Séminaire Laurent Schwartz

EDP et applications

Année 2021-2022

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Séminaire Laurent Schwartz — EDP et applications (2021-2022), Exposé n° II, 13 p.

https://doi.org/10.5802/slsedp.149

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Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org

Séminaire Laurent-Schwartz — EDP et applications
Institut des hautes études scientifiques, 2021-2022
Exposé n° II, 1-13

BLOW-UP DYNAMICS FOR THE SELF-DUAL CHERN–SIMONS–SCHRÖDINGER EQUATION

KIHYUN KIM

ABSTRACT. We review some recent progress on the blow-up dynamics for the self-dual Chern–Simons–Schrödinger equation within equivariance. We describe the results of the recent series of works [18, 19, 20, 22, 21] by the author, Kwon, and Oh. We in particular discuss soliton resolution and rotational instability for this model.

1. INTRODUCTION

The *self-dual Chern–Simons–Schrödinger* equation within m-*equivariance* is

$$i(\partial_t + iA_t[u])u + \partial_r^2 u + \frac{1}{r}\partial_r u - \Big(\frac{m + A_\theta[u]}{r}\Big)^2 u + |u|^2 u = 0, \tag{CSS}$$

where $u : I \times (0,\infty) \to \mathbb{C}$ (I is a time interval), $m \in \mathbb{Z}$ (called *equivariance index*), and the connection components $A_t[u]$ and $A_\theta[u]$ are given by

$$A_t[u] = -\int_r^\infty (m + A_\theta[u])|u|^2 \frac{dr'}{r'}, \qquad A_\theta[u] = -\frac{1}{2}\int_0^r |u|^2 r' dr'. \tag{1.1}$$

The Chern–Simons–Schrödinger equation was introduced by the physicists Jackiw and Pi [12] as a gauge-covariant cubic nonlinear Schrödinger equation on $\mathbb{R}^2$. It is a non-relativistic quantum mechanical model that describes the dynamics of interacting charged particles on the plane. The model (CSS) is derived after fixing the Coulomb gauge condition and imposing the equivariant symmetry on the scalar field $\phi : I \times \mathbb{R}^2 \to \mathbb{C}$:

$$\phi(t,x) = u(t,r)e^{im\theta},$$

where (r,θ) are the polar coordinates on $\mathbb{R}^2$. We refer to [12, 11, 13, 14, 6] for more physical backgrounds, and refer to the introduction of [18, 19, 20] for more details on this equivariant symmetry reduction.

What makes the model (CSS) fascinating is the *self-duality*, which equips the model with a special algebraic structure. It was observed by Jackiw and Pi in [12], who exploited the self-duality to connect the soliton equation to the Liouville equation, which is completely integrable, and were able to find an explicit family of solitons. We will discuss more on the self-duality below.

The long time dynamics of (CSS), we believe, makes the model more interesting. The goal of this report is to introduce the following two nonlinear dynamics: the strong rigidity in asymptotic behavior of solutions (*soliton resolution*) and the *rotational instability* mechanism for some finite-time blow-up solutions. The results presented here are based on the recent series of works [18, 19, 20, 22, 21] by the author, Kwon, and Oh.

2010 *Mathematics Subject Classification.* 35B44, 35Q55, 37K40.

We briefly describe various symmetries and conservation laws of (CSS). Among the most basic symmetries are the time translation and the phase rotation symmetries. Associated to these are the conservation laws for the *energy* and the *mass*[1]:

$$E[u] := \int \frac{1}{2}|\partial_r u|^2 + \frac{1}{2}\Big(\frac{m + A_\theta[u]}{r}\Big)^2 |u|^2 - \frac{1}{4}|u|^2, \tag{1.2}$$

$$M[u] := \int |u|^2, \tag{1.3}$$

where we denoted $\int f(r) = 2\pi \int f(r) r dr$. With this energy functional, (CSS) admits a *Hamiltonian* structure

$$\partial_t u = -i\nabla E[u],$$

where ∇ (acting on a functional) is the Fréchet derivative with respect to the real inner product $\int \mathrm{Re}(\overline{u}v)$. Of particular importance are the *L^2-scaling symmetry* and the *pseudoconformal symmetry*; if $u(t,r)$ is a solution to (CSS), then the functions u_λ and $\mathcal{C}u$ also solve (CSS):

$$u_\lambda(t,r) := \frac{1}{\lambda} u\Big(\frac{t}{\lambda^2}, \frac{r}{\lambda}\Big), \qquad \forall \lambda > 0, \tag{1.4}$$

$$[\mathcal{C}u](t,r) := \frac{1}{|t|} u\Big(-\frac{1}{t}, \frac{r}{|t|}\Big) e^{ir^2/4t}, \qquad \forall t \neq 0. \tag{1.5}$$

Associated to (1.4) and (1.5) are the *virial identities*:

$$\partial_t \int r^2 |u|^2 = 4\int \mathrm{Im}(\overline{u} \cdot r\partial_r u), \tag{1.6}$$

$$\partial_t \int \mathrm{Im}(\overline{u} \cdot r\partial_r u) = 4E[u]. \tag{1.7}$$

In this aspect, (CSS) shares many similarities with the cubic NLS

$$i\partial_t \psi + \Delta\psi + |\psi|^2 \psi = 0 \quad \text{on } \mathbb{R}^{1+2}. \tag{NLS}$$

A distinguished feature of (CSS) in comparison to NLS is the *self-duality*. Indeed, the energy functional can be written in the self-dual form

$$E[u] = \int \frac{1}{2} |\mathbf{D}_u u|^2, \tag{1.8}$$

where $\mathbf{D}_u$ is the (covariant) *Cauchy–Riemann operator* defined by

$$\mathbf{D}_u f := \partial_r f - \frac{m + A_\theta[u]}{r} f. \tag{1.9}$$

We call the operator $u \mapsto \mathbf{D}_u u$ the *Bogomol'nyi operator*. In particular, energy is always nonnegative. Due to (1.8) and the Hamitonian structure, any static solutions to (CSS) are given by solutions to the *Bogomol'nyi equation*:

$$\mathbf{D}_Q Q = 0. \tag{1.10}$$

For $m \geq 0$, there is an *explicit* m-equivariant static solution (Jackiw–Pi vortex) to the Bogomol'nyi equation which is unique up to the symmetries of the equation [11]:

$$Q(r) = \sqrt{8}(m+1)\frac{r^m}{1 + r^{2m+2}}, \qquad m \geq 0. \tag{1.11}$$

Note that we suppressed the m-dependences in $\mathbf{D}_u$ and Q for the simplicity of notation. Compared to the NLS case where the solitons exhibit exponential (spatial)

[1] The physical interpretation of the quantity $M[u]$ is the *total charge*, but in this paper we shall call it mass following the widespread convention for NLS.

decays, in the CSS case the solitons only have polynomially decaying tails. Applying the pseudoconformal transform (1.5) to Q, we obtain an explicit finite-time blow-up solution:

$$S(t,r) := \frac{1}{|t|}Q\Big(\frac{r}{|t|}\Big)e^{-ir^2/4|t|}, \qquad t<0,$$

which blows up at $t=0$ with the pseudoconformal blow-up rate $|t|$. We note that $S(t)$ has finite energy if and only if $m \geq 1$.

Let us briefly mention some known results on the Chern–Simons–Schrödinger equation without symmetry. The local well-posedness has been studied by many authors [2, 10, 27, 25], but the best known result by Liu–Smith–Tataru [27] still misses the critical L^2-space. There are also results on the long-term dynamics [2, 3, 37].

For the equivariant self-dual Chern–Simons–Schrödinger equation, i.e., (CSS), much more is known. Because there is no derivative nonlinearity, using only the Strichartz estimates, (CSS) can be shown to be well-posed in the critical space L^2 [26, Section 2]. Liu–Smith [26] proved the following *subthreshold theorem*: for $m \geq 0$, any m-equivariant L^2-solutions u with $M[u] < M[Q]$ scatter both forwards and backwards in time. At the threshold mass $M[u] = M[Q]$ (necessarily $m \geq 0$), the classification result of Li–Liu [24] says that either (i) u is a global scattering solution, (ii) $u(t) = Q$, or (iii) $u(t) = S(t)$ up to symmetries. The works [18, 19, 20, 22, 21] by the author, Kwon, and Oh studied the dynamics above the threshold, which are the main contents of this report.

2. Soliton resolution for (CSS)

In this section, we discuss the soliton resolution result [22] for (CSS). Most of the materials in this section are borrowed from [22].

It is widely believed that, for large data under generic assumptions, the maximal solutions asymptotically decompose into the sum of decoupled solitons and a radiation. This is referred to as the *soliton resolution conjecture*. This has been known for a wide range of completely integrable equations, but it is mostly open for *non-integrable* models. To our knowledge, the only known cases are the radial critical nonlinear wave equation and energy-critical equivariant wave maps [8, 9, 7, 5, 15, 16].

Our first result in this report is soliton resolution for the equivariant self-dual Chern–Simons–Schrödinger equation in a suitable weighted Sobolev class [22]. Let us denote the modulated soliton by

$$Q_{\lambda,\gamma}(r) := \frac{e^{i\gamma}}{\lambda}Q\Big(\frac{r}{\lambda}\Big), \qquad \lambda \in (0,\infty),\ \gamma \in \mathbb{R}/2\pi\mathbb{Z}.$$

We also denote by $H^{1,1}_m$ and H^1_m the (weighted) Sobolev spaces $H^{1,1}$ and H^1 restricted to m-equivariant functions.

Theorem 2.1 (Soliton resolution for equivariant $H^{1,1}$-data [22]). *Let $m \in \mathbb{Z}$. When $m \geq 0$, we have soliton resolution for $H^{1,1}_m$-solutions:*

- *(Finite-time blow-up solutions) If u is a H^1_m-solution to* (CSS) *that blows up forwards in time at $T < +\infty$, then $u(t)$ admits the decomposition*

$$u(t,\cdot) - Q_{\lambda(t),\gamma(t)} \to z^* \text{ in } L^2 \text{ as } t \to T^-, \tag{2.1}$$

  for some $\lambda(t) \in (0,\infty)$, $\gamma(t) \in \mathbb{R}/2\pi\mathbb{Z}$, and $z^* \in L^2$ with the following properties:
  - *(Further regularity of z^*) We have $\partial_r z^*, \frac{1}{r}z^* \in L^2$. Moreover, if u is a $H^{1,1}_m$ finite-time blow-up solution, then we also have $rz^* \in L^2$.*

– *(Bound on the blow-up speed) As $t \to T$, we have*

$$\lambda(t) \lesssim_{M[u]} \sqrt{E[u]}(T-t). \tag{2.2}$$

When $m = 0$, we further have the improved bound as $t \to T$

$$\lambda(t) \lesssim_{M[u]} \frac{\sqrt{E[u]}(T-t)}{|\log(T-t)|^{1/2}}. \tag{2.3}$$

- *(Global solutions) If u is a $H_m^{1,1}$-solution to* (CSS) *that exists globally forwards in time, then either $u(t)$ scatters forwards in time, or $u(t)$ admits the decomposition*

$$u(t,\cdot) - Q_{\lambda(t),\gamma(t)} - e^{it\Delta^{(-m-2)}}u^* \to 0 \text{ in } L^2 \text{ as } t \to +\infty, \tag{2.4}$$

for some $\lambda(t) \in (0,\infty)$, $\gamma(t) \in \mathbb{R}/2\pi\mathbb{Z}$, and $u^* \in L^2$ with the following properties:

– *(Further regularity of u^*) We have $\partial_r u^*, \frac{1}{r}u^*, ru^* \in L^2$.*

– *(Bound on the scale) As $t \to +\infty$, we have*

$$\lambda(t) \lesssim_{M[u]} \sqrt{E[\mathcal{C}u]}, \tag{2.5}$$

where $\mathcal{C}u$ is the pseudoconformal transform (1.5) *of u. When $m = 0$, we further have as $t \to +\infty$*

$$\lambda(t) \lesssim_{M[u]} \frac{\sqrt{E[\mathcal{C}u]}}{|\log t|^{1/2}}. \tag{2.6}$$

On the other hand, when $m < 0$, any $H_m^{1,1}$-solution to (CSS) *scatters forwards in time. Due to the time-reversal symmetry,*[2] *all the above statements also hold for backward-in-time evolutions.*

Remark 2.2 (The dynamics for $m \geq 0$ and $m < 0$). The dynamics of (CSS) for $m \geq 0$ and $m < 0$ are completely different. In fact, we will show that (CSS) for $m < 0$ is *defocusing* in the sense that energy is globally coercive:

$$E[u] \sim_{M[u]} \|u(t)\|_{\dot{H}_m^1}^2. \tag{2.7}$$

Hence there are no nontrivial Jackiw–Pi vortices for $m < 0$.

Remark 2.3 (Nonexistence of multi-solitons). It is remarkable that at most one soliton can appear in the resolution. This is a distinctive feature of (CSS). Indeed, as a consequence of the self-duality and non-locality, we observe a *defocusing* nature, i.e., the strict positivity of the energy, of (CSS) in the exterior of the soliton profile. Hence two solitons at different scales cannot exist simultaneously. We will obtain this defocusing nature by combining our two observations: (i) (CSS) at the exterior of soliton resembles (CSS) for $m < 0$ (observed in [18]) and (ii) the defocusing nature (2.7) when $m < 0$.

Even without equivariant symmetry, by essentially the same mechanism, we expect that there is no bubble tree (i.e., a multi-soliton separated *only* by scales) for the self-dual Chern–Simons–Schrödinger equation. However, multi-solitons separated by spatial distances may exist.

Remark 2.4 (Regularity assumptions on data). As seen in the above, we cover all H_m^1 finite-time blow-up solutions. For global solutions, we reduce the situation to the H_m^1 finite-time blow-up case using the spirit of the pseudoconformal transform,

[2] There is a time reversal symmetry for the covariant Chern–Simons–Schrödinger equation, but it *does not* flip the equivariance index. See [18] for more details. Thus the distinction of the cases $m \geq 0$ and $m < 0$ is not contradictory.

which requires the $H_m^{1,1}$-assumption. Note that $E[\mathcal{C}u]$ is well-defined for $H_m^{1,1}$-solutions u. Soliton resolution for any global H_m^1-solutions (or, more ambitiously L_m^2-solutions) is an interesting open problem.

Remark 2.5 (Comparison with (NLS)). For the finite-time blow-up case, there are similar results [38, 31] in (NLS) for solutions having slightly supercritical mass (i.e., $M[u] - M[Q] \ll 1$). Under this assumption, a standard variational argument in the blow-up scenario ensures that solutions eventually undergo the near-soliton dynamics in the L^2*-topology*. Note that in Theorem 2.1 we do not have L^2-proximity to solitons.

For the near-soliton dynamics of (NLS), it is known from [38] that any finite energy finite-time blow-up solutions satisfy either $\lambda(t) \sim ((T-t)/\log|\log(T-t)|)^{1/2}$ or $\lambda(t) \lesssim (T-t)$. The former log-log rate essentially arises from negative energy solutions, which is impossible for the self-dual (CSS). It is expected that such log-log rate is possible for the focusing non-self-dual (CSS) [3].

Remark 2.6 (Bounds for scaling parameter). When $m \geq 1$, the explicit blow-up solution $S(t)$ and the pseudoconformal blow-up solutions constructed in [18, 19] are finite energy finite-time blow-up solutions that saturate the bound (2.2). Similarly, the soliton Q itself saturates (2.5). It is an interesting open problem whether blow-up rates other than the pseudoconformal one is possible or not for finite energy finite-time blow-ups when $m \geq 1$. The analogous problem in (NLS) also remains open since the work of Merle and Raphaël.

When $m = 0$, the blow-up solution $S(t)$ and the soliton Q do not satisfy the bounds (2.3) and (2.6), respectively. This is consistent with Theorem 2.1 because Q does not belong to $H_0^{1,1}$ and the explicit blow-up solution $S(t)$ does not have finite energy, and hence Q and $S(t)$ are not covered by our theorem. Note that (2.6) says that any global-in-time nonscattering $H_0^{1,1}$-solution must blow up in infinite time. On the other hand, the authors [20, 21] construct finite energy finite-time blow-up solutions with the speed $\lambda(t) \sim (T-t)|\log(T-t)|^{-2}$ and $\lambda(t) \sim (T-t)^p|\log(T-t)|^{-1}$ for all $p > 1$, respectively (see Section 3.2 of this report). However, we do not know whether the upper bound (2.3) is sharp or not.

Remark 2.7 (On the phase rotation parameter). The phase rotation parameter does not necessarily stabilize as $t \to T$ (or $t \to +\infty$). Indeed, the finite-time blow-up solutions constructed in [21] for the $m = 0$ case exhibit infinite amount of phase rotations. The $m \geq 1$ case is open.

Strategy of the proof. Our key input is the *nonlinear coercivity of energy* (2.10) after extracting out the soliton profile, which holds for solutions with possibly large mass. As explained in Remark 2.3, this nonlinear coercivity is a consequence of the *self-duality* and *non-locality*, which are distinctive features of (CSS).

Step 1. Reduction to the finite-time blow-up case.

It suffices to consider the finite-time blow-up case, thanks to the $H^{1,1}$-assumption for the global case and the pseudoconformal transform. Now consider a H_m^1 finite-time blow-up solution u. By the blow-up criterion,

$$\|u(t)\|_{\dot{H}_m^1} \to +\infty \text{ but } E[u(t)] \text{ is conserved.} \tag{2.8}$$

For $m < 0$, we need to derive a contradiction. For $m \geq 0$, we need to show that u decomposes as in Theorem 2.1.

Step 2. Global coercivity of energy and contradiction for the $m < 0$ case.

We claim (2.7). Assuming this claim, we obtain a contradiction to (2.8). Thus it suffices to show (2.7). We first use the *self-duality* (1.8):

$$E[u] = \frac{1}{2}\int |\mathbf{D}_u u|^2 = \frac{1}{2}\int \Big|\Big(\partial_r - \frac{m + A_\theta[u]}{r}\Big)u\Big|^2.$$

We recall that $A_\theta[u] = -\frac{1}{2}\int_0^\infty |u| r' dr'$ is always *non-positive* with the bound

$$0 \leq -A_\theta[u] \leq \frac{M[u]}{4\pi}.$$

Note that m is also negative. Thus we can expect that the Cauchy–Riemann operator $\mathbf{D}_u$ enjoys a suitable Hardy inequality and this is in fact true:

$$\int \Big|\Big(\partial_r - \frac{m + A_\theta[u]}{r}\Big) f\Big|^2 \geq C(m, M[u]) \|f\|_{\dot{H}^1_m}^2, \qquad \forall f \in \dot{H}^1_m.$$

The point is that the constant C depends only on $M[u]$, which is a conserved quantity. See [22, Lemma 3.1] for a rigorous proof. This gives (2.7).

Step 3. Variational argument for $m \geq 0$.

We turn to the case of $m \geq 0$. By (2.8), the ratio $\sqrt{E[u(t)]}/\|u(t)\|_{\dot{H}^1_m}$ goes to zero. Using a soft argument and *the uniqueness of zero energy solutions*, one can show that $u(t)$ admits a decomposition

$$u(t) = [Q + \varepsilon(t, \cdot)]_{\lambda(t), \gamma(t)} \quad \text{with} \quad \|\varepsilon(t)\|_{\dot{H}^1} \to 0. \tag{2.9}$$

We remark that the L^2-norm cannot be used as a measure of proximity, because we do not assume that the mass of u is close to that of Q. We also remark that this decomposition heavily relies on the uniqueness of zero energy solutions (thanks to the self-duality), and hence this is not available for the NLS for arbitrary solutions with large mass.

Step 4. Nonlinear coercivity of energy.

For the proof of Theorem 2.1, the qualitative information $\|\varepsilon(t)\|_{\dot{H}^1} \to 0$ is not su□cient. Our next crucial input is the following *nonlinear coercivity of the energy*:

$$E[Q + \varepsilon] \gtrsim_{\|\varepsilon\|_{L^2}} \|\varepsilon\|_{\dot{H}^1}^2 \tag{2.10}$$

for ε satisfying the orthogonality conditions and $\|\varepsilon\|_{\dot{H}^1} \ll 1$. Here, the point is that *the coercivity holds even for* $\|\varepsilon(t)\|_{L^2} \gtrsim 1$. If we were to have L^2-smallness $\|\varepsilon(t)\|_{L^2} \ll 1$, then all the higher order terms of $E[Q+\varepsilon]$ are perturbative and (2.10) is merely a consequence of the linear coercivity (around Q). When $\varepsilon(t)$ has large L^2-norm, the higher order terms of $E[Q + \varepsilon]$ are no longer perturbative. Instead, we have (using the *self-duality* (1.8))

$$\begin{aligned} E[Q + \varepsilon] &= \frac{1}{2}\int |\mathbf{D}_{Q+\varepsilon}(Q + \varepsilon)|^2 \\ &\approx \frac{1}{2}\int |L_Q(\chi_R \varepsilon)|^2 + \Big|\Big(\partial_r - \frac{m + A_\theta[Q] + A_\theta[\varepsilon]}{r}\Big)\big((1 - \chi_R)\varepsilon\big)\Big|^2, \end{aligned}$$

where L_Q is the linearized Bogomol'nyi operator around Q. The interior term is simply handled by a localized version of the linear coercivity for L_Q. However, the exterior term contains non-perturbative higher order terms like $|\frac{A_\theta[\varepsilon]}{r}\varepsilon|^2$. At this point, we use the *non-locality* of the problem, particularly the fact that $m + A_\theta[Q] \approx -(m+2)$ is negative. Thus the exterior term can be viewed as the energy of ε for the $-(m+2)$-equivariant (CSS). Using the boundary condition $[(1-\chi_R)\varepsilon](R) = 0$ and the fact that both $m + A_\theta[Q]$ and $A_\theta[\varepsilon]$ are negative, we can prove unconditional coercivity for the exterior term, in the similar spirit with the proof in Step 2. As a result, the nonlinear coercivity of energy (2.10) follows.

Step 5. Completion of the proof.

The proof of (2.2) is standard and very similar to the pseudoconformal regime in Raphaël [38]. Indeed, a standard modulation analysis yields $|\lambda\lambda_t| \lesssim \|\varepsilon\|_{\dot{H}^1_m}$. Applying the nonlinear coercivity (2.10) of energy, we get

$$|\lambda_t| \lesssim \frac{1}{\lambda}\|\varepsilon\|_{\dot{H}^1_m} \lesssim \frac{1}{\lambda}\sqrt{E[Q + \varepsilon]} = \sqrt{E[u]},$$

whose integration yields the bound (2.2).

The proof of the improved bound (2.3) for $m = 0$ requires an additional trick, using the *logarithmic divergence* $\|\mathbf{1}_{y\leq R} yQ\|_{L^2} \sim \sqrt{\log R}$ and the generalized nullspace relations of the linearized operator $i\mathcal{L}_Q$ (of (CSS) around Q). We refer to [22] for this part.

Finally, the existence of the asymptotic profile z^* as in (2.1) as well as its regularity can be proved in a very similar manner to Merle–Raphaël [31]. To obtain z^* as the strong L^2-limit of $\varepsilon^\sharp(t) = u(t) - Q_{\lambda(t),\gamma(t)}$ as $t \to T^-$, we again take advantage of the nonlinear coercivity of energy in the form $\|\varepsilon^\sharp(t)\|_{H^1_m} \lesssim 1$. This means that $\varepsilon^\sharp(t)$ (and hence z^*) is not only controlled on the obvious soliton scale $r \lesssim \lambda$, but also up to scale $r \lesssim 1$.

3. Blow-up dynamics and rotational instability

In this section, we discuss the results of [18, 19, 20, 21] by the author, Kwon, and Oh on the blow-up dynamics for (CSS). The discussion here will be less rigorous and detailed than in the previous section. We borrow the materials from [18, 19, 20, 21].

From now on, we would like to study the refined description of the dynamics. We focus on finite-time blow-up for finite energy solutions. By Theorem 2.1, $m \geq 0$ necessarily and any such solution $u(t)$ decomposes as

$$u(t,r) - \frac{e^{i\gamma(t)}}{\lambda(t)} Q\Big(\frac{r}{\lambda(t)}\Big) \to z^*(r) \quad \text{as} \quad t \to T^-$$

with $z^* \in H^1$ and $\lambda(t)$ satisfying (2.2)–(2.3). In view of $S(t) \in H^1$ if and only if $m \geq 1$, we separate into two cases.

- High equivariance case $m \geq 1$: Section 3.1, following [18, 19].
- Radial case $m = 0$: Section 3.2, following [20, 21].

3.1. Pseudoconformal blow-up solutions and rotational instability.

In this subsection, we consider the high equivariance case $m \geq 1$. Recall that the explicit finite-time blow-up solution $S(t)$ has finite energy and satisfies

$$S(t,r) - \frac{1}{|t|} Q\Big(\frac{r}{|t|}\Big) \to 0 \quad \text{in } L^2 \quad \text{as } t \to 0.$$

We will study *pseudoconformal blow-up solutions*, namely, finite energy finite-time blow-up solutions $u(t)$ satisfying

$$u(t,r) - \frac{e^{i\gamma(t)}}{\lambda(t)} Q\Big(\frac{r}{\lambda(t)}\Big) \to z^*(r) \text{ in } L^2 \text{ as } t \to T^- \text{ with } \lambda(t) \approx C(u)\cdot(T-t),$$

and the dynamics near these blow-up solutions.

Our first result is on the construction and their instability mechanisms of pseudoconformal blow-up solutions.

Theorem 3.1 (Construction of pseudoconformal blow-up solutions [18]). *Given the asymptotic profile $z^*(r)$ that is small, smooth, and degenerate at the origin (i.e., $|z^*(r)| \lesssim r^K$ for K large), there exists a solution $u(t,r)$ such that*

$$u(t,r) - \frac{1}{|t|} Q\Big(\frac{r}{|t|}\Big) \to z^*(r) \text{ in } L^2 \text{ as } t \to 0.$$

The above theorem is a (CSS)-analogue of the Bourgain–Wang solutions for (NLS). However, an interesting dynamics diﬀering from the (NLS) one arises when we look at the instability mechanism of the pseudoconformal blow-up solutions.

Theorem 3.2 (Rotational instability [18]). *With the same hypothesis as above, there exists a continuous one-parameter family of solutions* $\{u^{(\eta)}\}_{\eta\geq 0}$ *such that*

$$u^{(\eta)}(t,r) \approx \frac{e^{i\gamma^{(\eta)}(t)}}{\lambda^{(\eta)}(t)} Q\Big(\frac{r}{\lambda^{(\eta)}(t)}\Big) + z^*(r) \text{ for } t \text{ near } 0,$$

where

- $u^{(0)}$ *is the pseudoconformal blow-up solution constructed in Theorem 3.1.*
- *If* $\eta \neq 0$, $u^{(\eta)}$ *scatters both forwards and backwards in time, and*

$$\lambda^{(\eta)}(t) = \sqrt{t^2+\eta^2},$$
$$\gamma^{(\eta)}(t) = \mathrm{sgn}(\eta)(m+1)\Big\{\tan^{-1}\Big(\frac{t}{|\eta|}\Big) - \frac{\pi}{2}\Big\}.$$

We believe that the same result holds true for $\eta \leq 0$.

Remark 3.3 (Rotational instability). Recall that the pseudoconformal blow-up solution $u^{(0)}$ does not exhibit any phase rotation. However, once $\eta \neq 0$, the solution $u^{(\eta)}$ stops concentrating at the scale $r \sim |\eta|$ but rather shows an abrupt phase rotation by the fixed amount of angle, $(m+1)\pi$, in a short time interval of length $\sim |\eta|$. It then spreads out like a backward pseudoconformal blow-up solution.

Rotational instability is not a unique feature of (CSS). In more general contexts, rotational instability is also expected for other critical geometric equations such as the wave maps, harmonic map heat flows, and Schrödinger maps from $\mathbb{R}^{1+2}$ into $\mathbb{S}^2$ [1, 33]. Theorem 3.2 rigorously constructs, up to our knowledge for the first time, a continuous curve of solutions exhibiting rotational instability.

Remark 3.4 (Comparison with (NLS)). The instability of Bourgain–Wang solutions for (NLS) was studied by Merle–Raphaël–Szeftel [35]. However, the instability mechanisms for (NLS) and (CSS) turn out to be completely different. In the (NLS) case, on one side (say $\eta > 0$) $u^{(\eta)}$ are global scattering solutions, but on the other side (say $\eta < 0$) $u^{(\eta)}$ blows up forwards in time under the log-log regime. The difference of the instability mechanisms is due to the difference of the structure of linearized operators around solitons in each case.

Remark 3.5 (Comments on the proof). The overall strategy is the method of backward construction with modulation analysis. One constructs a sequence of solutions $u^{(\eta)}$, proves *uniform* controls on the dynamics, and take the limit $\eta \to 0$ to obtain a blow-up solution. Of course, the continuity of the solution family in η should be proved separately. This kind of argument goes back to Merle [29], Martel [28], Raphaël–Szeftel [41], and Merle–Raphaël–Szeftel [35], where the last work is the most relevant to ours.

Compared to the (NLS) case, there are several additional difficulties in the (CSS) case. First, as the instability mechanism is completely different, one needs to construct new modified blow-up profiles (say $P(\cdot;b,\eta)$ satisfying $P(\cdot;b,\eta) \to Q$ as $b,\eta \to 0$) and find evolution laws for these additional modulation parameters b and η. In [18], motivated from the self-duality, we found a remarkable *nonlinear ansatz* for the modified profiles, which reduces a second-order nonlocal integro-differential equation to a first-order nonlocal ODE. Second, there are nontrivial interactions between the soliton $Q_{\lambda(t),\gamma(t)}$ and the radiation (i.e., a regular function $z(t,r)$ achieving $z(0,r) = z^*(r)$) from the nonlocal gauge potentials A. From the A_θ-potential, since $m + A_\theta[Q_{\lambda,\gamma} + z] \approx -(m+2) + A_\theta[z]$ (see the angular part of the covariant Laplacian in (CSS)), we need to evolve $z(t,r)$ using the $-(m+2)$-equivariant (CSS). The A_t-potential contains nontrivial phase correction to $Q_{\lambda,\gamma}$ from z. Finally, the Lyapunov functional which controls the remainder part of the solution needs a further correction (the *mass correction*). We refer to [18] for more details.

We complement our instability result by showing that the pseudoconformal blow-up can arise from a codimension one set of initial data.

Theorem 3.6 (Codimension one data set for pseudoconformal blow-up [19]). *There is a codimension one set of (smooth) initial data $u_0(r)$ such that the forward-in-time evolution $u(t,r)$ is a pseudoconformal blow-up solution.*

Remark 3.7 (Comparison with critical geometric equations). In the proof of Theorem 3.6, we found a simple but interesting identity, which we call the *linear conjugation identity*, that establishes an unexpected connection between (CSS) and other critical geometric equations such as the wave maps, harmonic map heat flows, and Schrödinger maps from $\mathbb{R}^{1+2}$ into $\mathbb{S}^2$. This identity takes the form

$$L_Q i L_Q^* = iH_Q, \tag{3.1}$$

where L_Q is the linearized operator of the Bogomol'nyi operator $u \mapsto \mathbf{D}_u u$ around Q and H_Q is the linearized operator arising in the aforementioned critical geometric equations. Experiences from these equations allow us to reveal a *hidden repulsivity* (or, monotonicity) in the linearized (CSS) dynamics.

Remark 3.8 (Comments on the proof). For the proof of Theorem 3.6, we again use modulation analysis. The overall scheme is the method of forward construction with repulsivity, developed through the works of Rodnianski–Sterbenz [42], Raphaël–Rodnianski [39], Merle–Raphaël–Rodnianski [33, 34], and Collot [4]. We employ the blow-up profile of [18], incorporate phase corrections from the nonlocal nonlinearities, and most importantly, exploit the repulsivity in Remark 3.7 to control the remainder part of the solution.

Theorems 3.2 and 3.6 naturally suggest the following *rotational instability conjecture*:

Conjecture 3.9 (Rotational instability conjecture). *Let $m \geq 1$. There is a codimension-one manifold $\mathcal{M}$ of (smooth) initial data with the following properties.*

- *If $u_0 \in \mathcal{M}$, then the associated forward-in-time solution $u(t)$ is a pseudoconformal blow-up solution.*
- *If $u_0 \notin \mathcal{M}$ but is near to $\mathcal{M}$, then the solution $u(t)$ does not blow up and rather exhibits rotational instability.*

If u_0 has mass slightly above the mass of Q, we further conjecture that $u(t)$ is a global scattering solution.

3.2. **Finite-time blow-up solutions in the radial case.** In this subsection, we consider the radial case $m = 0$. The radial case $(m = 0)$ turns out to be the most delicate due to the weakest spatial decay of Q and the weakest repulsivity associated with the linearized operator. However, it is also the case that is the most physically relevant because Q is the ground state without symmetry. Note that $S(t)$ is no longer a finite energy solution and the pseudoconformal blow-up is ruled out by Theorem 2.1. Thus it is natural to ask if finite energy finite-time blow-up solutions exist.

Our first result is the (forward) construction of smooth finite-time blow-up solutions. We note that the constructed solutions not only have finite energy but are also *smooth*. Moreover, by construction these solutions arise from a codimension one set of initial data.

Theorem 3.10 (Smooth finite energy finite-time blow-up solutions [20]). *There is a codimension one set of smooth finite energy radial initial data $u_0(r)$ such that the forward-in-time evolution $u(t,r)$ blows up in finite time (say $T < \infty$) and*

$$u(t,r) - \frac{e^{i\gamma^*}}{\lambda(t)} Q\Big(\frac{r}{\lambda(t)}\Big) \to z^* \text{ in } L^2, \quad \lambda(t) \approx \lambda^* \frac{T-t}{|\log(T-t)|^2}$$

as $t \to T$, for some $\gamma^* \in \mathbb{R}$, $\lambda^* \in \mathbb{R}_+$, and $z^* \in H^1$.

Remark 3.11. One can take $u_0 \in C_c^\infty$ and u_0 arbitrarily close to Q in the L^2-topology. Moreover, applying the pseudoconformal transform, one can construct infinite-time blow-up solutions with $\lambda(t) \sim (\log t)^2$ as $t \to +\infty$.

Remark 3.12 (Finite energy solution). A logarithmic derivation from the pseudoconformal rate stems from the fact that $S(t)$ has infinite energy. In the (NLS) context, the well-known log-log blow-up rate [30, 32], which deviates from the self-similar rate by a log-log factor, is due to the fact that the exact self-similar solutions to (NLS) barely fail to lie in L^2. A similar remark applies to the wave maps case [39].

Remark 3.13 (Comments on the proof). As for the proof of Theorem 3.6, we use the method of forward construction with repulsivity. On top of this existing road map, the key new input is a systematic use of nonlinear covariant conjugation identities. In short, this is a simple idea that we not only look at the equation for the original variable u, but do we also look at the system of evolution equations for the nonlinearly conjugated variables $u_1 = \mathbf{D}_u u$, and $u_2 = A_u \mathbf{D}_u u$, where $A_u = (\mathbf{D}_u - 1/y)$. Compared to the previously used linear adapted derivatives, these nonlinear conjugations better respect the full nonlinear evolution. Moreover, these variables enjoy *degeneracies*; indeed, proceeding to the variable $u_1 = \mathbf{D}_u u$ kills the $Q_{\lambda,\gamma}$-part in view of $\mathbf{D}_Q Q = 0$ and further proceeding to the variable $u_2 = A_u \mathbf{D}_u u$ even annihilates the generalized null modes. One of the most advantageous simplifications from these nonlinear conjugations and degeneracies is that the evolution equation for u_2 isolates the evolution of the remainder term (denoted by ε_2) very well, while keeping the structure simple. In particular, most of the non-perturbative terms in the original ε-equation disappear, and in the ε_2-equation there is only one non-perturbative term, which is moreover *local*. This enables us to identify a Morawetz-type correction term when we perform (modified) energy estimates. To put it differently, if one proceeds only with the variable ε, then the correction terms must be quite complicated, even involving some nonlocal (integral) expressions, that would be difficult to be found. We refer to [20, Section 1.4] for more detailed explanations.

On the other hand, by the method of backward construction, one can show that there is a continuum of possible blow-up rates for finite energy solutions.

Theorem 3.14 (Continuum of blow-up rates [21]). *For $q \in \mathbb{C} \setminus \{0\}$ and $\mathrm{Re}(\nu) > 0$, set*

$$z^*(r) = q r^\nu \chi(r).$$

Then, there exists a finite energy finite-time blow-up solution $u(t,r)$ such that

$$u(t,r) - \frac{e^{i\gamma_{q,\nu}(t)}}{\lambda_{q,\nu}(t)} Q\Big(\frac{r}{\lambda_{q,\nu}(t)}\Big) \to z^* \text{ in } L^2 \text{ as } t \to 0^-$$

with

$$\lambda_{q,\nu}(t) e^{i\gamma_{q,\nu}(t)} = c_\nu \cdot q \frac{|t|^{\frac{\nu}{2}+1}}{|\log|t||}, \qquad \text{in particular} \qquad \lambda_{q,\nu}(t) \sim_{q,\nu} \frac{|t|^{\frac{\mathrm{Re}(\nu)}{2}+1}}{|\log|t||}.$$

Remark 3.15. By Theorem 2.1, the range of ν is optimal. When $\mathrm{Im}(\nu) \neq 0$, the blow-up solution u exhibits infinite amount of phase rotation. Infinite-time blow-up solutions can be constructed by applying the pseudoconformal transform.

Remark 3.16 (Strongly interacting regime). The blow-up in Theorem 3.14 is driven by the strong soliton-radiation interaction. We exploit the fact that $Q \sim \langle y \rangle^{-2}$ has the slowest spatial decay in the $m = 0$ case, and the analysis does not work in the high equivariance case $m \geq 1$. Thus in the $m \geq 1$ case, the existence of blow-up rates other than the pseudoconformal one still remains as an interesting open problem.

Remark 3.17 (Regularity of the blow-up solution). Compared to Theorem 3.10, Theorem 3.14 provides more examples of blow-up rates for finite energy solutions. However, Theorem 3.14 does not tell about the regularity of blow-up solutions more than H^1. We believe that these solutions in general have limited regularity as in [23] and C^∞ finite-time blow-up solutions are only allowed to have quantized rates. We refer to [36, 40, 34].

Remark 3.18 (Comments on the proof). As in the proof of Theorem Theorem 3.1–3.2, we use the method of backward construction. The scheme of the proof is much inspired from the recent work of Jendrej–Lawrie–Rodriguez [17] for the 1-equivariant wave maps, where in particular profile modifications are not used. We extend their arguments to the NLS setting.

However, due to the lack of finite speed of propagation in the Schrödinger setting, the construction of the radiation $z(t, r)$ and the justification of necessary asymptotics from $z^*(r) = qr^\nu \chi(r)$ are nontrivial. More precisely, we need to justify the asymptotics $z(t, r) \approx qc_\nu |t|^{(\nu-2)/2} r^2$ in the self-similar region $r \lesssim |t|^{1/2}$. When $\mathrm{Re}(\nu)$ is large, as the leading term itself degenerates (a large power of t), we need to approximate $z(t, r)$ up to errors of su□ciently large powers of t. When $\mathrm{Re}(\nu) > 0$ is small, it is necessary to understand the *structure of the singular part* of $z(t, r)$. This is because $z^*(r)$ has limited regularity $H^{1+\nu-}$ but the justification of the asymptotics would require higher Sobolev regularity (e.g., H^{3+}). In this case, we explicitly identify the singular part $\widehat{z}$ of z and show that $z - \widehat{z}$ belongs to H^{3+}. We believe that this part of proof, namely the construction of the radiation, is of independent interest.

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