligned} &\Pi_x^N \varphi(\eta) = [\Pi_x^N \varphi](\langle \eta \rangle_{\mathcal{C}_x}) = \textstyle\int_{\Omega_{\langle \eta \rangle_{\mathcal{C}_x}}} \varphi(\widetilde{\eta}) \, \mathrm{d}\nu^{\ell, \langle \eta \rangle_{\mathcal{C}_x}}(\widetilde{\eta}) \\ &\Omega_m := \{ \widetilde{\eta} \, : \, \langle \widetilde{\eta} \rangle_{\mathcal{C}_x} = m \} \end{aligned} \right. \tag{4.3}$$

for a function φ on $\mathsf{X}^{\mathcal{C}_x}$. To estimate the first term J_t^1 we first approximate the measure $\vartheta_{f_s}^N$ on $\mathcal{C}_x$ by the equilibrium measure with local mass $f_t(x/N)$, and denote it by $\overline{\vartheta}_{f_s}$ (note that the approximation is made differently for each cube and depends on x, even if it is written explicitly). This produces an error $\mathcal{O}(\ell^{d+1}/N)$ (using the Lipschitz regularity of Φ_{t-s} and the exponential convergence $f_t \to \rho$ to get uniform in time bounds). We then apply the Poincaré inequality [LSV96, Theorem 1.1] in the cube $\mathcal{C}_x$ (whose constant is independent of the number of particles and proportional to the size of the cube) and the law of large number $\|\widehat{A}_x^N\|_{L^2(\overline{\vartheta}_{f_s}^N)} = \mathcal{O}(e^{-Cs}\ell^{-d/2})$:

$$\begin{aligned} J_t^1 &\le \sum_{x\in\mathcal{R}_N^d} \int_0^t \|\Phi_{t-s} - \Pi_x^N \Phi_{t-s}\|_{L^2(\overline{\vartheta}_{f_s}^N)} \|\widehat{A}_x^N\|_{L^2(\overline{\vartheta}_{f_s}^N)} \, \mathrm{d}s + \mathcal{O}\left(\ell^{d+1}/N\right) \\ &\square\ \ell^{1-d/2} \sum_{x\in\mathcal{R}_N^d} \int_0^t \sqrt{\overline{D}_x^\ell\left(\Phi_{t-s}\right)} e^{-Cs} \, \mathrm{d}s + \mathcal{O}\left(\ell^{d+1}/N\right) \\ &\square\ \ell^{1-d/2} N^{d/2} \int_0^t \Big(\sum_{x\in\mathcal{R}_N^d} \overline{D}_x^\ell\left(\Phi_{t-s}\right) \Big)^{1/2} e^{-Cs} \, \mathrm{d}s + \mathcal{O}\left(\ell^{d+1}/N\right) \end{aligned}$$

where $\overline{D}^\ell_x(\Phi)$ is the Dirichlet form on the cube $\mathcal{C}_x$ with respect to the measure $\overline{\vartheta}^N_{f_s}$:

$$\overline{D}^\ell_x(\Phi) := \sum_{y,z\in\mathcal{C}_x} \int_{\mathsf{X}_N} p(z-y) g(\eta_y) \left[\Phi(\eta^{yz}) - \Phi(\eta)\right]^2 \,\mathrm{d}\overline{\vartheta}^N_{f_s}.$$

Then we change back the measure $\overline{\vartheta}^N_{f_s}$ in each box, which produces (using the Lipschitz regularity of Φ) an error $\ell^{3/2}N^{-1/2}$), and we compute

$$\frac{1}{2N^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathsf{X}_N} \Phi_{t-s}(\eta)^2 \,\mathrm{d}\vartheta^N_{f_s} \le -\sum_{x\in\mathcal{R}^d_N} D^\ell_x\left(\Phi_{t-s}\right) + \mathcal{O}\left(1/N^2\right)$$

(with D^ℓ denoting the Dirichlet form for $\vartheta^N_{f_s}$), where the last error accounts for the small default of self-adjointness. We deduce (in dimension $d = 1$) that

$$\int_0^T J^1_t \,\mathrm{d}t \,\square\, T^{1/2}\left(\ell/N\right)^{1-d/2} + \mathcal{O}\left(T\ell^{d+1}/N\right).$$

To control the second term J^2_t, we first use the *equivalence of ensemble* in [KL99, Appendix II, Corollary 1.7] on the measure $\overline{\vartheta}^N_{f_s}$ (together with the exponential tail estimates on the local Gibbs measure) to get

$$\langle g(\eta)\rangle_{\mathcal{C}_x} = \sigma\left(\langle\eta\rangle_{\mathcal{C}_x}\right) + \mathcal{O}\left(1/\ell^d\right). \tag{4.4}$$

Second we remark that the Lipschitz regularity of Φ_{t-s} implies that $\Pi^N_x\Phi_{t-s} - \mathbf{E}_{\vartheta^N_{f_s}}[\Pi^N_x\Phi_{t-s}] = \mathcal{O}(\ell^d N^{-d})$, and since the average of $\widehat{A}^N_x$ with respect to $\vartheta^N_{f_s}$ is $\mathcal{O}(\ell/N)$, we can write

$$J^2_t = \sum_{x\in\mathcal{R}^d_N}\int_0^t\int_{\mathsf{X}_N}\left(\Pi^N_x\Phi_{t-s}[\langle\eta\rangle_{\mathcal{C}_x}] - \Pi^N_x\Phi_{t-s}\left[f_s\left(x/N\right)\right]\right)\widehat{A}^N_x \,\mathrm{d}\vartheta^N_{f_s} + \mathcal{O}\left(\ell/N\right).$$

Third, we remark that the Lipschitz regularity of Φ_{t-s} (with constant N^{-d}) implies a Lipschitz regularity of its averaged projection $\Pi^N_x\Phi_{t-s}$ with constant $\ell^d N^{-d}$, with respect to the local mass. Indeed, given $0 = m \le m' < +\infty$, pick any pair of configuration (η_0,ζ_0) with $\langle\eta_0\rangle_{\mathcal{C}_x} = m$, $\langle\zeta_0\rangle_{\mathcal{C}_x} = m'$ and $\eta_0 \le \zeta_0$ (such configuration trivially exists since $m \le m'$). Then we consider the initial coupling $\delta_{(\eta_0,\zeta_0)}$ on $\Omega_m \times \Omega_{m'}$ which has ℓ^1 cost $m'-m$. Then we evolve it along the flow of the coupling operator $e^{t\widetilde{\mathcal{L}}_N}\delta_{(\eta_0,\zeta_0)}$. The marginals respectively converge to $\nu^{\ell,m}$ and $\nu^{\ell,m'}$ (convergence to equilibrium of the oiriginal evolution). Since the evolution by the coupling operator does not increase the Wasserstein distance, we deduce $W_1(\nu^{\ell,m},\nu^{\ell,m'}) \le m'-m$. An optimal coupling Π associated to this distance thus satisfies

$$m'-m \le \int_{\Omega_m\times\Omega_{m'}}\left(\sum_{x\in\mathsf{T}^d_N}|\eta_x-\zeta_x|\right)\Pi(\eta,\zeta) \le m'-m$$

where the first inequality follows from Jensen's inequality. Thus the Jensen's inequality is saturated which implies that the cost does not change sign on the support of Π, i.e. $\eta \le \zeta$ in the support. We then compute

$$\begin{aligned}\Pi^N_x\Phi_{t-s}(m') - \Pi^N_x\Phi_{t-s}(m) &= \int_{\Omega_{m'}}\Phi_{t-s}(\zeta)\,\mathrm{d}\nu^{\ell,m'}(\zeta) - \int_{\Omega_m}\Phi_{t-s}(\eta)\,\mathrm{d}\nu^{\ell,m}(\eta)\\ &= \int_{\Omega_m\times\Omega_{m'}}\left[\Phi_{t-s}(\zeta) - \Phi(\eta)\right]\mathrm{d}\Pi(\eta,\zeta)\end{aligned}$$

and since $\eta \le \zeta$ on the support of Π, $\|\zeta - \eta\|_{\ell^1(\mathcal{C}_x)} = (m' - m)\ell^d$ and

$$\left|\Pi_x^N \Phi_{t-s}(m') - \Pi_x^N \Phi_{t-s}(m)\right| \le \frac{\ell^d}{N^d}|m' - m|.$$

We deduce (using (5.3))

$$\begin{aligned} J_t^2 \,\square\, & \frac{\ell^d}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} |\langle \eta \rangle_{\mathcal{C}_x} - f_s\left(x/N\right)| \times \\ & |\sigma(\langle \eta \rangle_{\mathcal{C}_x}) - \sigma\left(f_s\left(x/N\right)\right) - \sigma'\left(f_s\left(x/N\right)\right)\left[\langle \eta \rangle_{\mathcal{C}_x} - f_s\left(x/N\right)\right]| \,\mathrm{d}\vartheta_{f_s}^N e^{-Cs}\,\mathrm{d}s \\ & \qquad + \frac{1}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} |\langle \eta \rangle_{\mathcal{C}_x} - f_s\left(x/N\right)| \,\mathrm{d}\vartheta_{f_s}^N e^{-Cs}\,\mathrm{d}s + \mathcal{O}\left(\ell/N\right) \end{aligned}$$

which yields by Taylor formula, the approximation of $\vartheta_{f_s}^N$ by $\overline{\vartheta}_{f_s}^N$, and the law of large numbers

$$\begin{aligned} J_t^2 \,\square\, & \frac{\ell^d}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} |\langle \eta \rangle_{\mathcal{C}_x} - f_s\left(x/N\right)|^3 \,\mathrm{d}\vartheta_{f_s}^N e^{-Cs}\,\mathrm{d}s \\ & \qquad + \frac{1}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} |\langle \eta \rangle_{\mathcal{C}_x} - f_s\left(x/N\right)| \,\mathrm{d}\vartheta_{f_s}^N e^{-Cs}\,\mathrm{d}s + \mathcal{O}\left(\ell/N\right) e^{-Cs}\,\mathrm{d}s \\ \square\, & \frac{\ell^d}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} |\langle \eta \rangle_{\mathcal{C}_x} - f_s\left(x/N\right)|^3 \,\mathrm{d}\overline{\vartheta}_{f_s}^N e^{-Cs}\,\mathrm{d}s \\ & \qquad + \frac{1}{N^d} \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} |\langle \eta \rangle_{\mathcal{C}_x} - f_s\left(x/N\right)| \,\mathrm{d}\overline{\vartheta}_{f_s}^N e^{-Cs}\,\mathrm{d}s + \mathcal{O}\left(\ell/N\right) \\ \square\, & \mathcal{O}\left(\ell^{-3d/2}\right) + \mathcal{O}\left(\ell/N\right). \end{aligned}$$

Combining all estimates we get (optimizing $\ell := N^{1/4}$)

$$\frac{1}{T}\int_0^T I_t^N \,\mathrm{d}t \,\square\, \left(\frac{1}{N} + \frac{\ell^{1+d/2}}{N^{1-d/2}} + \frac{\ell^{1+2d}}{N} + \frac{1}{\ell^{d/2}} + \frac{\ell}{N}\right) \square\, \frac{1}{N^{1/8}}. \qquad \square$$

5. Proof for the GLK

In this section we prove Theorem 2.2 (hydrodynamic limit for the GLK). Note again that for this model $\mathcal{L}_N = \mathcal{L}_N^*$ is symmetric with respect to equilibrium measures. Given $f_t \in C^3(\mathsf{T}^d)$ and $\rho := \int_{\mathsf{T}^d} f \in \mathsf{R}$, the density of the local Gibbs measure relatively to the invariant measure with mass ρ is:

$$G_t^N(\eta) := \frac{\mathrm{d}\vartheta_f^N(\eta)}{\mathrm{d}\vartheta_\rho^N(\eta)} = \prod_{x \in \mathsf{T}_N^d} e^{[\sigma(f_t(x/N)) - \sigma(\rho)]\eta_x} \frac{Z(\sigma(\rho))}{Z\left(\sigma\left(f_t\left(x/N\right)\right)\right)}. \tag{5.1}$$

where the function $\sigma(r)$ is defined by $\langle n_{\sigma(r)}, \eta_x \rangle = r$ and the *partition function* $Z(\lambda) = \int_{\mathsf{R}} e^{\lambda r - V(r)}\,\mathrm{d}r$ is defined on R. The uniform convexity of V at infinity easily implies bounds on all moments of the invariant measure

$$\int_{\mathsf{X}_N} \sum_{x \in \mathsf{T}_N^d} \eta(x)^k d\vartheta_\rho^N(\eta) = C_k < \infty.$$

and it is known that **(HGLK)** implies that there exists $C > 0$ so that $0 < \frac{1}{C} \le \sigma' \le C < \infty$ (see [GOVW09, Lemma 41] and [DMOWa, Lemma 5.1]).

5.1. **Microscopic stability – hypothesis (H1).** We consider a coupling of two Ginzburg-Landau processes with generator $\widetilde{\mathcal{L}}_N : C_b(\mathsf{X}_N^2) \to C_b(\mathsf{X}_N^2)$ given by

$$
\begin{aligned}
\widetilde{\mathcal{L}}_N \Psi(\eta,\zeta) := \sum_{x\sim y} \Bigg(&\left[\left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^* \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right) \otimes 1 \right] \Psi(\eta,\zeta) \\
&+ \left[1 \otimes \left(\frac{\partial}{\partial \zeta_x} - \frac{\partial}{\partial \zeta_y} \right)^* \left(\frac{\partial}{\partial \zeta_x} - \frac{\partial}{\partial \zeta_y} \right) \right] \Psi(\eta,\zeta) \\
&+ K \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right) \otimes \left(\frac{\partial}{\partial \zeta_x} - \frac{\partial}{\partial \zeta_y} \right) \Psi(\eta,\zeta) \Bigg)
\end{aligned}
\tag{5.2}
$$

where $K > 0$ is a constant to be chosen later and the adjoint is taken in $L^2(\mathrm{d}\vartheta_\rho^N)$ so

$$
\begin{aligned}
\widehat{\mathcal{L}}_N &= \sum_{x\sim y} \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^2 - (V'(\eta_x) - V'(\eta_y)) \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right) \\
&= \sum_{x\sim y} \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right)^* \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_y} \right).
\end{aligned}
$$

Then for any $p \in (1,2]$ there is $K = K(p) > 0$ (depending on p) so that

$$
\begin{aligned}
\widetilde{\mathcal{L}}_N \Bigg(\sum_{x \in \mathsf{T}_N^d} |\eta_x - \zeta_x|^p \Bigg) = 2p(p-1)(2+4d) &\sum_{x \in \mathsf{T}_N^d} |\eta_x - \zeta_x|^{p-2} \\
&- 2(p-1) \sum_{x\sim y} \left[V_0'(\eta_x) - V_0'(\zeta_x) \right] (\eta_x - \zeta_x) |\eta_x - \zeta_x|^{p-1} \\
&- 2(p-1) \sum_{x\sim y} \left[V_1'(\eta_x) - V_1'(\zeta_x) \right] (\eta_x - \zeta_x) |\eta_x - \zeta_x|^{p-1} \\
&+ Kp(p-1)(2+4d) \sum_{x \in \mathsf{T}_N^d} |\eta_x - \zeta_x|^{p-2} \le 0
\end{aligned}
$$

by using the assumptions on the potential: V_0 uniformly strictly convex and $V_1 \in W^{1,\infty}$. This implies the weak contraction of the evolution in W_p (p-Wasserstein distance) for any $p \in (1,2]$, and thus by limit in W_1. By duality this implies that the evolution is weakly contractive for the dual Lipschitz norm.

5.2. **Macroscopic stability - hypothesis (H2).** The limit equation is a one-dimensional nonlinear diffusion equation with uniform ellipticity bounds, and standard elliptic theory shows that the solution exists globally and converges exponentially fast to a constant in $C^3(\mathsf{T}^d)$.

5.3. **Consistency estimate - hypothesis (H3).**

Proposition 5.1. *Given $d = 1$ and the solution $f_t \in C^3(\mathsf{T}^d)$ to* (2.4)*, and $\rho := \int_{\mathsf{T}^d} f$, and G_t^N defined in* (5.1)*, we have for every $\Phi \in \mathrm{Lip}(\mathsf{X}_N)$*

$$
\frac{1}{T} \int_0^T I_t^N \,\mathrm{d}t := \frac{1}{T} \int_0^T \int_0^t \left\langle \left(e^{(t-s)\mathcal{L}_N} \Phi \right), \left[\mathcal{L}_N G_s^N - \frac{\mathrm{d}}{\mathrm{d}s} G_s^N \right] \mathrm{d}\nu_\infty^N \right\rangle \mathrm{d}s \,\mathrm{d}t = \mathcal{O}\left(N^{-1/8} \right)
$$

where the constant depends on the estimates in **(H2)**.

Proof. The proof follows the same structure as for the ZRP. We start by computing

$$\mathcal{L}_N G_s^N - \frac{\mathrm{d}}{\mathrm{d}s} G_s^N = \sum_{x \in \mathsf{T}_N^d} A_x^N G_s^N$$

with (note again that $f_t \to \rho$ exponentially fast)

$$\begin{aligned}
A_x^N := \frac{N^2}{2} \sum_{y \sim x} \Bigg[& \left(\sigma\left(f_s\left(x/N\right)\right) - \sigma\left(f_s\left(y/N\right)\right)\right)^2 \\
& - \left(V'(\eta_x) - V'(\eta_y)\right)\left(\sigma\left(f_s\left(x/N\right)\right) - \sigma\left(f_s\left(y/N\right)\right)\right) \Bigg] \\
& - \sum_x \left(\eta_x - f_s\left(x/N\right)\right) \sigma'\left(f_s\left(x/N\right)\right) \Delta[\sigma(f)]\left(x/N\right) \\
= \frac{N^2}{2} \sum_{y \sim x} \Bigg[& 2\sigma\left(f_s\left(x/N\right)\right)\left(\sigma\left(f_s\left(x/N\right)\right) - \sigma\left(f_s\left(y/N\right)\right)\right) \\
& - 2V'(\eta_x)\sigma\left(f_s\left(x/N\right)\right) - \sigma\left(f_s\left(y/N\right)\right) \Bigg] \\
& - \sum_x \left(\eta_x - f_s\left(x/N\right)\right) \sigma'\left(f_s\left(x/N\right)\right) \Delta[\sigma(f)]\left(x/N\right) \\
= \Delta[\sigma(f)]\left(x/N\right) \Bigg[& V'(\eta_x) - \sigma\left(f_s\left(x/N\right)\right) \\
& - \sigma'\left(f_s\left(x/N\right)\right)\left(\eta_x - f_s\left(x/N\right)\right) \Bigg] + \mathcal{O}\left(e^{-Cs}/N\right)
\end{aligned}$$

for some $C > 0$. By conservation of mass we replace again $\Phi_{t-s} := e^{(t-s)\mathcal{L}_N}\Phi$ by

$$\widetilde{\Phi}_{t,s} := e^{(t-s)\mathcal{L}_N}\Phi - \mathbf{E}_{\vartheta^N_{f_s}}[e^{(t-s)\mathcal{L}_N}\Phi]$$

and use the Lipschitz bound **(H1)** on $e^{(t-s)\mathcal{L}_N}\Phi$ to get

$$I_t^N = \int_0^t \int_{\mathsf{X}_N} \widetilde{\Phi}_{t,s}(\eta) \Bigg(\sum_{x \in \mathsf{T}_N^d} \widetilde{A}_x^N \Bigg) \mathrm{d}\vartheta^N_{f_s} + \mathcal{O}\left(1/N\right)$$

with $\widetilde{A}_x^N$ defined by (note that it has zero average against $\mathrm{d}\vartheta^N_{f_s}$)

$$\widetilde{A}_x^N := \Delta[\sigma(f)]\left(x/N\right)\left[V'(\eta_x) - \sigma\left(f\left(x/N\right)\right) - \sigma'\left(f_s\left(x/N\right)\right)\left(\eta_x - f\left(x/N\right)\right)\right].$$

We again form sub-sum over non-overlapping cubes of size $\ell \in \{1, \dots, N\}$, with $\mathcal{R}_N^d \subset \mathsf{T}_N^d$ a net of centers of cubes $\mathcal{C}_x := \{y \in \mathsf{T}_N^d \ : \ \|x-y\|_\infty \le \ell\}$. Then

$$\begin{aligned}
I_t^N &= \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} \widetilde{\Phi}_{t,s}(\eta) \Bigg(\sum_{y \in \mathcal{C}_x} \widetilde{A}_y^N \Bigg) \mathrm{d}\vartheta^N_{f_s} + \mathcal{O}\left(1/N\right) \\
&= (2\ell+1)^d \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} \widetilde{\Phi}_{t,s}(\eta) \widehat{A}_x^N \, \mathrm{d}\vartheta^N_{f_s} + \mathcal{O}\left(1/N\right)
\end{aligned}$$

with the $\widehat{A}_x^N$ defined by (and $\langle F(\eta) \rangle_{\mathcal{C}_x}$ again denotes the average over the cube $\mathcal{C}_x$)

$$\widehat{A}_x^N := \Delta[\sigma(f)]\left(x/N\right)\left[\langle V'(\eta) \rangle_{\mathcal{C}_x} - \sigma\left(f\left(x/N\right)\right) - \sigma'\left(f_s\left(x/N\right)\right)\left(\langle \eta \rangle_{\mathcal{C}_x} - f\left(x/N\right)\right)\right].$$

(Note again that the average of $\widehat{A}_x^N$ against $\mathrm{d}\vartheta_{f_s}^N$ is $\mathcal{O}(e^{-Cs}\ell/N)$.) Then

$$\begin{aligned}
&\sum_{x\in\mathcal{R}_N^d}\int_0^t\int_{\mathsf{X}_N}\widetilde{\Phi}_{t,s}\widehat{A}_x^N\,\mathrm{d}\vartheta_{f_s}^N\\
&=\sum_{x\in\mathcal{R}_N^d}\int_0^t\int_{\mathsf{X}_N}\left(\widetilde{\Phi}_{t,s}-\Pi_x^N\widetilde{\Phi}_{t,s}\right)\widehat{A}_x^N\,\mathrm{d}\vartheta_{f_s}^N+\sum_{x\in\mathcal{R}_N^d}\int_0^t\int_{\mathsf{X}_N}\Pi_x^N\widetilde{\Phi}_{t,s}\widehat{A}_x^N\,\mathrm{d}\vartheta_{f_s}^N\\
&=\sum_{x\in\mathcal{R}_N^d}\int_0^t\int_{\mathsf{X}_N}\left(\Phi_{t-s}-\Pi_x^N\Phi_{t-s}\right)\widehat{A}_x^N\,\mathrm{d}\vartheta_{f_s}^N\\
&\qquad+\sum_{x\in\mathcal{R}_N^d}\int_0^t\int_{\mathsf{X}_N}\left(\Pi_x^N\Phi_{t-s}-\mathbf{E}_{\vartheta_{f_s}^N}[\Pi_x^N\Phi_{t-s}]\right)\widehat{A}_x^N\,\mathrm{d}\vartheta_{f_s}^N=:J_t^1+J_t^2
\end{aligned}$$

where Π_x^N again averages over Ω_m (and does not touch the other site) as in (4.3).

To estimate the first term J_t^1 we again approximate the measure $\vartheta_{f_s}^N$ on $\mathcal{C}_x$ by the equilibrium measure with local mass $f_t(x/N)$, and denote it by $\overline{\vartheta}_{f_s}$ (note that the approximation is made differently for each cube and depends on x, even if it is written explicitly). This produces an error $\mathcal{O}(\ell^{d+1}/N)$ (using the Lipschitz regularity of Φ_{t-s} and the exponential convergence $f_t\to\rho$ to get uniform in time bounds). We then apply the Poincaré inequality [LY93, Theorem 2] in the cube $\mathcal{C}_x$ (whose constant is independent of the number of particles and proportional to the size of the cube) and the law of large number $\|\widehat{A}_x^N\|_{L^2(\overline{\vartheta}_{f_s}^N)}=\mathcal{O}(e^{-Cs}\ell^{-d/2})$:

$$\begin{aligned}
J_t^1&\le\sum_{x\in\mathcal{R}_N^d}\int_0^t\|\Phi_{t-s}-\Pi_x^N\Phi_{t-s}\|_{L^2(\overline{\vartheta}_{f_s}^N)}\|\widehat{A}_x^N\|_{L^2(\overline{\vartheta}_{f_s}^N)}\,\mathrm{d}s+\mathcal{O}\left(\ell^{d+1}/N\right)\\
&\Box\,\ell^{1-\frac{d}{2}}\sum_{x\in\mathcal{R}_N^d}\int_0^t\sqrt{\overline{D}_x^\ell(\Phi_{t-s})}e^{-Cs}\,\mathrm{d}s+\mathcal{O}\left(\ell^{d+1}/N\right)\\
&\Box\,\ell^{1-\frac{d}{2}}N^{d/2}\int_0^t\left(\sum_{x\in\mathcal{R}_N^d}\overline{D}_x^\ell(\Phi_{t-s})\right)^{1/2}e^{-Cs}\,\mathrm{d}s+\mathcal{O}\left(\ell^{d+1}/N\right)
\end{aligned}$$

where $\overline{D}_x^\ell(\Phi)$ is the Dirichlet form on the cube $\mathcal{C}_x$ with respect to the measure $\overline{\vartheta}_{f_s}^N$:

$$\overline{D}_x^\ell(\Phi):=\sum_{y\sim z\in\mathcal{C}_x}\int_{\mathsf{X}_N}\left[\partial_{\eta_x}\Phi(\eta)-\partial_{\eta_y}\Phi(\eta)\right]^2\,\mathrm{d}\overline{\vartheta}_{f_s}^N.$$

Then we use the entropy production

$$\frac{1}{2N^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathsf{X}_N}\Phi_{t-s}(\eta)^2\,\mathrm{d}\vartheta_{f_s}^N\le-\sum_{x\in\mathcal{R}_N^d}D_x^\ell(\Phi_{t-s})+\mathcal{O}\left(1/N^2\right)$$

as before to deduce that

$$\int_0^T J_t^1\,\mathrm{d}t\,\Box\,T^{1/2}\left(\ell/N\right)^{1-d/2}+\mathcal{O}\left(T\ell^{d+1}/N\right).$$

To control the second term J_t^2, we first use the *equivalence of ensemble* in [LPY02, Corollary 5.3] on the measure $\overline{\vartheta}_{f_s}^N$ (together with the exponential tail estimates on the local Gibbs measure) to get

$$\langle V'(\eta)\rangle_{\mathcal{C}_x}=\sigma\left(\langle\eta\rangle_{\mathcal{C}_x}\right)+\mathcal{O}\left(1/\ell^d\right). \tag{5.3}$$

Second we remark that the Lipschitz regularity of Φ_{t-s} implies that $\Pi_x^N \Phi_{t-s} - \mathbf{E}_{\vartheta^N_{f_s}}[\Pi_x^N \Phi_{t-s}] = \mathcal{O}(\ell^d N^{-d})$, and since the average of $\widehat{A}_x^N$ with respect to $\vartheta^N_{f_s}$ is $\mathcal{O}(\ell/N)$, we can write

$$J_t^2 = \sum_{x \in \mathcal{R}_N^d} \int_0^t \int_{\mathsf{X}_N} \left(\Pi_x^N \Phi_{t-s}[\langle \eta \rangle_{\mathcal{C}_x}] - \Pi_x^N \Phi_{t-s}\left[f_s\left(x/N \right) \right] \right) \widehat{A}_x^N \, \mathrm{d}\vartheta^N_{f_s} + \mathcal{O}\left(\ell/N \right).$$

Third, we prove again that the Lipschitz regularity of Φ_{t-s} (with constant N^{-d}) implies a Lipschitz regularity of its averaged projection $\Pi_x^N \Phi_{t-s}$ with constant $\ell^d N^{-d}$, with respect to the local mass. Indeed, given $0 = m < m' < +\infty$, pick any pair of configuration (η_0, ζ_0) with $\langle \eta_0 \rangle_{\mathcal{C}_x} = m$, $\langle \zeta_0 \rangle_{\mathcal{C}_x} = m'$ and $\eta_0 < \zeta_0$ (such configuration trivially exists since $m < m'$). Then consider the coupling on $\Omega_m \times \Omega_{m'}$ produced by a product of localised smooth distribution around δ_{η_0} and δ_{ζ_0}, so that the support only contains strictly ordered $\eta < \zeta$. Then we evolve it along the flow of the coupling operator $e^{t \widetilde{\mathcal{L}}_N} \delta_{(\eta_0, \zeta_0)}$. The marginals respectively converge to $\nu^{\ell,m}$ and $\nu^{\ell,m'}$ (convergence to equilibrium of the oiriginal evolution). Arguing as for the ZRP, we deduce that $W_1(\nu^{\ell,m}, \nu^{\ell,m'}) = m' - m$, and a corresponding optimal coupling Π associated to this distance is so that the cost does not change sign on its support, i.e. $\eta \le \zeta$ in the support. We deduce as for the ZRP that $\Pi_x^N \Phi_{t-s}$ is $\ell^d N^{-d}$-Lipschitz.

We deduce (using (5.3)), the Taylor formula, the approximation of $\vartheta^N_{f_s}$ by $\overline{\vartheta}^N_{f_s}$, and the law of large numbers, the same estimate on J_t^2 as for the ZRP, and finally the same conclusion follows (optimizing $\ell := N^{1/4}$)

$$\frac{1}{T} \int_0^T I_t^N \, \mathrm{d}t \;\Box\; \left(\frac{1}{N} + \frac{\ell^{1+d/2}}{N^{1-d/2}} + \frac{\ell^{1+2d}}{N} + \frac{1}{\ell^{d/2}} + \frac{\ell}{N} \right) \;\Box\; \frac{1}{N^{1/8}}. \qquad \Box$$

References

[DMOWa] D. Dizdar, G. Menz, F. Otto, and T. Wu. The quantitative hydrodynamic limit of the Kawasaki dynamics. arXiv:1807.09850.

[DMOWb] D. Dizdar, G. Menz, F. Otto, and T. Wu. Toward a quantitative theory of the hydrodynamic limit. arXiv:1807.09857.

[Fat13] M. Fathi. A two-scale approach to the hydrodynamic limit part II: local Gibbs behavior. *ALEA Lat. Am. J. Probab. Math. Stat.*, 10(2):625–651, 2013.

[GOVW09] Ñ. Grunewald, F. Otto, C. Villani, and M. Westdickenberg. A two-scale approach to logarithmic Sobolev inequalities and the hydrodynamic limit. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(2):302–351, 2009.

[GPV88] M. Z. Guo, G. C. Papanicolaou, and S. R. S. Varadhan. Nonlinear diffusion limit for a system with nearest neighbor interactions. *Comm. Math. Phys.*, 118(1):31–59, 1988.

[KL99] C. Kipnis and C. Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[Lig85] T. Liggett. *Interacting Particle Systems*. Springer Berlin Heidelberg, 1985.

[LPY02] C. Landim, G. Panizo, and H. T. Yau. Spectral gap and logarithmic Sobolev inequality for unbounded conservative spin systems. *Ann. Inst. H. Poincaré Probab. Statist.*, 38(5):739–777, 2002.

[LSV96] C. Landim, S. Sethuraman, and S. Varadhan. Spectral gap for zero-range dynamics. *Ann. Probab.*, 24(4):1871–1902, 1996.

[LY93] Sheng Lin Lu and Horng-Tzer Yau. Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics. *Comm. Math. Phys.*, 156(2):399–433, 1993.

[MMM22] D. Marahrens, A. Menegaki, and C. Mouhot. Quantitative hydrodynamic limit of interacting particle systems on lattices. soon on the ArXiv, 2022.

[Rez91] F. Rezakhanlou. Hydrodynamic limit for attractive particle systems on $\mathbf{Z}^d$. *Comm. Math. Phys.*, 140(3):417–448, 1991.

[Yau91] H.-T. Yau. Relative entropy and hydrodynamics of Ginzburg-Landau models. *Lett. Math. Phys.*, 22(1):63–80, 1991.

Institut des hautes études Scientifiques, 35 Rte de Chartres, 91440, Bures-sur-Yvette, France
Email address: Menegaki@ihes.fr

DPMMS, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK
Email address: C.Mouhot@dpmms.cam.ac.uk