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Decompositions of high-frequency Helmholtz solutions and application to the finite element method

D. Lafontaine*
(based on joint works with J. Galkowski, E. Spence, and J. Wunsch)

Abstract

This paper presents joint works with Jeffrey Galkowski, Euan Spence, and Jared Wunsch [16, 12]. It corresponds to the talk the author gave at IHES for the Séminaire Laurent Schwartz in April 2022.

Over the last ten years, results of Melenk and Sauter [20, 21] decomposing high-frequency Helmholtz solutions into an analytic part and a well-behaved in frequency part have had a large impact in the numerical analysis of the Helmholtz equation. These results have been proved for the constant-coefficients Helmholtz equation outside an analytic Dirichlet obstacle or an interior domain with an impedance boundary condition.

In [16], we obtained an analogous decomposition for the Helmholtz equation with C^∞ variable coefficients in R^d, then in [12], analogous decompositions for scattering problems fitting into the very general black-box scattering framework of Sjöstrand and Zworski [26], thus covering Helmholtz problems with variable coefficients, impenetrable obstacles, and penetrable obstacles all at once. These results allowed us to prove new sharp frequency-explicit convergence results for (i) the hp-finite-element method (hp-FEM) applied to the C^∞ variable-coefficient Helmholtz equation in R^d, (ii) the hp-FEM applied to the variable-coefficient Helmholtz equation in the exterior of an analytic Dirichlet obstacle, where the coefficients are analytic in a neighborhood of the obstacle, and (iii) the h-FEM applied to the Helmholtz penetrable-obstacle transmission problem. In this expository paper, we show how to obtain the decomposition from [16], and the main ideas behind the general result of [12].

1 Introduction

We are interested in the Helmholtz equation

$$\Delta u + k^2 u = f \quad \text{in } \mathsf{R}^d \backslash O$$

in the exterior of an obstacle O – one of the simplest wave models, obtained for example as the Fourier transform in time of the wave equation. We ask for the solution to verify the Sommerfeld radiation condition at infinity

$$\partial_r u - iku = o(r^{-(d-1)/2}),$$

corresponding to the fact that we are looking for an *outgoing* wave (in other words, propagating from the obstacle toward infinity), with, for example, Dirichlet boundary condition at the boundary of the obstacle

$$u = 0 \text{ on } \partial O.$$

A very flexible and popular method to solve *numerically* such an equation is the finite element method, where one approximates the solution by piecewise polynomial functions. We are in particular interested in the so called hp-finite element method (hp-FEM), where one decreases the meshsize h and increases the polynomial degree p of the approximation, both depending on the frequency k of the solution. A natural question in this framework is the following: *what is a*

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condition on h, p, and k for these methods to converge? A first intuition is that, as the solution oscillates at scale k^{-1}, we should need at least a number of degrees of freedom $\#\mathrm{DOF} \gtrsim k^d$. Is it enough ?

In a very influential series of paper [20, 21], Melenk and Sauter brought an answer when the obstacle *O* is *analytic* (see also [19] and [10] for the corresponding interior impedance problem). They have shown that, under the conditions

$$\frac{hk}{p} \leq C_1, \qquad p \geq C_2 \log k,$$

the solution to the discrete problem exists, is unique, and is quasi-optimal (that is, it is the best possible approximation of the solution by a piecewise polynomial, up to a multiplicative constant). This condition is sharp with respect to the number of degrees of freedom of the problem: one can construct h and p so that the number of degrees of freedom of the problem verifies

$$\#\mathrm{DOF} \quad \left(\frac{p}{h}\right)^d \sim k^d.$$

In other words, hp-FEM applied to this setting does not suffer the *pollution effect* that plagues the h-FEM (where h decreases depending on k and p is left constant), for which one needs more degrees of freedom than k^d [1]. The proof of Melenk and Sauter [20, 21] is based on a decomposition of the Helmholtz solutions

$$u = u_{H^2} + u_A, \tag{$\star$}$$

where u_{H^2} verifies *better* estimates in the frequency k than u, and u_A verifies the same estimates in k as u but is *analytic*. The idea is that u_{H^2} contains the high frequencies ($\gtrsim k$) of the solution, and u_A the low frequencies ($\lesssim k$). Their proof of the decomposition ($\star$) is based on explicit computations (relying on Bessel and Hankel functions) in $\mathbb{R}^d$, and therefore, cannot be generalised in a straightforward way to more general problems, such as the Helmholtz equation with variables coefficients, despite the large interest in this problem in the numerical analysis community.

In the joint works [16, 12] with Jeffrey Galkowski, Euan Spence and Jared Wunsch, we tackled the question of *understanding the frequency-decomposition ($\star$) in the most general possible situation.* We obtained the following results.

1. In [16], we obtained the decomposition ($\star$) for the variable C^∞ coefficients equation in $\mathbb{R}^d$.
2. Then, in [12], we have shown such a decomposition in the very general *black-box scattering* framework of Sjöstrand-Zworski.

One can apply our results to obtain new frequency-explicit convergence estimates in the free space and for penetrable and non-penetrable obstacles for the equation with variable coefficients. In particular, theses applications show that the hp-FEM applied to the Hemlholtz equation does not suffer from the pollution effect in the two following situations:

1. For the equation without obstacle and with variable C^∞ coefficients.
2. In the exterior of an analytic obstacle, for the equation with variable C^∞ coefficients which are analytic near the obstacle.

In the sequel of this expository paper, we will show how to obtain the decomposition from [16], and the main ideas behind the general result of [12]. In particular, we will focus on the decomposition ($\star$), and we refer to [16] and [12] for the application of this decomposition to convergence estimates for hp-FEM. §2 is dedicated to the exposition of the result of [16], §3 to the general, abstract decomposition obtained in [12], and §4 to its application to concrete Helmholtz problems.

2 The splitting for the equation with C^∞ coefficients in $\mathbb{R}^d$

2.1 Our result

Let us restate the problem in the semiclassical analysis framework. We will use the semiclassical small parameter $\hbar := k^{-1}$ (not to be confused with the meshsize h coming from the finite element method), and the semiclassical Sobolev norms

$$\|v\|_{H^m_\hbar} := \|v\|_{L^2} + \hbar^m \|v\|_{\dot{H}^m}.$$

We are interested in the semiclassical Helmholtz equation

$$P_\hbar u = f, \qquad P_\hbar := -\hbar^2 \nabla \cdot (A \nabla \cdot) - n,$$

with f compactly supported, where $n, A \in C^\infty$, supp$(1-n)$, supp$(1-A)$ are compact, with

$$n_{\min} \le n \le n_{\max}, \quad A_{\min} \le A \le A_{\max}. \tag{2.1}$$

The main assumption we will make is that the resolvent of the problem is *polynomially bounded* on a certain set of frequencies, for which we will obtain the decomposition. In other words, we assume that there exists $\mathsf{H} \subset (0, \hbar_0]$ so that P_h satisfies a polynomial resolvent estimate for $\hbar \in \mathsf{H}$: if $\chi \in C_c^\infty$, then for any outgoing Helmholtz solution u we have

$$\|\chi u\|_{H^1_\hbar} \le C \hbar^{-1-M} \|f\|_{L^2}, \quad \forall h \in \mathsf{H}. \tag{2.2}$$

Resolvent estimates have been intensively studied in the past decades. In particular, in any non-trapping geometry, where all the optical trajectories are escaping to infinity, such an estimate always holds with $M = 0$ and $\mathsf{H} = (0, \hbar_0]$ (see [22], [17], [3], [13]...). Whereas a polynomial bound fails in strongly trapping geometries (see e.g. [2], [4],[23]), it always holds, no matter the strength of the trapping, outside an arbitrary small set of frequencies, as shown in [15]: therefore, we see this assumption as non-restricting in the numerical setting our applications are aimed to.

We can now state the main result of [16].

Theorem 2.1 (Main result of [16]) *Under the previous assumptions, if $R > 0$ and u is an outgoing solution of $P_\hbar u = f$, there exists u_{H^2}, u_A so that*

$$u|_{B_R} = u_{H^2} + u_A,$$

where $u_{H^2} \in H^2(B_R)$ verifies

$$\|u_{H^2}\|_{H^2_\hbar(B_R)} \lesssim \|f\|_{L^2(B_R)} \quad \forall \hbar \in \mathsf{H},$$

and $u_A \in C^\omega(B_R)$ verifies, for some $\lambda > 1$,

$$\|\partial^\alpha u_A\|_{L^2(B_R)} \lesssim \hbar^{-1-M-|\alpha|} \lambda^{|\alpha|} \|f\|_{L^2(B_R)} \quad \forall \alpha, \hbar \in \mathsf{H}.$$

Such a result is applied in [16] to sharp convergence estimates (i.e., no pollution effect) for the hp-FEM in this setting. Before pursuing to the proof of the decomposition, let us make the following comments:

1. The bound on u_{H^2} is better than, and does not depend on, the bound on u given by the polynomial resolvent estimate. In this sense u_{H^2} is better behaved in frequency than u.
2. The $\hbar$-dependency in the bound on u_A is the one given by the polynomial resolvent estimate, hence the same as for u. The dependency in the order of derivation $|\alpha|$ shows that u_A is analytic.

Concerning the second point above, it is worth noting that a bound in $C(□)□^{-|\alpha|}|\alpha|!\lambda^{|\alpha|}$ would have been sufficient to obtain analyticity. Melenk and Sauter [21] actually obtain, outside an analytic obstacle for the equation with constant coefficients, the intermediate bound $C(□)\max(|\alpha|, □^{-1})^{|\alpha|}\lambda^{|\alpha|}$. This zoology of $(|\alpha|, □)$ dependencies in the u_A-type bounds can be better understood by making the connection with the radius of analyticity of u_A, as recapped in Figure 2.1. The bound in Theorem 2.1 is similar to the one in the constant coefficients case

Bound	Radius of convergence
$C(□)□^{-|\alpha|}\lambda^{|\alpha|}$	$+\infty$
$C(□)□^{-|\alpha|}|\alpha|!\lambda^{|\alpha|}$	$□$
$C(□)\max(|\alpha|, □^{-1})^{|\alpha|}\lambda^{|\alpha|}$	1

Figure 2.1: Estimate on u_A and radius of analyticity

[20], and the bound of [21], implying a radius of convergence independent of the frequency, is sufficient to obtain sharp convergence estimates for hp-FEM.

The rest of the section is devoted to (a sketch of) the proof of Theorem 2.1.

2.2 The frequency splitting

The idea is the following:

- u_A will contain the "low frequencies" of the solution. It will be analytic because compactly supported in frequencies.
- u_{H^2} will contain the "high frequencies" of the solution. In particular, it will be supported in frequencies where $P_□$ is semiclassically elliptic.

In practice, let $\mu \gg 1$ to be fixed later, $\psi_\mu \in C_c^\infty(\mathbb{R})$ be supported in $B(0, 2\mu)$ with $\psi_\mu = 1$ on $B(0, \mu)$, and define the low- and high-frequency projections Π_{low} and Π_{high} as Fourier multipliers

$$\widehat{\Pi_{\mathrm{low}} v}(\zeta) := \psi_\mu(□^2|\zeta|^2)\hat{v}(\zeta), \qquad \Pi_{\mathrm{high}} := \mathrm{Id} - \Pi_{\mathrm{low}}.$$

Hence Π_{low} localises in Fourier variables $|\zeta| □ □^{-1}\sqrt{\mu}$, and Π_{high} in Fourier variables $|\zeta| □ □^{-1}\sqrt{\mu}$. Finally, with $\varphi \in C_c^\infty(\mathbb{R}^d)$ so that $\varphi = 1$ in $B(0, R)$, set

$$u_{H^2} := \Pi_{\mathrm{high}}(\varphi u), \quad u_A := \Pi_{\mathrm{low}}(\varphi u).$$

Hence the decomposition $u|_{B_R} = u_{H^2} + u_A$ holds by definition and we are left with verifying the bounds on u_{H^2} and u_A.

2.3 Low frequencies

The bound on the low-frequency component u_A is an immediate consequence of its frequency localisation, together with the Parseval identity:

$$\begin{aligned}
\|\partial^\alpha u_A\|_{L^2} &= \|\partial^\alpha(\Pi_{\mathrm{low}}\varphi u)\|_{L^2} = \frac{1}{(2\pi)^d}\|\zeta^\alpha \widehat{\Pi_{\mathrm{low}}\varphi u}\|_{L^2} \\
&= \frac{1}{(2\pi)^d}\Big\|\zeta^\alpha \underbrace{\psi_\mu(□^2|\zeta|^2)}_{\substack{\text{supported in} \\ |\zeta| \leq □^{-1}\sqrt{2\mu}}}\widehat{\varphi u}(\zeta)\Big\|_{L^2} \\
&\leq \frac{1}{(2\pi)^d}(□^{-1}\sqrt{2\mu})^{|\alpha|}\|\widehat{\varphi u}(\zeta)\|_{L^2} = (□^{-1}\sqrt{\mu})^{|\alpha|}\|\varphi u\|_{L^2} \\
&□ (□^{-1}\sqrt{2\mu})^{|\alpha|} h^{-1-M}\|f\|_{L^2},
\end{aligned}$$

where we used the polynomial resolvent bound (2.2) on the last line.

2.4 High frequencies

A quick reminder on semiclassical ellipticity

In order to deal with u_{H^2}, we prefer to see Π_{high} as a semiclassical pseudo-differential operator and write it as

$$\Pi_{\text{high}} = \mathrm{Op}_\hbar\left(1 - \psi_\mu(|\xi|^2)\right),$$

where, for $a \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$,

$$\left(\mathrm{Op}_\hbar(a)v\right)(x) := (2\pi\hbar)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi/\hbar}\, a(x,\xi)v(y)\, dy d\xi.$$

We refer to [16, §4] for the notions of semiclassical pseudo-differential operators used here, and to [27] and [8, Appendix E] for a comprehensive presentation on the topic. It will be sufficient for the present discussion to know that

- One says that a is a symbol of order m and writes $a \in S^m$ if
$$|\partial^x_\alpha \partial^\xi_\beta a(x,\xi)| \lesssim \langle \xi \rangle^{m-\beta}, \quad \langle \xi \rangle := \sqrt{1+|\xi|^2}.$$
- If $a \in S^m$, we write $\mathrm{Op}_\hbar(a) \in \Psi^m$; if $A \in \Psi := \cup_m \Psi^m$, we denote $\sigma(A)$ its semiclassical principal symbol.
- If $A \in \Psi^{m_1}$ and $B \in \Psi^{m_2}$ then $AB \in \Psi^{m_1+m_2}$, and for any $s \in \mathbb{R}$, A is bounded uniformly in $\hbar$ as an operator from $H^s_\hbar$ to $H^{s-m_1}_\hbar$.
- $P_h \in \Psi^2$ with $\sigma(P_h) = \langle A\xi, \xi\rangle - n(x)$.

In order to introduce the notion of semiclassical ellipticity we will use to control u_{H^2}, we need to define the operator-wavefront set $\mathrm{WF}_\hbar A$ of an operator $A \in \Psi$: it is the subset of the phase-space where the action of A is not negligible at high frequencies. Whereas a precise definition can be found for example in [16, §4], it will be sufficient for our purposes to know that

$$\mathrm{WF}_\hbar(\mathrm{Op}_\hbar(a)) \subset \mathrm{supp}\, a.$$

Now, one says that $B \in \Psi^m$ is *elliptic* on $X \subset \mathbb{R}^d_x \times \mathbb{R}^d_\xi$ if

$$\langle \xi \rangle^{-m}\left|\sigma(B)(x,\xi)\right| \geq c > 0, \quad \forall (x,\xi) \in X,\ 0 < \hbar \leq \hbar_0.$$

The importance of this notion lies in the fact that semiclassical pseudo-differential operators are always microlocally invertible where they are elliptic, up to negligible high-frequency errors. In other words, if $B \in \Psi^m$ is semiclassically elliptic on $\mathrm{WF}_\hbar A$, $A \in \Psi^l$ then

$$A = QB + E = BQ' + E',$$

(see for example [8, Proposition E.32]) where $Q, Q' \in \Psi^{m-l}$ and E, E' are negligible:

$$\forall N, \quad \|E\|_{H^{-N}_\hbar \to H^N_\hbar} \leq C_N \hbar^N.$$

One writes that E, E' are $O(\hbar^\infty)$.

Controlling u_{H^2}

We are now in position to obtain the bound on u_{H^2} and end the proof of Theorem 2.1. As advertised previously, the idea is to define u_{H^2} in such a way that it will be supported where $P_\hbar$ is semiclassically elliptic. But

$$\sigma(P_\hbar)(x,\xi) = \langle A\xi, \xi\rangle - n(x),$$

hence one can take $\mu \gg 1$ large enough so that, by (2.1)

$$|\xi| \geq \mu \implies \langle\xi\rangle^{-2}\sigma(P_\hbar)(x,\xi) \geq c_0 > 0,$$

in such a way that $P_\hbar$ is elliptic on

$$\{|\xi| \geq \mu\} \supset \operatorname{supp}(1-\psi_\mu) \supset \mathrm{WF}_\hbar \Pi_{\text{high}} \supset \mathrm{WF}_\hbar \Pi_{\text{high}}\varphi.$$

Therefore, there exists $Q \in \Psi^{-2}$ so that

$$\Pi_{\text{high}}\varphi = QP_\hbar + O(\hbar^\infty).$$

As a first attempt at a proof, applying u to the previous identity, we get

$$\begin{aligned} u_{H^2} = \Pi_{\text{high}}\varphi u &= QP_\hbar u + O(\hbar^\infty)u \\ &= Qf + O(\hbar^\infty)u, \end{aligned}$$

and as $Q \in \Psi^{-2}$,

$$\|Qf\|_{H^2_\hbar} \lesssim \|f\|_{L^2}.$$

It would remain to deal with the error $O(\hbar^\infty)u$: the idea is that u grows as most polynomially in $\hbar^{-1}$ via the resolvent estimate (2.2), so that this term should be indeed negligible. But, observe that the polynomial resolvent estimate $\|\chi u\|_{H^1_\hbar} \lesssim \hbar^{-1-M}\|f\|_{L^2}$ is a *truncated* estimate (hence the appearance of the cut-off χ): u is indeed a priori not globally L^2.

Hence, to make a rigorous proof of this first attempt, we introduce enough spatial cut-offs to get a truncated error term: let φ_1, φ_2, φ_3 be compactly supported and so that $\varphi_{i+1} = 1$ on the support of φ_i, with $\varphi_0 := \varphi$. As $\varphi_2 P_\hbar$ is semiclassically elliptic on $\mathrm{WF}_\hbar\varphi_1\Pi_{\text{high}}\varphi$, there exists $Q \in \Psi^{-2}$ so that

$$\begin{aligned} \varphi_1\Pi_{\text{high}}\varphi &= Q\varphi_2 P_\hbar + O(\hbar^\infty) \\ &= Q\varphi_2 P_\hbar + O(\hbar^\infty)\varphi_3, \end{aligned}$$

where we multiplied on the right by φ_3 to obtain the second identity. Now, applying u to the above we get

$$\begin{aligned} u_{H^2} = \varphi_1\Pi_{\text{high}}\varphi u + (1-\varphi_1)\Pi_{\text{high}}\varphi u &= Q\varphi_2 P_\hbar u + O(\hbar^\infty)\varphi_3 u \\ &= Q\varphi_2 f + O(\hbar^\infty)\varphi_3 u, \end{aligned}$$

where $(1-\varphi_1)\Pi_{\text{high}}\varphi = O(\hbar^\infty)$ by the pseudo-locality of the pseudo-differential semiclassical calculus as $\operatorname{supp}(1-\varphi_1)\cap\operatorname{supp}\varphi = \emptyset$ (see e.g. [16, (4.14)-(4.15)-(4.16)]), and hence is $= O(\hbar^\infty)\varphi_3$ by multiplying by φ_3 on the right. As $Q \in \Psi^{-2}$,

$$\|Q\varphi_2 f\|_{H^2_\hbar} \lesssim \|f\|_{L^2},$$

and the error $O(\hbar^\infty)\varphi_3 u$ is now indeed negligible as $\|\varphi_3 u\|_{H^1_\hbar} \lesssim \hbar^{-1-M}\|f\|_{L^2}$ via the polynomial resolvent estimate (2.2). We conclude

$$\|u_{H^2}\|_{H^2_\hbar} \lesssim \|f\|_{L^2}.$$

3 An abstract splitting for black-box scattering problems

3.1 The main idea: use the functional calculus

When trying to extend the proof of Theorem 2.1 presented in §2 to a setup with boundaries – for example, to the Helmholtz equation in the exterior of an obstacle – one runs into the issue to try to extend the solution to the whole space in a suitable way in order to be able to use frequency projections given by Fourier multipliers.

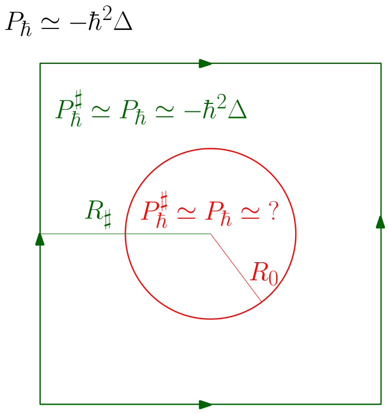

Figure 3.1: Black-box scattering operator and reference operator

The main idea of [12] was to rather use *the functional calculus*, and define low- and high-frequency projections as

$$\begin{cases} \Pi_{\mathrm{high}} := (1-\psi_\mu)(P_\hbar), \\ \Pi_{\mathrm{low}} := \psi_\mu(P_\hbar), \end{cases}$$

where $P_\hbar$ is the semiclassical Helmholtz operator associated with the equation. In addition to not needing to extend the Helmholtz solutions to the whole space, this point of view has two immediate advantages: these operations commute with the equation, and, as their formulation is rather abstract, we can try to take $P_\hbar$ as general as possible and handle a variety of Helmholtz problems all at once.

3.2 Black-box scattering of Sjöstrand-Zworski

The general setup we will work with is the one given by the *black-box scattering of Sjöstrand and Zworski*, introduced in [26]. We refer to [12, §2] for an overview of black-box scattering in our setting, to the comprehensive presentation in [8, Chapter 4] for more details, and to [15, §2] for a brief overview with an emphasis on the counting of resonances.

We work in an Hilbert space with the orthogonal decomposition

$$H = L^2(\mathbb{R}^d \setminus B_{R_0}) \quad H_{R_0},$$

were H_{R_0} is an Hilbert space. A semiclassical black-box scattering operator $P_\hbar$ is a self-adjoint operator on H that coincides with $-\hbar^2\Delta$ outside the "black-box" $B_{R_0} := B(0, R_0)$, where it is left unspecified. In order to compactify the problem, and, in particular, been able to define a functional calculus, we define the corresponding *reference operator* $P_\hbar^\sharp$ by placing the black-box operator in a large reference torus $\mathbb{T}^d_{R^\sharp} := \mathbb{R}^d/(2R_\sharp\mathbb{Z})^d$ with $R_\sharp > R_0$, instead of $\mathbb{R}^d$. Hence $P_\hbar^\sharp$ acts on

$$H^\sharp = L^2(\mathbb{T}^d_{R^\sharp} \setminus B_{R_0}) \quad H_{R_0},$$

with domain $D^\sharp \quad H^\sharp$. Whereas we refer to [12, §2] and [8, Chapter 4] for a careful definition of these operators, the two main assumptions in this framework are that $\mathbf{1}_{B_{R_0}}(P_\hbar + i)^{-1}$ is compact from H to H, and that the eigenvalues counting-function of $P_\hbar^\sharp$ grows at most polynomially (this can be seen as a weak Weyl law).

This very general setup includes most of, if not all reasonable Helmholtz scattering problems one can think of, such as scattering by a Lipschitz Dirichlet or Neumann obstacle, a Lipschitz penetrable obstacle, a metric perturbation, a potential... (see for example [8, §4.1]).

3.3 Our result

Before stating a precise version of our result [12], we give an informal version of it.

Theorem 3.1 **([12], informal version)** *Let $P_\hbar$ be a black-box scattering operator of Sjöstrand-Zworski. We make the two following assumptions.*

(H1) One has a polynomial resolvent estimate.

(H2) One has an estimate quantifying the regularity of $P_\hbar$ "inside the black-box" $B(0, R_0)$.

Then, any solution u of the Helmholtz equation $(P_\hbar - 1)u = f$, can be decomposed as

$$u = u_{H^2} + u_A,$$

where

- *u_{H^2} verifies a black-box version of the estimate $\|u_{H^2}\|_{H^2_\hbar} \lesssim \|f\|_{L^2}$,*
- *u_A verifies the same estimates in $\hbar$ as u but is* regular*. This regularity is dictated by the regularity of the underlying problem as measured by (H2).*

The Assumption (H1) is the same as the one encountered in Theorem 2.1. It is shown in [15] that it holds in the black-box framework for "most" frequencies: the key point, therefore, to apply the above result to specific situations is to check that an estimate of the type (H2) holds. In the applications we will present later, this estimate (H2) corresponds to an heat-flow estimate, an elliptic estimate, and regularity of the eigenfunctions of the Laplace operator on the torus. This Theorem could be applied to a range of other specific situations, provided an estimate of type (H2) is at hand.

Let us now give a more precise version of this result. In the following, $\mathcal{D}_{\text{loc}}$ denotes the set of functions locally in the domain of $P_\hbar$ (see [12, §2] for a precise definition), $\mathcal{D}_\hbar^{\sharp,\infty}$ the set of functions belonging to the domains of all the iterates of $P_\hbar^\sharp$, and $C_0(\mathbb{R})$ the set of continuous functions f so that $\lim f(\lambda) = 0$ as $\lambda \to \pm\infty$.

Theorem 3.2 **([12], simplified version)** *Let $P_\hbar$ be a black-box scattering operator of Sjöstrand-Zworski. Then, there exists $\Lambda > 0$ so that the following holds. We assume that:*

(H1) There exists $H \subset (0, \hbar_0]$, $M \geq 0$, and $\mathcal{D}_{\text{out}} \subset \mathcal{D}_{\text{loc}}$, so that if $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi = 1$ near B_{R_0}, for any $v \in \mathcal{D}_{\text{out}}$ solution of $(P_\hbar - 1)v = \chi g$ we have

$$\|\chi v\|_{\mathcal{H}} \lesssim \hbar^{-M-1}\|g\|_{\mathcal{H}} \quad \forall h \in H.$$

(H2) There exists $E \in C_0(\mathbb{R})$ verifying $E > 0$ on $[-\Lambda, \Lambda]$ and an α-family of operators $D(\alpha)$ coinciding with ∂^α outside B_{R_0} so that

$$\|D(\alpha)E(P_\hbar^\sharp)v\|_{\mathcal{H}^\sharp} \leq C_E(\alpha, \hbar)\|v\|_{\mathcal{H}^\sharp} \quad \forall v \in \mathcal{D}_\hbar^{\sharp,\infty}, \ \hbar \in H.$$

Then, if $u \in \mathcal{D}_{\text{out}}$ is solution of $(P_\hbar - 1)u = f$ with $\operatorname{supp} f \subset B_R$, $R_0 < R < R_\sharp$, we have

$$u|_{B_R} = (u_{H^2} + u_A)|_{B_R},$$

with $u_{H^2} \in \mathcal{D}^\sharp$, $u_A \in \mathcal{D}_\hbar^{\sharp,\infty}$ and

$$\|u_{H^2}\|_{\mathcal{H}^\sharp} + \left\|P_\hbar^\sharp u_{H^2}\right\|_{\mathcal{H}^\sharp} \lesssim \|f\|_{\mathcal{H}} \quad \forall \hbar \in H, \tag{3.1}$$

$$\|D(\alpha)u_A\|_{\mathcal{H}^\sharp} \lesssim C_E(\alpha, \hbar)\hbar^{-M-1}\|f\|_{\mathcal{H}} \quad \forall \hbar \in H, \ \alpha. \tag{3.2}$$

We make the following comments.

- The assumption (H1) is a polynomial resolvent estimate bound as announced. The set D_{out} D_{loc} of functions verifying this estimate can be thought of as a set of "outgoing" functions (but observe that no notion of outgoingness is actually needed here).

- The assumption (H2), measuring the regularity of the underlying problem ("the content of the black-box"), will be the key assumption to check when wanting to apply this result to concrete situations. The appearance of the abstract operator $D(\alpha)$ is due to the fact that the derivation doesn't a-priori make sense inside the black-box, but in most of the applications we aim, $D(\alpha) := \partial^\alpha$. In practice, one has to identify the function E to take in order to have a good estimate. As we only require E to be non-vanishing in an interval $[-\Lambda, \Lambda]$, we often refer to the estimate in (H2) as "the low-energy estimate".

- When $P_\square$ is, for example, the Dirichlet Laplacian, the bound (3.1) on u_{H^2} is a bound on u_{H^2} $_{H^2_\square}$ by elliptic regularity. Hence, this bound is the exact analog of the bound on u_{H^2} in Theorem 2.1. In particular, it is better, and does not depend on, the resolvent-estimate bound (H1).

- u_A is regular because belongs to the domains of all the iterates of the reference operator u_A $D^{\sharp,\infty}_\square$. This regularity is further quantified by (3.2), which is, in practice, an estimate on the derivatives $\partial^\alpha u_A$. It depends, on the one hand, on the resolvent estimate $\square^{-M-1}$, and, on the other hand, on the regularity of the underlying problem as measured by (H2) by the quantity $C_E(\alpha, \square)$.

We give in [12] a stronger version of Theorem 3.2, that will be mentioned in the applications presented in §4: see [12, Theorem A]. Let us just mention here that the main result of [12] allows us

- to relax the assumption (H2) to having an estimate on some $D(\alpha)E$ where E is only given by the functional calculus up to negligible high-frequency errors,

- to localise the assumption (H2) near the black-box in order to be able to use local analytic estimates (the problem cannot possibly be analytic everywhere as $P_\square$ has constant coe□cients outside a compact set),

- and to use a *family* of estimates as (H2), in order to allow tuning the estimate we use depending on □ and α.

3.4 A functional calculus for $P_\square$

We will work with $P^\sharp_\square$ on the reference torus, and would like to define low- and high-frequency projections by a functional calculus for this operator. As $P^\sharp_\square$ with domain $D^\sharp$ is self-adjoint with compact resolvent, we can describe its Borel functional calculus [24, Theorem VIII.6] explicitly in terms of the orthonormal basis of eigenfunctions $\phi^\sharp_j$ $H^\sharp$ (with eigenvalues $\lambda^\sharp_j$, appearing with multiplicity): for f a real-valued Borel function on R, $f(P^\sharp_\square)$ is self-adjoint with domain

$$D_f := \left\{ \sum a_j \phi^\sharp_j \quad H^\sharp \ : \ \sum \left| f(\lambda^\sharp_j) a_j \right|^2 < \infty \right\},$$

and if $v = \sum a_j \phi^\sharp_j$ D_f then

$$f(P^\sharp_\square)(v) := \sum a_j f(\lambda^\sharp_j) \phi^\sharp_j.$$

As usual, the map $f \to f(P^\sharp_\square)$ is an algebra morphism, with, for f bounded

$$f(P^\sharp_\square) \ _{L(H^\sharp)} \leq \sup_{\lambda \ \mathrm{R}} |f(\lambda)|.$$

However, this very general formulation does not give us a-priori informations on the *structure* of the calculus. To this account, it is useful to recall the Hel□er–Sjöstrand construction of the functional calculus [14], [7, §2.2] (which can also be used to prove the spectral theorem to begin with; see [6]): if f is smooth and sufficiently decaying, we define

$$f(P_\hbar^\sharp) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \widetilde{f}}{\partial \overline{z}}(P_\hbar^\sharp - z)^{-1}\, dxdy,$$

where $\widetilde{f}$ is an almost-analytic extension of f:

$$\tilde{f}_{|\mathbb{R}} = f, \quad \frac{\partial \tilde{f}}{\partial \overline{z}} = O(|\operatorname{Im} z|^n),\ n > 0.$$

This definition can be shown to agree with the operators defined by the Borel functional calculus for well-behaved f; see [6, Theorems 2-5], [7, Lemmas 2.2.4-2.2.7]. The key refinement we will use, under this formulation, and essentially due to Sjöstrand [25, §4], is that

$$\text{\textit{Outside the black-box, the functional calculus coincides, modulo negligible terms, with the semiclassical pseudo-differential calculus on the torus.}} \tag{3.3}$$

In other words, for $f \in C_c^\infty(\mathbb{R})$, if $\chi \in C^\infty(\mathbb{T}^d_{R_\sharp})$ vanishes near B_{R_0} then

$$\chi f(P_\hbar^\sharp)\chi = \chi f(-\hbar^2\Delta)\chi + O(\hbar^\infty),$$

and in addition, $f(-\hbar^2\Delta) = \operatorname{Op}_\hbar^{\mathbb{T}^d_{R^\sharp}}\left(f(|\xi|^2)\right)$. The negligible term $O(\hbar^\infty)$ is defined similarly as in §2 (see [12] for a precise formulation).

3.5 Elements of proof

We now present the ideas behind the proof of Theorem 3.2. We refer to [12] for the details, as well as for the statement and the proof of the more general [12, Theorem A].

The splitting

Now armed with a functional calculus for $P_\hbar^\sharp$, we can define low- and high-frequency projections as announced. For $\psi_\mu \in C_c^\infty(\mathbb{R})$ supported in $B(0, 2\mu)$, $\psi_\mu = 1$ on $B(0, \mu)$, $\mu \ll 1$, we define

$$\Pi_{\text{high}} := (1 - \psi_\mu)(P_\hbar^\sharp), \qquad \Pi_{\text{low}} := \psi_\mu(P_\hbar^\sharp).$$

And with

$$\varphi = 1 \text{ in } B_R \setminus B_{R_0}, \qquad \operatorname{supp}\varphi \subset B_{R_\sharp},$$

one takes

$$u_{H^2} := \Pi_{\text{high}}\varphi u, \qquad u_A := \Pi_{\text{low}}\varphi u,$$

and we are left with verifying the bounds on u_{H^2} and u_A.

High frequencies

The bound on u_{H^2} comes from an abstract ellipticity manipulation near the black-box, and semiclassical ellipticity away from it, thanks to the refinement of Sjöstrand (3.4). Indeed, as Π_{high} and $P_\hbar^\sharp$ commute, using the equation $(P_\hbar - 1)u = f$ and the fact that $P_\hbar$ and $P_\hbar^\sharp$ coincide on $\operatorname{supp}\varphi$

$$(P_\hbar^\sharp - 1)(\Pi_{\text{high}}\varphi u) = \Pi_{\text{high}}(P_\hbar^\sharp - 1)(\varphi u) = \Pi_{\text{high}}\varphi f + \Pi_{\text{high}}[P_\hbar^\sharp, \varphi]u. \tag{3.4}$$

But, taking a slightly sharper cut-off $\tilde{\psi}_\mu \in C_c^\infty(\mathbb{R})$, equal to zero on $\operatorname{supp}(1 - \psi_\mu)$ and so that $1 - \tilde{\psi}_\mu$ is supported away from $\lambda = 1$, and using the morphism property of the functional calculus

$$u_{H^2} = \Pi_{\text{high}}\varphi u = (1 - \psi_\mu)(P_\hbar^\sharp)\varphi u$$

$$= \Big[(1-\tilde{\psi}_\mu(\lambda))(\lambda-1)^{-1}(\lambda-1)(1-\psi_\mu(\lambda))\Big](P^\sharp_\hbar)\varphi u$$
$$= \Big[\underbrace{(1-\tilde{\psi}_\mu(\lambda))(\lambda-1)^{-1}}_{\text{bounded}}\Big](P^\sharp_\hbar)(P^\sharp_\hbar-1)(\Pi_{\text{high}}\varphi u),$$

where $(1-\tilde{\psi}_\mu(\lambda))(\lambda-1)^{-1}(P^\sharp_\hbar)$ is bounded. Hence, using the equation satisfied by $\Pi_{\text{high}}\varphi u$ (3.4),

$$\|u_{H^2}\|_{H^\sharp} \lesssim \|(P^\sharp_\hbar-1)(\Pi_{\text{high}}\varphi u)\|_{H^\sharp} \leq \|\Pi_{\text{high}}\varphi f\|_{H^\sharp} + \|\Pi_{\text{high}}[P^\sharp_\hbar,\varphi]u\|_{H^\sharp}$$
$$\lesssim \|f\|_{H^\sharp} + \|\Pi_{\text{high}}[P^\sharp_\hbar,\varphi]u\|_{H^\sharp}.$$

Therefore, using (3.4) once again, the same bound holds on $P^\sharp_\hbar u_{H^2}$ and

$$\|u_{H^2}\|_{H^\sharp} + \|P^\sharp_\hbar u_{H^2}\|_{H^\sharp} \lesssim \|f\|_{H^\sharp} + \|\Pi_{\text{high}}[P^\sharp_\hbar,\varphi]u\|_{H^\sharp}.$$

It now only remains to control the commutator term $\Pi_{\text{high}}[P^\sharp_\hbar,\varphi]u$. But $[P^\sharp_\hbar,\varphi]$ is supported away from the black-box B_{R_0}, hence, by Sjöstrand (3.4), it is up to negligible $O(\hbar^\infty)$ terms a frequency cut-off defined by the semiclassical pseudo-differential calculus on the torus ! Therefore, we can use semiclassical ellipticity to control

$$\|\Pi_{\text{high}}[P^\sharp_\hbar,\varphi]u\|_{H^\sharp} \lesssim \|f\|_{H^\sharp},$$

for $\mu \gg 1$ fixed large enough, in the same way as in the $\mathbb{R}^d$ case presented in §2.4.

Low frequencies

The low frequency estimate on u_A is a simple consequence of the assumption (H2) together with the morphism property of the calculus. Taking indeed $\Lambda > 0$ so that $\operatorname{supp}\psi_\mu \subset [-\Lambda,\Lambda]$, as $E > 0$ on $\operatorname{supp}\psi_\mu$ we can decompose

$$D(\alpha)u_A = D(\alpha)\psi_\mu(P^\sharp_\hbar)\varphi u = D(\alpha)E(P^\sharp_\hbar)\big(\psi_\mu\frac{1}{E}\big)(P^\sharp_\hbar)\varphi u.$$

Therefore, using assumption (H2),

$$\|D(\alpha)u_A\|_{H^\sharp} \leq C_E(\alpha,\hbar)\left\|\left(\frac{1}{E}\psi_\mu\right)(P^\sharp_\hbar)\varphi u\right\|_{H^\sharp}$$
$$\leq C_E(\alpha,\hbar)\sup_{\lambda\in\mathbb{R}}\left|\frac{1}{E(\lambda)}\psi_\mu(\lambda)\right|\|\varphi u\|_{H^\sharp}$$
$$\lesssim C_E(\alpha,\hbar)\sup_{\lambda\in\mathbb{R}}\left|\frac{1}{E(\lambda)}\psi_\mu(\lambda)\right|\hbar^{-M-1}\|f\|_{H},$$

where we used the polynomial resolvent estimate (H1) on the last inequality.

4 Applications of the abstract splitting to concrete Helmholtz scattering problems

4.1 Recovering the splitting for C^∞ coefficients in $\mathbb{R}^d$

As a first application, it is reinsuring to check that we can recover Theorem 2.1. In order to do so, we need a slightly stronger version of Theorem 3.2, mentioned before: one can relax the assumption (H2) to having an estimate on some $D(\alpha)E_0$ where E_0 is only given by the functional calculus up to negligible high-frequencies errors. In other words, it is sufficient to assume that there is E_0 so that $E_0 = E(P^\sharp_\hbar) + O(\hbar^\infty)$ with $E > 0$ in $[-\Lambda,\Lambda]$ and

$$\|D(\alpha)E_0 v\|_{H^\sharp} \leq C_E(\alpha,\hbar)\|v\|_{H^\sharp}, \qquad v \in D^{\sharp,\infty}_{\hbar,\hbar} \cap H.$$

Now, for the Helmholtz equation with variable C^∞ coefficients in $\mathbb{R}^d$, $P_\hbar = -\hbar^2 c^2 \nabla\cdot(A\nabla\cdot)$, and $P_\hbar^\sharp$ is the same operator but on the torus. We simply take, for $\Lambda > 0$ given by (the more general version of) Theorem 3.2

$$E = 1 \text{ in } [-\Lambda, \Lambda], \qquad \operatorname{supp} E \subset [-2\Lambda, 2\Lambda].$$

By the coercivity of A, it is not difficult to check that $E(-c^2\hbar^2\nabla\cdot(A\nabla\cdot))$ is cutting-off in Fourier variables modulo negligible terms: in other words, taking

$$\varphi = 1 \text{ in } [-\Lambda', \Lambda'], \qquad \operatorname{supp} \varphi \subset [-2\Lambda', 2\Lambda'],$$

for $\Lambda' \gg 1$ large enough, we have

$$E(-c^2\hbar^2\nabla\cdot(A\nabla\cdot)) = \varphi(-\hbar^2\Delta)E(-c^2\hbar^2\nabla\cdot(A\nabla\cdot)) + O(\hbar^\infty),$$

and taking

$$E_0 := \varphi(-\hbar^2\Delta)E(-c^2\hbar^2\nabla\cdot(A\nabla\cdot)),$$

the Parseval identity together with Fourier localisation implies in the exact same way as in §2.3 for Theorem 2.1

$$\|\partial^\alpha E_0 v\|_{L^2} \lesssim \hbar^{-|\alpha|}\lambda^{|\alpha|}\|v\|_{L^2},$$

and we recover the decomposition given by Theorem 2.1.

4.2 Toward Dirichlet: localizing the analyticity near the black-box

We now would like to apply Theorem 3.2 to the exterior of an analytic Dirichlet obstacle for the equation with variable coefficients. As the coefficients are constants outside a compact set, the most analyticity we can ask for is for them to be smooth and analytic near the obstacle. In this setting, we would like (H2) to be an analytic estimate. However, this assumption as stated in Theorem 3.2 is a global estimate, and we cannot hope for a global analytic estimate to be at hand as the problem is not globally analytic. For this reason, we want to be able to use as (H2) an analytic estimate *locally near the scatterer*.

We can indeed, as mentioned before, show a more general version of Theorem 3.2 allowing such a local estimate. The black-box scattering operator $P_\hbar$ is now a compact perturbation of $-\hbar^2\Delta$ outside B_{R_0} (instead of $= -\hbar^2\Delta$) and the estimate in (H2) becomes

$$\|\rho D(\alpha)E(P_\hbar^\sharp)v\|_{H^\sharp} \leq C_E(\alpha,\hbar)\|v\|_{H^\sharp}, \qquad v \in \mathcal{D}_\hbar^{\sharp,\infty}, \hbar \in \mathrm{H},$$

for some $\rho \in C^\infty$ with $\rho = 1$ near B_{R_0}.

Under this relaxed assumption, u_A doesn't verify a global regular estimate: rather, it can be splited as

$$u_A = u_A^{R_0} + u_A^\infty,$$

where (see Figure 4.1, and [12, Theorem A] for a precise formulation)

- $u_A^{R_0}$ verifies (3.2) near the black-box and is negligible away from it,
- u_A^∞ is negligible near the black-box and is entire away from it.

4.3 Analytic Dirichlet obstacle with locally analytic coefficients

We can now apply the more general version of Theorem 3.2 presented in §4.2 to the Helmholtz equation in the exterior of an analytic Dirichlet obstacle with C^∞ variable coefficients that are analytic near the obstacle.

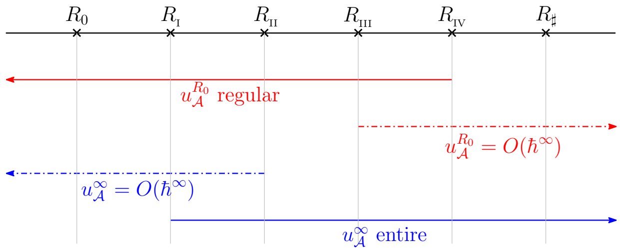

Figure 4.1: The splitting $u_A = u_A^{R_0} + u_A^\infty$

As a low-energy estimate (H2), we will use a local heat-flow estimate. In our setting, the following folklore analytic heat-flow estimate, that can be attributed to Friedman [11],

$$\|\rho\partial^\alpha e^{tP}\|_{L^2\to L^2} \lesssim \lambda^{|\alpha|}|\alpha|!t^{-|\alpha|/2}, \tag{4.1}$$

where P is the (non-semiclassical) variable coefficients Laplacian outside the obstacle and ρ is supported where the coefficients are analytic, is at hand. Taking $E(\lambda) := e^{-\lambda}$ and $t := \hbar^2$, the above translates in our framework to

$$\|\rho\partial^\alpha E(P_\hbar^\sharp)v\|_{L^2} \lesssim \lambda^{|\alpha|}\hbar^{-|\alpha|}|\alpha|!\,\|v\|_{L^2},$$

and we obtain a splitting with $u_A^{R_0}$ verifying where it is not negligible the following analytic estimate

$$\|\partial^\alpha u_A^{R_0}\|_{L^2(B(0,R_{\rm III}))} \lesssim \lambda^{|\alpha|}\hbar^{-|\alpha|-1-M}|\alpha|!.$$

Observe how we lost $|\alpha|!$ with respect to the $\mathbb{R}^d$ case. While it is enough for the analyticity of $u_A^{R_0}$ where it is not negligible, this is not enough for the sharp convergence of the hp-FEM: using this result, one indeed only obtains convergence for a number of degrees of freedom $\#\text{DOF} \sim \left(\frac{p}{h}\right)^d \lesssim k^d \log k$, a logarithm of the frequency away from the conjectured sharp result (no pollution effect).

To obtain the sharp result, we use in combination with Friedman's estimate (4.1) the more recent heat-flow estimate due to Escauriaza, Montaner and Zhang [9]

$$\|\rho\partial^\alpha e^{tP}\|_{L^2\to L^2} \lesssim \lambda^{|\alpha|}|\alpha|!\exp\frac{1}{t}. \tag{4.2}$$

The key point here is that $|\alpha|$ can be arbitrarily large, hence, there is regimes where (4.2) becomes *better* than (4.1). Using the most-general version [12, Theorem A] of Theorem 3.2 allowing us to use as (H2) a *family* of estimates, interpolating between (4.2) and (4.1) and using the best possible t depending on $\hbar$ and α, we obtain a splitting with

$$\|\partial^\alpha u_A^{R_0}\|_{L^2(B(0,R_{\rm III}))} \lesssim \lambda^{|\alpha|}\hbar^{-1-M}\max(\hbar^{-1},|\alpha|)^{|\alpha|}.$$

Observe how we went from a radius of analyticity proportional to $\hbar$, to a radius of analyticity independent of $\hbar$ (see Figure 2.1). This is actually the analog of Melenk and Sauter estimate outside an analytic obstacle for the equation with constant coefficients [21], and we can apply it to the sharp-convergence of hp-FEM (no pollution effect) in our variable coefficients setting, that is, with a number of degrees of freedom

$$\#\text{DOF} \sim \left(\frac{p}{h}\right)^d \lesssim k^d.$$

4.4 Penetrable obstacles

As a last application, we treat the less regular *transmission problem*. The equation is now posed in $\mathbb{R}^d$, with variable coefficients having a *jump* on an interface O, that we can see as a penetrable obstacle. We assume that the coefficients away from the interface, and the interface itself, have finite C^{2m} regularity. In this setting, we will use as (H2) the standard elliptic regularity estimate (see e.g. [18, Theorem 4.20], [5, Theorem 5.2.1, Part (i)])

$$\|v\|_{H^{2m}(B_R\setminus O)\oplus H^{2m}(O)} \le C(m)\sum_{\ell=0}^{m} \|(c^2\nabla\cdot(A\nabla))^\ell v\|_{L^2(B_R\setminus O)\oplus L^2(O)}.$$

Taking $E(\lambda) := \langle\lambda\rangle^{-2m}$, the above can be translated in the framework of Theorem 3.2, and we obtain a splitting with $u_A = (u_A^+, u_A^-) \in C^\infty(B_R\setminus O) \times C^\infty(O)$ verifiying

$$\|\partial^\alpha u_A^\pm\|_{L^2} \le C(m)\hbar^{-|\alpha|-M-1}\|f\|_{L^2}, \qquad |\alpha| \le 2m.$$

As the problem is far less regular, it is of no surprise we obtain a less regular splitting, where in particular the dependency on $|\alpha|$ in the above estimate is not explicit. While this is not enough to obtain convergence estimates for hp-FEM, we can apply this result to obtain sharp convergence estimates for the h-FEM method, where only the meshsize h decreases as the frequency k increases, and the polynomial degree p of the approximation is left constant.

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