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# CRIME PAYS; HOMOGENIZATION FOR LONG TIMES

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Abstract. This article examines the accuracy for large times of the asymptotic expansions in periodic homogenisation of wave equations. Of particular interest is the precision for anormally long times of approximations constructed from asymptotic crimes committed on the standard two-scale expansions. We first prove that the standard two-scale asymptotic expansion provides an accurate approximation of the exact solution for all times $t$ less than $C\varepsilon^{-2+\delta}$ for any $C, \delta > 0$ where $\varepsilon \ll 1$ denotes the period of the coefficients. Second, for longer times, we show that the criminal two-scale asymptotic expansion, first proposed by Bakhavalov and Panasenko in the elliptic setting because they mix various powers of $\varepsilon$ in the same equations, yields approximations of the exact solution with error $\leq C\,\varepsilon^N$ for $t \leq \varepsilon^{-N}$ with $N$ as large as one likes.

**Key words.** homogenisation, secular growth, dispersive effects, asymptotic crimes, wave equations

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## 1. $\varepsilon$-periodic wave equation

Study the family $u^\varepsilon$ of solutions to

$$\rho(x/\varepsilon)\,\partial_t^2 u^\varepsilon \;-\; \big(\mathrm{div}\, a(x/\varepsilon)\,\mathrm{grad}\big)\, u^\varepsilon \;=\; f(t,x), \tag{1}$$

$$u^\varepsilon = 0 \quad \text{for} \quad t < 0,$$

$$\mathrm{supp}\, f \;\subset\; \big\{0 \leq t \leq T\big\} \quad \text{and} \quad \forall \alpha, \quad \partial_{t,x}^\alpha f \in L^2(\mathbb{R}^{1+d}).$$

The solutions $u^\varepsilon$ are real valued as is $\rho \in L^\infty(\mathbb{T}^d)$ ($\mathbb{T}^d :=$ the unit torus). The coefficient $a \in L^\infty(\mathbb{T}^d)$ has values that are real symmetric matrices. The coefficients are positive definite,

$$\rho(y) \;\geq\; m_1 > 0\,, \qquad \text{and,} \qquad a(y)\xi\cdot\xi \;\geq\; m_1|\xi|^2.$$

The **parent operator** on the left of (1) has $\varepsilon$-periodic coefficients. They vary rapidly *in $x$ only.* Order one oscillations in time on scale $\varepsilon$ would lead to explosive growth.

## 2. The Homogenization Problem

The standard energy estimate implies that

$$\forall j \geq 0, \quad \sup_t \ \left\| \partial_t^j \, \nabla_{t,x} \, u^\varepsilon(t) \right\|_{L^2(\mathbb{R}^d)} \ < \ \infty. \tag{2}$$

PROBLEM. *Describe the solutions for $t \sim \varepsilon^{-N}$ without having to solve equations that require discretization on scale $\sim \varepsilon$.*

The coefficients $\rho(x/\varepsilon)$ and $a(x/\varepsilon)$ tend weakly to their mean values. Even for times $t \sim 1$, the approximations do NOT simply replace $\rho$ and $a$ by their average values.

**Example 1.** Consider the simple problem

$$\rho(x/\varepsilon)u^\varepsilon = f, \qquad u^\varepsilon = \frac{1}{\rho(x/\varepsilon)} \, f \approx Av(1/\rho) \, f := \underline{u} \, .$$

$\underline{u}$ satisfies the homogenized equation

$$\rho^h \underline{u} = f, \qquad \rho^h := \frac{1}{Av(1/\rho)} \neq Av(\rho) \, .$$

## 3. The two scale *ansatz*

Classical Answer. There is a constant coefficient homogenized wave operator $a_2^h(\partial_t, \partial_x)$ so that on bounded time intervals $u^\varepsilon$ differs by $O(\varepsilon)$ from the solution of $a_2^h(\partial_t, \partial_x)w = f$.

Bensoussan-Lions-Papanicolaou 1978 take as approximate solution, $U(\varepsilon, t, x, x/\varepsilon)$, where

$$U(\varepsilon, t, x, y) \ \sim \ \sum_{n=0}^{\infty} \varepsilon^n \, u_n(t, x, y) \, , \qquad 1 - \text{periodic in } y \, . \tag{3}$$

The right hand side is a formal power series in $\varepsilon$. It usually diverges.

Compute

$$\Big[\rho(x/\varepsilon) \, \partial_t^2 \ - \ \operatorname{div} a(x/\varepsilon) \operatorname{grad}\Big] U(\varepsilon, t, x, x/\varepsilon) \, .$$

Use $\partial_x \to \partial_x + \varepsilon^{-1}\partial_y$. Introduce

$$\begin{aligned}
\mathcal{A}_{yy} \ &:= \ \operatorname{div}_y a(y) \operatorname{grad}_y, \\
\mathcal{A}_{xx} \ &:= \ \operatorname{div}_x a(y) \operatorname{grad}_x, \\
\mathcal{A}_{xy} \ &:= \ \operatorname{div}_x a(y) \operatorname{grad}_y \ + \ \operatorname{div}_y a(y) \operatorname{grad}_x \, .
\end{aligned}$$

Find,

$$\Big[\rho(x/\varepsilon) \, \partial_t^2 \ - \ \operatorname{div} a(x/\varepsilon) \operatorname{grad}\Big] U(\varepsilon, t, x, x/\varepsilon) \ = \ W(\varepsilon, t, x, x/\varepsilon),$$

where
$$W(\varepsilon,t,x,y) \ := \ \Big[\rho(y)\partial_t^2 - \frac{1}{\varepsilon^2}\mathcal{A}_{yy} - \frac{1}{\varepsilon}\mathcal{A}_{xy} - \mathcal{A}_{xx}\Big] U(\varepsilon,t,x,y).$$
The profile $W$ has the expansion,
$$W(t,x,y) \ \sim \ \sum_{n\geq -2} \varepsilon^n\, w_n(t,x,y),$$
with
$$w_k \ = \ \rho(y)\partial_t^2 u_k - (\mathcal{A}_{yy}u_{k+2} \ + \ \mathcal{A}_{xy}u_{k+1} \ + \ \mathcal{A}_{xx}u_k).$$
Study map $\{u_n\} \mapsto \{w_n\}$.

## 4. Oscillator parts

The operator $\mathcal{A}_{yy}$ is a bijection from the mean zero elements of $H^1(\mathbb{T}^d)$ to mean zero elements of $H^{-1}(\mathbb{T}^d)$. Its inverse is denoted $A_{yy}^{-1}$. The nullspace of $A_{yy}$ acting on $L^2(\mathbb{T}^d)$ consists of the constant functions. Denote by $\pi$ the orthogonal projection of $L^2(\mathbb{T}^d)$ onto the nullspace. Then
$$u_n(t,x,y) = \pi u_n + (I-\pi)u_n, \quad (\pi u_n)(t,x) := \int_{\mathbb{T}^d} u_n(t,x,y)\,\frac{dy}{|\mathbb{T}^d|}.$$
$\pi u_n :=$ the *non oscillating part.* $(I-\pi)u_n :=$ the *oscillating part.*

Characterize power series $U$ so that the corresponding $W$ has *no oscillatory part*, that is, $(I-\pi)w_n = 0$ for all $n \geq -2$. This is equivalent to the sequence of equations,
$$(I-\pi)\Big[\rho(y)\partial_t^2 u_k - \mathcal{A}_{yy}u_{k+2} \ - \ \mathcal{A}_{xy}u_{k+1} - \mathcal{A}_{xx}u_k\Big] = 0. \tag{4}$$
Solve for $(I-\pi)u_{k+2}$ in terms of earlier profiles. Find that $(I-\pi)w_n = 0$ for all $n \iff$
$$(I-\pi)u_{k+2} = -\,\mathcal{A}_{yy}^{-1}\,\big(I-\pi\big)\Big[\mathcal{A}_{xy}u_{k+1} + (\mathcal{A}_{xx} - \rho(y)\partial_t^2)u_k\Big].$$
The oscillatory part of $u_{k+2}$ is expressed in terms of earlier profiles.

Write each earlier profile as $u_j = \pi u_j + (1-\pi)u_j$.

Express their oscillatory $(1-\pi)u_j$ parts in terms of still earlier profiles. *The oscillatory parts are eliminated entirely.*

**Definition 2.** Set $\chi_{-1} := 0$, $\chi_0 := I$. For $k \geq 1$ define partial differential operators mapping functions of $t,x$ to functions of $t,x,y$ by
$$\chi_k(y,\partial_t,\partial_x) := -\mathcal{A}_{yy}^{-1}(I-\pi)\Big[\mathcal{A}_{xy}\chi_{k-1} + (\mathcal{A}_{xx} - \rho(y)\partial_t^2)\chi_{k-2}\Big],$$
$$\chi_k(y,\partial_t,\partial_x) \ = \ \sum_{|\beta|=k} c_{\beta,k}(y)\,\partial_{t,x}^{\beta}, \qquad c_{\beta,k} \in (I-\pi)H^1(\mathbb{T}^d).$$
In particular, $\chi_k$ is a homogeneous operator of degree $k$ in $\partial_{t,x}$.

**Theorem 3.** *The following are equivalent.*

**i.** *For all $j$*

$$(I-\pi)w_j = 0\,. \tag{5}$$

**ii.** *As formal power series in $\varepsilon$, one has*

$$\sum_{n=0}^{\infty} \varepsilon^n\, u_n \;=\; \Big(\sum_{\ell=0}^{\infty} \varepsilon^\ell\, \chi_\ell\Big)\,\Big(\sum_{k=0}^{\infty} \varepsilon^k\, \pi\, u_k\Big)\,.$$

## 5. The nonoscillatory hierarchy

With the oscillatory parts controlled turn to the nonoscillatory parts that must satisfy,

$$\pi w_k \;=\; \pi\Big[\rho(y)\partial_t^2 u_k \;-\; \mathcal{A}_{xy} u_{k+1} \;-\; \mathcal{A}_{xx} u_k\Big]. \tag{6}$$

**Definition 4.** For $n \geq 2$, the $n^{th}$ **order homogenized operator** is

$$a_n^h(\partial_t, \partial_x) \;:=\; \pi\Big(\big(\rho(y)\partial_t^2 - \mathcal{A}_{xx}\big)\chi_{n-2} - \mathcal{A}_{xy}\chi_{n-1}\Big). \tag{7}$$

**Theorem 5.**

$$\pi w_j \;=\; \sum_{n=0}^{j} a_{n+2}^h(\partial_t, \partial_x)\, \pi\, u_{j-n}\,. \tag{8}$$

**Theorem 6. (Leap Frog Theorem)** *For odd $n \geq 1$, $a_n^h = 0$.*

Bahkvalov and Panasenko 1989 found the elliptic versions of the hierarchies. The hyperbolic versions have striking consequences.

## 6. Classical homogenisation and secular growth

For the approximation to be infinitely accurate for all sufficiently small $\varepsilon$, the coefficients of all powers of $\varepsilon$ must vanish. Yields the classical algorithm, $\pi w_0 = f$ and $\pi w_j = 0.$ for $j \neq 0$.

$$\begin{aligned}
\varepsilon^0 :&\quad a_2^h(\partial)\pi u_0 = f\\
\varepsilon^1 :&\quad a_2^h(\partial)\pi u_1 = 0\\
\varepsilon^2 :&\quad a_2^h(\partial)\pi u_2 = -a_4^h(\partial)\pi u_0\\
\varepsilon^3 :&\quad a_2^h(\partial)\pi u_3 = -a_4^h(\partial)\pi u_1\\
\varepsilon^4 :&\quad a_2^h(\partial)\pi u_4 = -a_4^h(\partial)\pi u_2 - a_6^h(\partial)\pi u_0\\
\varepsilon^5 :&\quad a_2^h(\partial)\pi u_5 = -a_4^h(\partial)\pi u_3 - a_6^h(\partial)\pi u_1\\
\varepsilon^6 :&\quad a_2^h(\partial)\pi u_6 = -a_4^h(\partial)\pi u_4 - a_6^h(\partial)\pi u_2 - a_8^h(\partial)\pi u_0\\
\varepsilon^7 :&\quad a_2^h(\partial)\pi u_7 = -a_4^h(\partial)\pi u_5 - a_6^h(\partial)\pi u_3 - a_8^h(\partial)\pi u_1
\end{aligned}$$

Odd and even decouple. $\pi u_j = 0$ for $j$ odd. The first line is a recipe for $\pi u_0$.

## 7. Homogenization 101

• The operator $a_2^h(\partial_t, \partial_x)$ is not obvious.

• Even though $\pi u_1 = 0$, $u_1(t,x,y)$ has an oscillating part. $\varepsilon\, u_1(t,x,x/\varepsilon)$ is small in $H^s(\mathbb{R}^d)$ for all $s < 1$ but is $O(1)$ in $H^1$.

*Need $u_1$ to describe the energy.* The function $u_1$ is given by $u_1 = (1-\pi)u_1 = \chi_1 \pi u_0$ in terms of $\pi u_0$.

## 8. Secular growth

The $n^{th}$ term is $\varepsilon^n\, u_n$. For large times the $u_n$ can grow. Their rate of growth determines for how long the term $\varepsilon^n\, u_n$ remains small.

The growth is slower than one might suspect because $a^h_{\rm odd} = 0$.

Classic energy estimate shows that $\partial^\alpha u_0(t) = \partial^\alpha \pi u_0(t)$ is uniformly bounded in $L^2(\mathbb{R}^d)$ for all $\alpha \neq 0$. Therefore the source term for $\pi u_2$ is bounded as $t \to \infty$. Duhamel's formula shows that $\pi u_2$ grows at most linearly. Then $\pi u_4$ grows at most quadratically. Continuing yields the following result. Note especially the $2k$ on the left and the $k$ on the right.

**Theorem 7.** *For $|\alpha| \geq 1$,*

$$\left\| \partial^\alpha_{t,x}\, \pi u_{2k}(t) \right\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \;\leq\; C(\alpha, k)\, \langle t \rangle^k\,, \qquad \langle t \rangle := (1+t^2)^{1/2}.$$

## 9. The classical *ansatz* for long times

**Theorem 8.** *Define profiles, constructed from the first $k+1$ non oscillating profiles $\pi u_0, \pi u_2, \ldots, \pi u_{2k}$ by*

$$U^k(\varepsilon, t, x, y) := \sum_{n=0}^{2k} \varepsilon^n u_n + \varepsilon^{2k+1}(I-\pi)u_{2k+1} + \varepsilon^{2k+2}(I-\pi)u_{2k+2}.$$

*The approximate solution is $U^k(\varepsilon, t, x, x/\varepsilon)$. Then*

$$\left\| \nabla_{t,x} \left[ u^\varepsilon(t,x) \;-\; U^k(\varepsilon, t, x, x/\varepsilon) \right] \right\|_{L^2(\mathbb{R}^d_x)} \;\leq\; C\, \varepsilon^{2k+1} \langle t \rangle^{k+1}\,.$$

**Discussion i.** The $(2n)^{\rm th}$ term in the ansatz (3) is of size $\varepsilon^{2n}\, t^n$. $t \sim 1/\varepsilon^2$ is a critical time scale.

**Discussion ii.** *For any $N, \delta > 0$ choosing a sufficiently large number of terms in the traditional ansatz guarantees that the error is $O(\varepsilon^N)$ for times $t \leq 1/\varepsilon^{2-\delta}$.* The standard assertion is that it is this accurate on bounded time intervals.

**Discussion iii.** $\pi u_n$ grows no faster than $t^{n/2}$. Without the leap frog structure one would have found $t^n$. For systems of conservative wave equations one finds non zero odd order homogenized operators and this faster secular growth. The critical time scale is $t \sim \varepsilon^{-1}$.

**Discussion iv.** That $a_3^h$ can be non zero had not, to our knowledge, been observed before.

## 10. The first asymptotic crime for longer times

For critical times *and beyond*, abandon the classical *ansatz.*

**Main (criminal) idea.** *Change the choice of the nonoscillatory parts $\pi u_n$.* **For each** *$\varepsilon$, construct a different $\pi u_n$. The oscillatory parts $(1-\pi)u_n$ are given in terms of the nonoscillatory parts from the classical hierarchy.*

The profiles are NOT those of classical homogenisation. The new $\varepsilon$-dependent $u_n$ are denoted $v_n(\varepsilon, t, x, y)$ and

$$V(\varepsilon, t, x, y) \;\sim\; \sum_{n=0}^{\infty} \varepsilon^n \, v_n(\varepsilon, t, x, y)\,.$$

Letting $v_n$ depend on $\varepsilon$ is our *first asymptotic crime.* When the $v_n$ depend on $\varepsilon$ one loses unique determination of expansions.

## 11. The second asymptotic crime

The residual $\sum \varepsilon^n w_n$ is computed as before. We do not set the $w_n = 0$. This is the second asymptotic crime.

We set $(1-\pi)w_n = 0$ for all $n$. That choice yields the oscillatory hierarchy determining $(1-\pi)v_n$ from the $\pi v_n$. Next choose the $\pi v_n$.

The nonoscillatory hierachy implies that $W - f = \pi(W - f)$ is equal to *the sum of the lines*

$$\begin{aligned}
&\varepsilon^0\big[a_2^h(\partial)\pi v_0 \;-\; f\big]\\
&\varepsilon^1\big[a_2^h(\partial)\pi v_1\big]\\
&\varepsilon^2\big[a_2^h(\partial)\pi v_2 \;+\; a_4^h(\partial)\pi v_0\big]\\
&\varepsilon^3\big[a_2^h(\partial)\pi v_3 \;+\; a_4^h(\partial)\pi v_1\big]\\
&\varepsilon^4\big[a_2^h(\partial)\pi v_4 \;+\; a_4^h(\partial)\pi v_2 \;+\; a_6^h(\partial)\pi v_0\big]\\
&\varepsilon^5\big[a_2^h(\partial)\pi v_5 \;+\; a_4^h(\partial)\pi v_3 \;+\; a_6^h(\partial)\pi v_1\big]
\end{aligned}$$

To avoid secular growth, set $\pi v_n = 0$ for $n \geq 1$.

With that choice, *the sum of the lines vanishes if and only if*

$$\Big[a_2^h(\partial_{t,x}) + \varepsilon^2 a_4^h(\partial_{t,x}) + \varepsilon^4 a_6^h(\partial_{t,x}) + \cdots\Big]\pi\, v_0 \;=\; f\,. \tag{9}$$

**Bad news 1.** The terms $\varepsilon^{2j-2}\, a^h_{2j}(\partial_{t,x})$ are typically of order $2j$ in $\partial_t$. The more terms one keeps the higher order is the equation.

**Bad news 2.** When one keeps only a finite number of terms, the truncated operators usually define ill posed initial value problems.

**i.** Equation (9) is a sort of exact solution recipe. To the extent that one finds a $\pi v_0$ that solves with small error, that generates a good approximation to $u^\varepsilon$.

**ii.** The $\varepsilon^{2k-2} a^h_{2k}(\partial_{t,x})$ are dispersive correctors. The correctors improve the approximation to $u^\varepsilon$. For $a^h_4$, Santosa-Symes 1991 showed improved numerics and Lamacz 2011 proved $L^2$-error $\leq \varepsilon$ for $t \sim \varepsilon^{-2}$.

**iii.** A very interesting approach to long times is pursued by Gloria et.al 2017-2020 in random media. They construct approximate solutions from high order accurate eigenvalues of the elliptic part. The analogue of secular growth is that their corrector terms grow as $|x| \to \infty$. When that growth can be controlled they construct waves in higher dimensions that do not have Anderson localisation.

## 12. An elimination algorithm to make sense of (9)

Eliminate time derivatives other than those in $a^h_2(\partial_{t,x})$.

**Proposition 9.** *There are uniquely determined homogeneous partial differential operators $R_{2j}(\partial_{t,x})$ and $\widetilde{a}_{2j}(\partial_x)$ of degree $2j$, the latter involving only $\partial_x$, so that in the sense of formal power series,*

$$\Big[1 + \sum_{j=1}^{\infty} R_{2j}\big(\partial_{t,x}\big)\Big]\Big[\sum_{j=1}^{\infty} a^h_{2j}\big(\partial_{t,x}\big)\Big] \;=\; a^h_2\big(\partial_{t,x}\big) \;+\; \sum_{j=2}^{\infty} \widetilde{a}_{2j}\big(\partial_x\big).$$

Multiplying (9) by $1 + \sum_{j=1}^{\infty} R_{2j}\big(\varepsilon\partial_{t,x}\big)$ yields

$$\Big[a^h_2(\partial_{t,x}) \;+\; \sum_{j=2}^{\infty} \varepsilon^{2j-2}\; \widetilde{a}_{2j}(\partial_x)\Big]\pi\, v_0 = \Big[1 + \sum_{j=1}^{\infty} \varepsilon^{2j} R_{2j}\big(\partial_{t,x}\big)\Big] f.$$

**This** equation for $\pi v_0$ is second order in time.

## 13. Truncation to a finite number of terms

Drop all but a finite number of terms. The unknown function is denoted $\pi\, \underline{v}^k_0$. The truncated equation is, with $R^k := \sum_{j=1}^{k} R_{2j}$,

$$\Big[a^h_2(\partial_{t,x}) + \sum_{j=2}^{k+1} \varepsilon^{2j-2}\; \widetilde{a}_{2j}(\partial_x)\Big]\pi\, \underline{v}^k_0 = \Big[1 + R^k\big(\varepsilon\partial_{t,x}\big)\Big] f. \tag{10}$$

**Bad news 3.** The initial value problem (10) is usually ill posed.

**Good news.** $f$ is smooth. The Fourier transform of the right hand side decays rapidly.

## 14. Filtering

Choose cutoffs $\psi_j \in C_0^\infty(\mathbb{R}^d)$, $j = 1, 2$, with $\psi_1 = 1$ on a neighborhood of the origin and $\psi_2 = 1$ on a neighborhood of $\operatorname{supp} \psi_1$. Define $D := (1/i)\partial_x$. Choose $0 < \alpha < 1$. Filter the right hand side to obtain,

$$\big[a_2^h(\partial_t, \partial_x) + \varepsilon^2 \widetilde{a}_4(\partial_x) + \cdots + \varepsilon^{2k} \widetilde{a}_{2k+2}(\partial_x)\big] \pi \, \underline{v}_0^k \;=\; \psi_1(\varepsilon^\alpha D)\big(1 + R^k(\varepsilon \partial_{t,x})\big) f.$$

Fourier transformation shows that there are tempered solutions with transform supported in $\varepsilon^{-\alpha}\{\operatorname{supp} \psi_1\}$. Such a solution satisfies $\psi_2(\varepsilon^\alpha D)\underline{v}_0^k = \underline{v}_0^k$. This yields

$$\Big[a_2^h(\partial_t, \partial_x) + \big(\varepsilon^2 \widetilde{a}_4(\partial_x) + \cdots + \varepsilon^{2k} \widetilde{a}_{2k+2}(\partial_x)\big)\psi_2(\varepsilon^\alpha D)\Big] \pi \, \underline{v}_0^k \;=\; \psi_1(\varepsilon^\alpha D)\big(1 + R^k(\varepsilon \partial_{t,x})\big) f.$$

## 15. Criminal Approximation

The last section derived the recipe,

$$\Big(a_2^h(\partial_t, \partial_x) + M(\varepsilon, k, \partial_x)\Big) \pi v_0^k \;=\; \psi_1(\varepsilon^\alpha D)\big(1 + R^k(\varepsilon \partial_{t,x})\big) f, \tag{11}$$

$$M := \sum_{j=2}^{k+1} \varepsilon^{2j-2} \, \widetilde{a}_{2j}(\partial_x) \, \psi_2(\varepsilon^\alpha D).$$

The filtered operator $M$ is small and real. Equation (11) generates uniformly stable initial value problems. Solving determines $\pi \, \underline{v}_0^k$ and therefore $V^k(\varepsilon, t, x, y)$. This can be done efficiently by FFT. Even for $t \gg 1$. The approximate solution is $V^k(\varepsilon, t, x, x/\varepsilon)$. *This is the Criminal Algorithm.*

Having sidestepped secular growth, this algorithm constructs approximate solutions with residuals smaller than any desired power of $\varepsilon$ *uniformly in time*. The standard duhamel arguement yields error estimates $\varepsilon^n \langle t \rangle$ with $n$ as large as one likes. That yields the following theorem.

**Theorem 10.** *The error in energy satisfies*

$$\big\| \nabla_{t,x} \big(u^\varepsilon(t, x) \;-\; V^k(\varepsilon, t, x, x/\varepsilon)\big) \big\|_{L^2(\mathbb{R}^d_x)} \;\leq\; C \, \varepsilon^{2k+1} \, \langle t \rangle \, .$$

**Example 11.** Choose $k$ with $2k + 1 \geq 2N$. Then the error is $\leq C_k \, \varepsilon^N$ for $t \leq \varepsilon^{-N}$.

**Algorithm summary.** Two asymptotic crimes. Eliminate time derivatives. Filter. Solve for $\pi v_0$. Add the oscillatory parts.

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