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NON LINEAR STABILITY OF SPHERICAL GRAVITATIONAL SYSTEMS DESCRIBED BY THE VLASOV-POISSON EQUATION

MOHAMMED LEMOU

Abstract. In this work, we prove the nonlinear stability of galaxy models derived from the three dimensional gravitational Vlasov Poisson system, which is a canonical model in astrophysics to describe the dynamics of galactic clusters. The stability of the so-called spherical models (which are radially symmetric steady states solutions) is a major question in astrophysics and, for decades, this problem has been the subject of a considerable amount of works in both mathematical and physics communities. A well known conjecture [6] is the stability of spherical models which are nonincreasing functions of their microscopic energy. This conjecture was proved at the linear level by several authors in the continuation of the breakthrough work by Antonov [2] in 1961. In a previous work [30], we proved the stability of anisotropic spherical models under *radially symmetric perturbations* using fundamental monotonicity properties of the Hamiltonian under suitable generalized symmetric rearrangements first observed in the physics literature [36, 12, 50, 1]. In a more recent work [31], we show how this approach combined with a *new generalized* Antonov type coercivity property implies the orbital stability of spherical isotropic models under general perturbations. In this paper, we summarize the results obtained in this work and give the main lines of the proofs.

Joint work with Florian Méhats and Pierre Raphaël

1. Introduction and statement of the results

1.1. The gravitational Vlasov Poisson system. Kinetic theory is commonly used to statistically describe galactic systems (such as stars clusters) since the N-body description through Newton equations is not understood for systems with a large number of components. In these gravitational systems the collisions between the stars may be neglected and only long-range interactions due to the gravitational field have to be taken into account. When relativistic effects do not enter in play, such systems are commonly described by the three dimensional gravitational Vlasov-Poisson system:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla\phi_f\cdot\nabla_v f = 0, & (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3 \\ f(t=0,x,v) = f_0(x,v)\geq 0, \end{cases} \tag{1.1}$$

where

$$\rho_f(x) = \int_{\mathbb{R}^3} f(x,v)\,dv \quad\text{and}\quad \phi_f(x) = -\frac{1}{4\pi|x|}*\rho_f \tag{1.2}$$

are the density and the gravitational Poisson field associated to f. This nonlinear transport equation is a well known model in astrophysics for the description of the mechanical state of a stellar system subject to its own gravity, see for instance [6, 11].

The global Cauchy problem has been solved in [35, 42, 44] where unique global classical solutions $f(t)$ in $\mathcal{C}^1_c$, the space of $\mathcal{C}^1$ compactly supported functions, are derived. The nonlinear transport flow (1.1) satisfies the two following fundamental properties:

(i) The preservation of the total Hamiltonian

$$\mathcal{H}(f(t)) = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f(t,x,v)dxdv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi_f(t,x)|^2 dx = \mathcal{H}(f(0)), \tag{1.3}$$

(ii) The preservation of all the so-called Casimir functions: $\forall G \in \mathcal{C}^1([0,+\infty),\mathbb{R}^+)$ such that $G(0)=0$,

$$\int_{\mathbb{R}^6} G(f(t,x,v))\, dxdv = \int_{\mathbb{R}^6} G(f_0(x,v))\, dxdv\,. \tag{1.4}$$

This property is equivalent to the following conservation law:

$$\forall t \geq 0, \quad \mu_{f(t)} = \mu_{f_0}, \tag{1.5}$$

where μ_f is the distribution function associated to f:

$$\forall s \geq 0, \quad \mu_f(s) = \operatorname{meas}\left\{(x,v)\in\mathbb{R}^6 : \, f(x,v) > s\right\}, \tag{1.6}$$

In particular the system has an infinite and uncountable set of invariant quantities.

In this work, we will deal with weak solutions in the natural energy space

$$\mathcal{E} = \left\{f \geq 0 \text{ with } f \in L^1\cap L^\infty(\mathbb{R}^6) \text{ and } |v|^2 f \in L^1(\mathbb{R}^6)\right\}. \tag{1.7}$$

For all $f_0 \in \mathcal{E}$, (1.1) admits a weak solution $f(t)$, constructed for instance in [4, 22, 23], which is also a renormalized solution, see [7, 8]. Moreover, this solution still satisfies (1.4), belongs to $\mathcal{C}([0,+\infty), L^1(\mathbb{R}^6))$ and the energy conservation (1.3) is replaced by an inequality:

$$\forall t \geq 0, \quad \mathcal{H}(f(t)) \leq \mathcal{H}(f_0). \tag{1.8}$$

This text is a summary of [31]. Here we just give the main steps of the proofs and underline the main arguments. Complete and detailed proofs of our claims can be found in [31].

1.2. **Known results.** Steady states solutions to (1.1) are functions $Q(x,v)$ satisfying

$$v\cdot\nabla_x Q - \nabla\phi_Q\cdot\nabla_v Q = 0. \tag{1.9}$$

The complete determination of all the solutions to this equation is not known, but a particular family of solutions can easily be exhibited. Indeed any function of the form

$$Q(x,v) = F(e,\ell), \tag{1.10}$$

where e,ℓ are respectively the microscopic energy and the kinetic momentum

$$e(x,v) = \frac{|v|^2}{2} + \phi_Q(x), \quad \ell = |x\wedge v|^2, \tag{1.11}$$

is a solution to (1.9). Note that if we restrict ourselves to radially symmetric solutions, the Jean's theorem [5] gives a complete classification of the solutions to (1.9). Recall that the radial symmetry in our context means that the distribution functions have the form $f(x,v) \equiv f(|x|,|v|,x\cdot v)$. In this case the *only* solutions to (1.9) are of the form (1.10). In fact the microscopic energy and the kinetic momentum (given by (1.11)) are the *only* invariants of the radially symmetric characteristic flow associated with the transport operator $\tau = v \cdot \nabla_x - \nabla \phi_Q \cdot \nabla_v$.

A canonical problem which has attracted a considerable amount of works both in the physical and the mathematical communities is the question of the *nonlinear stability of steady states models.* We may summarize these stability issues in the two following conjectures:

(i) **Conjecture 1,** *anistropic steady states*: Any steady state of the form $Q(x,v) = F(e(x,v), \ell(x,v))$ which is nonincreasing with respect to the microscopic energy ($\partial_e F < 0$ on the support of Q) is stable under spherically symmetric perturbations.

(ii) **Conjecture 2,** *isotropic steady states*: Any steady state of the form $Q(x,v) = F(e(x,v))$ which is nonincreasing with respect to the microscopic energy ($F'(e) < 0$ on the support of Q) is stable under general perturbations.

At the *linear* level, these two conjectures have been proved by many authors [10], [13, 24, 46], following the pioneering work by Antonov in the 60's [2, 3]. This analysis is based on some coercivity properties of the linearized Hamiltonian under constraints formally arising from the linearization of the Casimir conservation laws (1.4), see Lynden-Bell [36], known as Antonov's coercivity property.

At the nonlinear level, the full orbital stability in the natural energy space $\mathcal{E}$ has been obtained for specific subclasses of steady states using variational techniques [51, 14, 16, 17, 18, 9, 45, 26, 27, 28, 43]. The key tool in such analyses is the Lions' concentration compactness principle [33, 34]. This powerful strategy however only applies to specific models which are *global* minimizers of the Hamiltonian (1.3) under at most two Casimir type conservation laws, see [27, 28] for a more complete introduction. Unfortunately, important physical models cannot be covered by such variational methods.

A non variational approach has been explored as a first attempt to treat the general case and to use the full rigidity provided by the *continuum* of conservation laws (1.4) [19], [15]. The outcome of this approach is a first result of stability against radially symmetric perturbations for the King model $F(e) = (\exp(e_0 - e) - 1)_+$. The method is based on Antonov's coercivity property and a direct linearization of the Hamiltonian near the King profile.

To completely solve conjecture 1, we proposed in [30] a different approach based on fine monotonicity properties of the Hamiltonian under suitable generalized symmetric rearrangements as first observed in pioneering breakthrough works in the physics literature, see in particular Lynden-Bell [36], Gardner [12], Wiechen, Ziegler, Schindler [50], Aly [1]. This approach avoids the delicate step of linearization of the Hamiltonian and reduces the stability problem for the full distribution function f to *a minimization problem for a generalized energy involving the Poisson field ϕ_f only.* The main outcome is the *radial* stability of nonincreasing anisotropic models, solving in this way the first stability conjecture (see [30] for the proof):

Theorem 1.1 (Radial stability of nonincreasing anisotropic models, [30])**.** *Let $Q(x,v) = F(e,\ell)$ be a continuous, nonnegative compactly supported steady state solution to* (1.1)*. Assume that Q is nonincreasing in the following sense: there exists $e_0 < 0$ such that F is $\mathcal{C}^1$ on $\mathcal{O} = \{(e,\ell) \in \mathbb{R} \times \mathbb{R}_+ \ : \ F(e,\ell) > 0\} \subset (-\infty, e_0) \times \mathbb{R}_+$*

and

$$\frac{\partial F}{\partial e} < 0 \quad on \quad \mathcal{O}.$$

Then Q is stable in the energy norm by radially symmetric perturbations, ie: for all $M > 0$, for all $\varepsilon > 0$, there exists $\eta > 0$ such that given $f_0 \in \mathcal{C}^1_c$ radially symmetric with

$$\|f_0 - Q\|_{L^1} \leq \eta, \quad \|f_0\|_{L^\infty} \leq \|Q\|_{L^\infty} + M, \quad |\mathcal{H}(f_0) - \mathcal{H}(Q)| \leq \eta, \tag{1.12}$$

the corresponding global strong solution $f(t)$ to (1.1) *satisfies:*

$$\forall t \geq 0, \quad \|(1 + |v|^2)(f(t) - Q)\|_{L^1} \leq \varepsilon. \tag{1.13}$$

Finally, let us also mention the recent and remarkable work by Mouhot and Villani [39, 40] on the Landau damping, where asymptotic stability results have been proved for spatially homogeneous steady states of the Vlasov-Poisson system.

1.3. **Statement of the result.** Our aim in this paper is to solve the second stability conjecture (conjecture 2 above) by extending the stability result of Theorem 1.1 to the full set of non radial perturbations of isotropic models. We recall that the radial problem enjoys an additional rigidity because for $f(x, v)$ radially symmetric, the Casimir conservation laws (1.4) can be extended as follows: $\forall G(h, \ell) \geq 0$, $\mathcal{C}^1$ with $G(0, \ell) = 0$,

$$\int_{\mathbb{R}^6} G(f(t, x, v), |x \wedge v|^2) dx dv = \int_{\mathbb{R}^6} G(f_0(x, v), |x \wedge v|^2) dx dv. \tag{1.14}$$

This additional conservation law is fundamental in the proof of Theorem 1.1, and at the linear level, it is intimately connected to Antonov's coercivity property which is essentially equivalent to the coercivity of the Hessian of the Hamiltonian (1.3) under the *full set of linearized constraints generated by* (1.14).

For the full non radial problem, (1.14) is lost. However, we claim that the strategy developed in [30] coupled with a *new generalized Antonov coercivity property* allows us to derive the classical conjecture (conjecture 2) of orbital stability of non-increasing isotropic models.

Theorem 1.2 (Orbital stability of nonincreasing isotropic models)**.** *Let Q be a continuous, nonnegative, non zero, compactly supported steady solution to* (1.1)*. Assume that Q is a nonincreasing spherical model in the following sense: there exists a continuous function $F : \mathbb{R} \to \mathbb{R}_+$ such that*

$$\forall (x, v) \in \mathbb{R}^6, \quad Q(x, v) = F\left(\frac{|v|^2}{2} + \phi_Q(x)\right), \tag{1.15}$$

and there exists $e_0 < 0$ such that $F(e) = 0$ for $e \geq e_0$, F is $\mathcal{C}^1$ on $(-\infty, e_0)$ and

$$F' < 0 \quad on \quad (-\infty, e_0). \tag{1.16}$$

Then Q is orbitally stable in the energy norm by the flow (1.1)*: for all $M > 0$, for all $\varepsilon > 0$, there exists $\eta > 0$ such that, given $f_0 \in \mathcal{E}$ with*

$$\|f_0 - Q\|_{L^1} \leq \eta, \quad \mathcal{H}(f_0) \leq \mathcal{H}(Q) + \eta, \quad \|f_0\|_{L^\infty} \leq \|Q\|_{L^\infty} + M, \tag{1.17}$$

for any weak solution $f(t)$ to (1.1)*, there exists a translation shift $z(t)$ such that $\forall t \geq 0$,*

$$\|(1 + |v|^2)(f(t, x, v) - Q(x - z(t), v))\|_{L^1(\mathbb{R}^6)} \leq \varepsilon. \tag{1.18}$$

We can see that the assumptions we make on Q are very general. Note that we allow F' to blow up on the boundary $e \to e_0$ which is known to happen for many standard models. We in particular extract from [6] two models of physical relevance which fit into our analysis:

– The generalized polytropic models:

$$F(e) = \sum_{0 \le i \le N} \alpha_i (e_0 - e)_+^{q_i}, \quad 0 < q_i < \frac{7}{2}, \quad \alpha_i \ge 0.$$

– The King model:

$$F(e) = (\exp(e_0 - e) - 1)_+ \quad \text{for some} \quad e_0 < 0.$$

1.4. **Strategy of the proof.** We first give a brief insight into the strategy of the proof of Theorem 1.2 which extends the approach introduced in [30]. More details are given in the next sections but the complete proofs are not addressed here and can be found in [31].

Step 1. Monotonicity of the Hamiltonian under generalized symmetric rearrangements.

We recall that the standard Schwarz symmetrization of f is defined by

$$f^*(s) = \inf\{\tau \ge 0 \quad : \quad \mu_f(\tau) \le s\}, \tag{1.19}$$

where μ_f is defined by (1.6). The rearrangement f^* is also the unique nonincreasing function on $\mathbb{R}_+$ satisfying the equimeasurability property:

$$\mu_f = \mu_{f^*}.$$

Given a potential ϕ in a suitable "Poisson field" class, we define the generalized symmetric nonincreasing rearrangement of f, denoted by $f^{*\phi}$, with respect to the microscopic energy $e = \frac{|v|^2}{2} + \phi(x)$ as the unique function of e which is equimeasurable to f, explicitely

$$f^{*\phi}(x,v) = f^* \circ a_\phi(e(x,v)), \quad a_\phi(e) = \text{meas}\{(x,v) \in \mathbb{R}^6, \quad \frac{|v|^2}{2} + \phi(x) < e\}. \tag{1.20}$$

Any nonincreasing spherical steady state solution to (1.1) is a fixed point of this transformation when generated by its own Poisson field:

$$Q^{*\phi_Q} = Q. \tag{1.21}$$

Moreover, the Hamiltonian (1.3) enjoys a nonlinear monotonicity property which was first observed in the physics literature, see in particular Aly [1]:

$$\mathcal{H}(f) \ge \mathcal{H}(f^{*\phi_f}). \tag{1.22}$$

For perturbations which are equimeasurable to Q ie

$$f^* = Q^*, \tag{1.23}$$

we can more precisely lower bound the Hamiltonian by a functional which depends on the Poisson field only:

$$\mathcal{H}(f) - \mathcal{H}(Q) \ge \mathcal{J}(\phi_f) - \mathcal{J}(\phi_Q) \tag{1.24}$$

where $\mathcal{J}$ can be interpreted as a generalized energy, [36]:

$$\mathcal{J}(\phi_f) = \mathcal{H}(Q^{*\phi_f}) + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi_{Q^{*\phi_f}} - \nabla\phi_f|^2.$$

Step 2. Coercivity of the Hessian: a Poincaré inequality.

The first step shows that the Hamiltonian controls a functional $\mathcal{J}$ of the potential field only. Therefore we are led to study this reduced functional and analyse how it controls the potential field itself. We then linearize this functional $\mathcal{J}$ at ϕ_Q. The linear term drops thanks to the Euler-Lagrange equation (1.21) and the Hessian takes the following remarkable form

$$D^2\mathcal{J}(\phi_Q)(h,h) = \int_{\mathbb{R}^3} |\nabla h|^2 dx - \int_{\mathbb{R}^6} |F'(e)|(h - \Pi h)^2 dxdv \tag{1.25}$$

where Π, defined by (3.6), denotes after a suitable *phase space* change of variables the projection of h onto the functions which depend only on the microscopic energy e. A similar structure occured in [30] where the corresponding quadratic form was

$$\int_{\mathbb{R}^3} |\nabla h|^2 - \int_{\mathbb{R}^6} \left|\frac{\partial F}{\partial e}(e,\ell)\right| (h - \Pi_{e,\ell} h)^2 dxdv \tag{1.26}$$

and where $\Pi_{e,\ell}$ corresponds to the projection onto functions which depend on (e,ℓ) only (e and ℓ being defined by (1.11)). The strict coercivity of the quadratic form (1.26) was then equivalent to Antonov's stability result, but this statement is no longer sufficient in our setting as (1.26) is lower bounded by (1.25).

We now claim the positivity of (1.25) for spherical models

$$D^2\mathcal{J}(\phi_Q)(h,h) = \int_{\mathbb{R}^3} |\nabla h|^2 - \int_{\mathbb{R}^6} |F'(e)|(h - \Pi h)^2 dxdv \geq 0, \tag{1.27}$$

and in fact the quadratic form is coercive up to the degeneracy induced by translation invariance[1]. For this, we reinterpret (1.27) as a generalized Poincaré inequality, and we claim that the classical approach developed by Hörmander [20, 21] for the proof of weighted L^2 Poincaré inequalities:

$$d\mu = e^{-V(x)}dx, \quad \int_{\mathbb{R}^N} (f - \overline{f})^2 d\mu \lesssim \int_{\mathbb{R}^N} |\nabla f|^2 d\mu, \quad \overline{f} = \frac{\int_{\mathbb{R}^N} f d\mu}{\int_{\mathbb{R}^N} d\mu}$$

under the convexity assumption $\nabla^2 V \gtrsim 1$, can be adapted to our setting.

Step 3. Compactness up to translations.

From standard continuity arguments, the conservation law (1.5) and the inequality (1.8) ensure that Theorem 1.2 is now equivalent to the relative compactness in the energy space up to translation of generalized minimizing sequences:

$$f_n^* \to Q^* \text{ in } L^1 \text{ and } \limsup_{n\to+\infty} \mathcal{H}(f_n) \leq \mathcal{H}(Q).$$

The two last steps yield the control of the potential in terms of the functional $\mathcal{J}$. A slight improvement of the inequality (1.24) now gives

$$\mathcal{H}(f_n) - \mathcal{H}(Q) + \|\phi_{f_n}\|_\infty \|f_n^* - Q^*\|_{L^1} \geq \mathcal{J}(\phi_{f_n}) - \mathcal{J}(\phi_Q). \tag{1.28}$$

This implies the relative compactness up to translations

$$\nabla\phi_{f_n}(\cdot + x_n) \to \nabla\phi_Q \text{ in } L^2(\mathbb{R}^3).$$

[1] see Proposition 3.2 for precise statements

The strong convergence in the energy norm of the full distribution function now follows from a further use of the extra terms in the monotonicity property (1.22) which yields:

$$\int (1+|v|^2)|f_n(x+x_n,v)-Q(x,v)|dxdv \to 0 \quad \text{as} \quad n\to+\infty$$

and enables to conclude the proof of Theorem 1.2.

Now we give a more detailed summary of the main steps of the proof and underline the main arguments. Complete and detailed proofs of our claims are not given here, and we refer the reader to [31] for a complete presentation.

2. Generalized symmetrization and reduction to a functional of the gravitational potential

In this section, we construct the new rearrangement $f^{*\phi}$ with respect to a given Poisson type field, and show the monotonicity of the Hamiltonian under the corresponding transformation. This allows to find out a new functional of the potential field only (to be analyzed later on) which is controlled by the Hamiltonian. Our approach extends the one we developed in [30] to the case of non radial potentials, and most arguments are in fact simplified by the absence of kinetic momentum.

2.1. **Symmetrization with respect to the microscopic energy.** We start by defining a suitable convex set of "Poisson type" potentials:

$$\mathcal{X} = \left\{\phi \in \mathcal{C}(\mathbb{R}^3) \text{ such that } \phi \le 0,\ \lim_{|x|\to+\infty} \phi(x)=0,\ \nabla\phi\in L^2(\mathbb{R}^3) \text{ and } m(\phi)>0\right\}$$

where

$$m(\phi) := \inf_{x\in\mathbb{R}^3}(1+|x|)|\phi(x)|. \tag{2.1}$$

Note that if $f\in\mathcal{E}$ is nonzero and ϕ_f is its Poisson field given by (1.2), then $\phi_f\in\mathcal{X}$. We associate to any $\phi\in\mathcal{X}$ the following Jacobian function $a_\phi : \mathbb{R}_-^* \to \mathbb{R}^+$ as:

$$\forall e<0,\quad a_\phi(e) = \text{meas}\left\{(x,v)\in\mathbb{R}^6 : \frac{|v|^2}{2}+\phi(x) < e\right\}.$$

This may also be explicited by the formula

$$\forall e<0,\quad a_\phi(e) = \frac{8\pi\sqrt{2}}{3}\int_{\mathbb{R}^3}(e-\phi(x))_+^{3/2}\,dx. \tag{2.2}$$

In particular, $a_\phi(e)=0$ for all $e<\min\phi$; Moreover a_ϕ is $\mathcal{C}^1$ on $(-\infty,0)$ and is a strictly increasing $\mathcal{C}^1$ diffeomorphism from $[\min\phi,0)$ onto $\mathbb{R}_+$.

We then introduce the generalized rearrangement of f with respect to a Poisson field $\phi\in\mathcal{X}$. For given $f\in L^1_+\cap L^\infty$ and $\phi\in\mathcal{X}$, we define the rearrangement of f with respect to the microscopic energy $\frac{|v|^2}{2}+\phi(x)$ as follows:

$$f^{*\phi}(x,v) = \begin{cases} f^*\left(a_\phi\left(\frac{|v|^2}{2}+\phi(x)\right)\right) & \text{if} \quad \frac{|v|^2}{2}+\phi(x)<0 \\ 0 & \text{if} \quad \frac{|v|^2}{2}+\phi(x)\ge 0 \end{cases} \tag{2.3}$$

on $\mathbb{R}^6$, where a_ϕ is defined by (2.2). Then:

(i) $f^{*\phi}$ is equimeasurable with f, i.e.

$$f^{*\phi} \in \mathrm{Eq}(f) = \{g \in L^1_+ \cap L^\infty \ \text{ with } \ \mu_f = \mu_g\}. \tag{2.4}$$

(ii) $f^{*\phi}$ belongs to the energy space, i.e. $f^{*\phi} \in \mathcal{E}$ with

$$\int_{\mathbb{R}^6} \frac{|v|^2}{2} f^{*\phi} dxdv \leq C\|\nabla\phi\|_{L^2}^{4/3} \|f\|_{L^1}^{7/9} \|f\|_{L^\infty}^{2/9}. \tag{2.5}$$

We now reinterpret the assumptions on Q in Theorem 1.2 and claim that spherical models are fixed points of the $f \to f^{*\phi_f}$ transformation[2].

$$F(e) = Q^* \circ a_{\phi_Q}(e), \quad \forall e \in [\phi_Q(0), 0), \qquad \text{and} \qquad Q^{*\phi_Q} = Q \quad \textit{on } \mathbb{R}^6. \tag{2.6}$$

2.2. **Monotonicity of the Hamiltonian under symmetric rearrangement.** We are now in position to derive the monotonicity of the Hamiltonian under the generalized rearrangement which is the first key to our analysis and was already observed in the physics literature, see [1] and references therein. Given $f \in \mathcal{E} \setminus \{0\}$, we have $\phi_f \in \mathcal{X}$ and we will note to ease notation:

$$\widehat{f} = f^{*\phi_f}. \tag{2.7}$$

Given $\phi \in \mathcal{X}$, we define the functional

$$\mathcal{J}_{f^*}(\phi) = \mathcal{H}(f^{*\phi}) + \frac{1}{2}\|\nabla\phi - \nabla\phi_{f^{*\phi}}\|_{L^2}^2 \tag{2.8}$$

which is well defined from (2.5). We claim:

Proposition 2.1 (Monotonicity of the Hamiltonian under the $f^{*\phi_f}$ rearrangement)**.** *Let $f \in \mathcal{E} \setminus \{0\}$ and $\widehat{f}$ given by (2.7), then:*

$$\mathcal{H}(f) \geq \mathcal{J}_{f^*}(\phi_f) \geq \mathcal{H}(\widehat{f}). \tag{2.9}$$

Moreover, $\mathcal{H}(f) = \mathcal{H}(\widehat{f})$ if and only if $f = \widehat{f}$.

Proof. First compute for all $f, g \in \mathcal{E}$:

$$\begin{aligned}
\mathcal{H}(f) &= \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi_f|^2 \\
&= \int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi_f\right)(f-g) + \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 g + \int_{\mathbb{R}^3} \phi_f g + \frac{1}{2}\int |\nabla\phi_f|^2 \\
&= \mathcal{H}(g) + \frac{1}{2}\|\nabla\phi_f - \nabla\phi_g\|_{L^2}^2 + \int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi_f(x)\right)(f-g).
\end{aligned} \tag{2.10}$$

Replacing g by $\widehat{f} = f^{*\phi_f}$ yields from (2.8):

$$\mathcal{H}(f) = \mathcal{J}_{f^*}(\phi_f) + \int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi_f(x)\right)(f - f^{*\phi_f})\, dxdv, \tag{2.11}$$

and hence (2.9) follows from

$$\int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi_f(x)\right)(f - \widehat{f})\, dxdv \geq 0, \tag{2.12}$$

[2]Note that this is essentially a characterization of spherical models

with equality if and only if $f = \widehat{f}$. The proof of (2.12) is reminiscent from a standard inequality for symmetric rearrangement, see [32]. Indeed, use the layer cake representation

$$f(x,v) = \int_{t=0}^{\|f\|_{L^\infty}} \mathbb{1}_{t<f(x,v)} dt$$

and Fubini to derive:

$$\begin{aligned}
&\int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi_f \right) (f - \widehat{f})\, dxdv \\
&\qquad = \int_{t=0}^{\|f\|_{L^\infty}} dt \int_{\mathbb{R}^6} \left(\mathbb{1}_{t<f(x,v)} - \mathbb{1}_{t<\widehat{f}(x,v)} \right) \left(\frac{|v|^2}{2} + \phi_f \right) dxdv \\
&\qquad = \int_{t=0}^{\|f\|_{L^\infty}} dt \int_{\mathbb{R}^6} \left(\mathbb{1}_{\widehat{f}(x,v)\le t<f(x,v)} - \mathbb{1}_{f(x,v)\le t<\widehat{f}(x,v)} \right) \left(\frac{|v|^2}{2} + \phi_f \right) dxdv \\
&\qquad = \int_{t=0}^{\|f\|_{L^\infty}} dt \left(\int_{S_1(t)} \left(\frac{|v|^2}{2} + \phi_f \right) dxdv - \int_{S_2(t)} \left(\frac{|v|^2}{2} + \phi_f \right) dxdv \right) \qquad (2.13)
\end{aligned}$$

with

$$S_1(t) = \{\widehat{f}(x,v) \le t < f(x,v)\}, \qquad S_2(t) = \{f(x,v) \le t < \widehat{f}(x,v)\}.$$

Observe from $\widehat{f} \in \mathrm{Eq}(f)$ that:

$$\text{for a.e. } t > 0, \qquad \mathrm{meas}(S_1(t)) = \mathrm{meas}(S_2(t)). \qquad (2.14)$$

Thus: $\forall t \in (0, \|f\|_{L^\infty})$,

$$\int_{S_2(t)} \left(\frac{|v|^2}{2} + \phi_f(x) \right) dxdv \le \mathrm{meas}(S_2(t))(f^* \circ a_{\phi_f})^{-1}(t) = \int_{S_1(t)} (f^* \circ a_{\phi_f})^{-1}(t) dxdv,$$

where $f^* \circ a_{\phi_f}^{-1}$ is a pseudo-inverse of the nonincreasing function $f^* \circ a_{\phi_f}$. Injecting this into (2.13) yields:

$$\begin{aligned}
&\int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi_f \right) (f - \widehat{f})\, dxdv \ge \\
&\qquad \int_0^{\|f\|_{L^\infty}} dt \int_{S_1(t)} \left(\frac{|v|^2}{2} + \phi_f(x) - (f^* \circ a_{\phi_f})^{-1}(t) \right) dxdv \ge 0
\end{aligned}$$

and (2.12) is proved. We also have the analogous inequality for $S_2(t)$. From this chain of inequalities one may easily derive the case of equality and concludes the proof of Proposition 2.1. □

3. Study of the reduced functional $\mathcal{J}$

In this section, we study the following functional $\mathcal{J}$ defined on $\mathcal{X}$:

$$\mathcal{J}(\phi) = \mathcal{J}_{Q^*}(\phi) = \mathcal{H}(Q^{*\phi}) + \frac{1}{2}\|\nabla\phi - \nabla\phi_{Q^{*\phi}}\|_{L^2}^2. \qquad (3.1)$$

We claim that locally near ϕ_Q, $\mathcal{J}(\phi) - \mathcal{J}(\phi_Q)$ is equivalent to the distance of ϕ to the manifold of translated Poisson fields $\phi_Q(\cdot + x)$, $x \in \mathbb{R}^3$.

We first recall the definition of the homogeneous Sobolev space $\dot{H}^1$, which is the completion of the space of functions of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\nabla\phi\|_{L^2}$, and define $\dot{H}^1_{rad}$ as the subset of radial functions of $\dot{H}^1$.

Proposition 3.1 (Coercive behavior of $\mathcal{J}$ near ϕ_Q)**.** *There exist universal constants $c_0, \delta_0 > 0$ and a continuous map $\phi \mapsto z_\phi$ from $(\dot{H}^1, \|\cdot\|_{\dot{H}^1}) \to \mathbb{R}^3$ such that the following holds true. Let $\phi \in \mathcal{X}$ with*

$$\inf_{z\in\mathbb{R}^3} (\|\phi - \phi_Q(\cdot - z)\|_{L^\infty} + \|\nabla\phi - \nabla\phi_Q(\cdot - z)\|_{L^2}) < \delta_0, \tag{3.2}$$

then:

$$\mathcal{J}(\phi) - \mathcal{J}(\phi_Q) \geq c_0\|\nabla\phi - \nabla\phi_Q(\cdot - z_\phi)\|^2_{L^2}. \tag{3.3}$$

This section will be devoted to a sketched proof of Proposition 3.1. This proof relies first on the second order Taylor expansion of $\mathcal{J}$ at ϕ_Q, and then on the coercivity of the Hessian which is the second main key to our analysis, Proposition 3.2, and corresponds to a generalized Antonov's coercivity property. Again the detailed proof can be found in [31]. We present the main lines of the proof in several steps.

3.1. **Taylor expansion for $\mathcal{J}$.** In this step, we proved that the functional $\mathcal{J}$ defined on $\mathcal{X}$ satisfies the following properties.

(i) *Differentiability of $\mathcal{J}$.* Let $\phi, \widetilde{\phi} \in \mathcal{X}$, then the function

$$\lambda \mapsto \mathcal{J}(\phi + \lambda(\widetilde{\phi} - \phi))$$

is twice differentiable on $[0, 1]$.

(ii) *Taylor expansion of $\mathcal{J}$ near ϕ_Q.* There holds the Taylor expansion near ϕ_Q: $\forall\phi \in \mathcal{X}$,

$$\mathcal{J}(\phi)-\mathcal{J}(\phi_Q) = \frac{1}{2}D^2\mathcal{J}(\phi_Q)(\phi-\phi_Q, \phi-\phi_Q)+\eta\left(\|\phi - \phi_Q\|_{L^\infty}\right) \|\nabla\phi-\nabla\phi_Q\|^2_{L^2} \tag{3.4}$$

where

$$\eta(\delta) \to 0 \text{ as } \delta \to 0.$$

Moreover, the second derivative of $\mathcal{J}$ at ϕ_Q in the direction h is given by

$$\begin{aligned} &D^2\mathcal{J}(\phi_Q)(h, h) \\ &\quad= \int_{\mathbb{R}^3} |\nabla h|^2\, dx - \int_{\mathbb{R}^6} \left|F'\left(\frac{|v|^2}{2} + \phi_Q(x)\right)\right| (h(x) - \Pi h(x, v))^2\, dxdv\,, \end{aligned} \tag{3.5}$$

where Πh is the projector:

$$\Pi h(x, v) = \frac{\displaystyle\int_{\mathbb{R}^3} \left(\frac{|v|^2}{2} + \phi_Q(x) - \phi_Q(y)\right)^{1/2}_+ h(y)dy}{\displaystyle\int_{\mathbb{R}^3} \left(\frac{|v|^2}{2} + \phi_Q(x) - \phi_Q(y)\right)^{1/2}_+ dy}. \tag{3.6}$$

3.2. **A new Antonov type inequality.** Now we come to an important step in our analysis which is the coercivity of the Hessian $D^2\mathcal{J}(\phi_Q)$ up to the degeneracies induced by the translation invariance. This can be seen as a generalization of the celebrated Antonov's stability property –see Proposition 4.1 in [30] for a precise statement–:

Proposition 3.2 (Generalized Antonov's stability property)**.** *Let Q satisfy the assumptions of Theorem* 1.2 *and consider the linear operator generated by the Hessian* (3.5)*:*

$$\mathcal{L}h = -\Delta h - \int_{\mathbb{R}^3} |F'(e)|(h - \Pi h)dv.$$

Then $\mathcal{L}$ is a compact perturbation of the Laplacian operator on $\dot{H}^1$ and is positive:

$$\forall h \in \dot{H}^1, \quad (\mathcal{L}h, h) = D^2\mathcal{J}(\phi_Q)(h,h) \geq 0. \tag{3.7}$$

Moreover,

$$Ker(\mathcal{L}) = \{h \in \dot{H}^1 \quad with \quad \mathcal{L}h = 0\} = Span(\partial_{x_i}\phi_Q)_{1\leq i\leq 3}.$$

In particular, there exists $c_0 > 0$ such that

$$\forall h \in \dot{H}^1, \quad (\mathcal{L}h, h) \geq c_0\|\nabla h\|^2_{L^2} - \frac{1}{c_0}\sum_{i=1}^{3}\left(\int_{\mathbb{R}^3} h\Delta(\partial_{x_i}\phi_Q)dx\right)^2. \tag{3.8}$$

We sketch the proof of this result in few steps.

Step 1. Positivity away from radial modes.

Let $h \in \dot{H}^1$, and let h_0 be its orthogonal projection (in L^2) onto the space of radial functions $\dot{H}^1_{rad}$. We write the decomposition

$$h = h_0 + h_1, \quad h_0 \in \dot{H}^1_{rad}, \quad h_1 \in (\dot{H}^1_{rad})^\perp.$$

The angular integration in (3.6) ensures $\Pi h_1 = 0$ and thus

$$(\mathcal{L}h, h) = (\mathcal{L}h_0, h_0) + \int_{\mathbb{R}^3}|\nabla h_1|^2 - \int_{\mathbb{R}^3} V_Q\, h_1^2$$

with

$$V_Q(r) = \int_{\mathbb{R}^3}|F'(e)|dv = 4\pi\sqrt{2}\int_{\phi_Q(0)}^{0}|F'(e)|\,(e-\phi_Q(r))_+^{1/2}\,de.$$

Since V_Q is continuous and compactly supported, the Schrödinger operator $-\Delta - V_Q$ is a compact perturbation of the Laplacian on $\dot{H}^1$.

Now the space-translation invariance of $\mathcal{L}$ attests that its kernel is non trivial and that

$$\mathcal{L}(\partial_{x_i}\phi_Q) = 0, \qquad i = 1,2,3. \tag{3.9}$$

We now deduce from standard argument (based on spherical harmonics expansion and the monotonicity of the radial function ϕ_Q) that this implies the positivity of $\mathcal{L}$ away from radial modes, see [49] for related statements:

$$\forall h \in \left(\dot{H}^1_{rad}\right)^\perp, \quad (\mathcal{L}h, h) \geq 0, \tag{3.10}$$

and

$$\{h \in (\dot{H}^1_{rad})^\perp \quad \text{with} \quad \mathcal{L}h = 0\} = \text{Span}(\partial_{x_i}\phi_Q)_{1\leq i\leq 3}. \tag{3.11}$$

This classically yields the coercivity of $D^2\mathcal{J}(\phi_Q)$ on $(\dot{H}^1_{rad})^\perp \cap [\text{Span}(\partial_{x_i}\phi_Q)_{1\leq i\leq 3}]^\perp$.

Step 2. Coercivity on the radial modes: A new Antonov-Poincaré inequality.

First we have the relative compactness of $\mathcal{L}$ with respect to Δ in $\dot{H}^1$ (see [31] for the proof). Hence, it is sufficient to prove that

$$\forall h \in \dot{H}^1_{rad}, \quad h \neq 0, \quad (\mathcal{L}h, h) > 0. \tag{3.12}$$

Our main observation is now from (3.5) that (3.12) is nothing but a *Poincaré inequality*, and we now claim that we can adapt the celebrated proof by Hörmander

[20, 21] to our setting. Hörmander's approach involves two key steps: the introduction of a self-adjoint operator adapted to the projection involved, and a suitable convexity property. The operator will be given by

$$Tf(e,r) = \frac{1}{r^2\sqrt{2(e-\phi_Q(r))}}\partial_r f \tag{3.13}$$

which essentially satisfies the requirement

$$\Pi h = 0 \ \text{ implies } \ h \in \mathrm{Im}(T),$$

and the convexity will correspond to the lower bound:

$$-\frac{T^2 g}{g} \geq \frac{3}{(r\sqrt{2(e-\phi_Q(r))})^4}\left(\rho_Q(r) + \frac{\phi_Q'(r)}{r}\right) \tag{3.14}$$

with

$$g(r,e) = \left(r\sqrt{2(e-\phi_Q(r))}\right)^3.$$

Note that the original proof of Antonov's stability criterion can be revisited as well using the transport operator $\tau = v \cdot \nabla_x - \nabla_x \phi_Q \cdot \nabla_v$ in the radial case as differential operator and whose image can be realized in the radial setting as the kernel of the *full* projection including the kinetic momentum ℓ, see [19], [30] for more details.

For a detailed proof of this new Antonov inequality (3.12), we refer to [31] .

Step 3. End of the proof of Proposition 3.1

This step is a classical consequence of the modulation theory coupled with the coercivity estimate (3.8). It consists in adjusting the translation parameter in space in order to satisfy the orthogonality conditions that are necessary to get the coercivity of Hessian $D^2\mathcal{J}(\phi_Q)$. An implicit function theorem is used in this step from which the continuity $\phi \to z_\phi$ is derived. This, combined with the Taylor expansion (3.4) ends the proof of Proposition 3.1. We again refer to [31] for a detailed proof.

4. Compactness of local minimizing sequences of the Hamiltonian

The aim of this section is to prove the following compactness result which is the heart of the proof of Theorem 1.2.

Proposition 4.1 (Compactness of local minimizing sequences)**.** *Let $\delta_0 > 0$ be as in Proposition* 3.1*. Let $\phi \mapsto z_\phi$ the continuous map from $(\dot{H}^1, \|\cdot\|_{\dot{H}^1}) \to \mathbb{R}^3$ build in Proposition* 3.1*. Let f_n be a sequence of functions of $\mathcal{E}$, bounded in L^∞, such that*

$$\inf_{z\in\mathbb{R}^3} \left(\|\phi_{f_n} - \phi_Q(\cdot - z)\|_{L^\infty} + \|\nabla\phi_{f_n} - \nabla\phi_Q(\cdot - z)\|_{L^2}\right) < \delta_0, \tag{4.1}$$

and

$$\limsup_{n\to+\infty} \mathcal{H}(f_n) \leq \mathcal{H}(Q), \qquad f_n^* \to Q^* \ \text{in } L^1(\mathbb{R}_+) \quad \text{as } n \to +\infty. \tag{4.2}$$

Then

$$\int (1+|v|^2)|f_n - Q(x - z_{\phi_{f_n}})| \to 0 \quad \text{as} \quad n \to +\infty. \tag{4.3}$$

Proof. Step 1: Compactness of the potential

We first claim the following quantitative lower bound which generalizes the monotonicity formula (2.9): let $f \in \mathcal{E}$ such that ϕ_f satisfies (3.2), let z_{ϕ_f} given by Proposition 3.1, then

$$\mathcal{H}(f) - \mathcal{H}(Q) + \|\phi_f\|_{L^\infty} \|f^* - Q^*\|_{L^1} \geq c_0 \|\nabla \phi_f - \nabla \phi_Q(\cdot - z_{\phi_f})\|_{L^2}^2. \tag{4.4}$$

Indeed,

$$\mathcal{H}(f) - \mathcal{H}(Q) \geq \mathcal{J}_{f^*}(\phi_f) - \mathcal{J}(\phi_Q) = \mathcal{J}_{f^*}(\phi_f) - \mathcal{J}(\phi_f) + \mathcal{J}(\phi_f) - \mathcal{J}(\phi_Q), \tag{4.5}$$

where we have used that $\mathcal{H}(Q) = \mathcal{J}(\phi_Q)$. Now, we recall that

$$\mathcal{J}_{f^*}(\phi) = \int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi \right) f^{*\phi}(x, v) dx dv + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx.,$$

and deduce from a suitable change of variables that

$$\mathcal{J}_{f^*}(\phi_f) - \mathcal{J}(\phi_f) = \int_0^{+\infty} a_{\phi_f}^{-1}(s) \left(f^*(s) - Q^*(s) \right) ds.$$

Since $|a_{\phi_f}^{-1}(s)| \leq -\min\, \phi_f = \|\phi_f\|_{L^\infty}$, we have

$$\mathcal{J}_{f^*}(\phi_f) - \mathcal{J}(\phi_f) \geq -\|\phi_f\|_{L^\infty} \|f^* - Q^*\|_{L^1}.$$

Inserting this estimate into (4.5) and using Proposition 3.1 yields (4.4) .

Let us now consider a sequence $f_n \in \mathcal{E}$ satisfying the assumptions of Proposition 4.1, then (4.4) applied to f_n ensures:

$$\|\nabla \phi_{f_n}(. + z_{\phi_{f_n}}) - \nabla \phi_Q\|_{L^2} \to 0, \qquad \text{as } n \to \infty. \tag{4.6}$$

Step 2: Strong convergence of f_n to Q

To ease notations, we shall still denote by f_n the translated function $f_n(. + z_{\phi_{f_n}}, v)$. We then observe the identity:

$$\mathcal{H}(f_n) - \mathcal{H}(Q) + \frac{1}{2} \|\nabla \phi_{f_n} - \nabla \phi_Q\|_{L^2}^2 = \int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi_Q(x) \right) (f_n - Q) dx dv \tag{4.7}$$

which implies, from (4.2) and (4.6), that

$$\int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi_Q(x) \right) (f_n - Q) dx dv \to 0, \qquad \text{as } n \to \infty. \tag{4.8}$$

Since we have $\|f_n^* - Q^*\|_{L^1} \to 0$ from (4.2), we derive the following convergence

$$0 \leq T_n = \int_{\mathbb{R}^6} \left(\frac{|v|^2}{2} + \phi_Q(x) \right) (f_n - f_n^{*\phi_Q}) dx dv \to 0, \qquad \text{as } n \to \infty. \tag{4.9}$$

This positive quantity T_n is then shown to control the norm $\|f_n^{*\phi_Q} - f_n\|_{L^1}$ in some sense, and then $\|f_n^{*\phi_Q} - f_n\|_{L^1}$ goes to 0, see [31] for the detailed proof of this claim. Using again the convergence $\|f_n^* - Q^*\|_{L^1} \to 0$ and the fact that Q is a fixed point of our rearrangement, we finally deduce that $\|f_n - Q\|_{L^1} \to 0$. Combining this with the boundeness of the Hamiltonian and the strong convergence of the potential, we also deduce that

$$\int_{\mathbb{R}^6} |v|^2 |f_n - Q| dx dv \to 0.$$

This concludes the proof of Proposition 4.1.

□

5. Non linear stability of Q

We now turn to the proof of the nonlinear stability result stated in Theorem 1.2, which is a direct consequence of Proposition 4.1 and the known regularity of weak solutions to the Vlasov-Poisson system.

Proof of Theorem 1.2.

Step 1. Continuity claim for weak solutions

Let $f_0 \in \mathcal{E}$ and let $f(t) \in \mathcal{E}$ be a corresponding weak solution to (1.1). By the properties of weak solutions of the Vlasov-Poisson system [7, 8], we have

$$\forall t \geq 0, \qquad f(t) \in \mathrm{Eq}(f_0), \qquad \mathcal{H}(f(t)) \leq \mathcal{H}(f_0), \tag{5.1}$$

and we may prove that

$$\phi_f \in \mathcal{C}([0, +\infty), L^\infty(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)). \tag{5.2}$$

Now this implies from Proposition 3.1 that

$$t \mapsto z_{\phi_{f(t)}} \quad \text{is continuous.} \tag{5.3}$$

Step 2: Conclusion.

An equivalent reformulation of Proposition 4.1 is the following: for all $\varepsilon > 0$ small enough, there exists $\eta > 0$ such that if $f \in \mathcal{E}$ with

$$\|f^* - Q^*\|_{L^1} \leq \eta, \quad \|f\|_{L^\infty} \leq \|Q\|_{L^\infty} + M, \quad \mathcal{H}(f) \leq \mathcal{H}(Q) + \eta \tag{5.4}$$

and

$$\inf_{z \in \mathbb{R}^3} (\|\phi_f - \phi_Q(\cdot - z)\|_{L^\infty} + \|\nabla \phi_f - \nabla \phi_Q(\cdot - z)\|_{L^2}) < \delta_0, \tag{5.5}$$

then

$$\|(1 + |v|^2)(f - Q(\cdot - z_{\phi_f}))\|_{L^1} \leq \varepsilon. \tag{5.6}$$

Let $\varepsilon > 0$ and let $\eta > 0$ be the associated constant. We consider an initial data $f_0 \in \mathcal{E}$ with

$$\|f_0 - Q\|_{L^1} < \eta, \quad \|f_0\|_{L^\infty} \leq \|Q\|_{L^\infty} + M \quad \text{and} \quad \mathcal{H}(f_0) \leq \mathcal{H}(Q) + \eta$$

and a corresponding weak solution $f(t)$ of (1.1). Observe that, by the contractivity of the symmetric rearrangement in L^1 (see [32]), we have

$$\|f_0^* - Q^*\|_{L^1} \leq \|f_0 - Q\|_{L^1} \leq \eta. \tag{5.7}$$

This implies from interpolation inequalities that, for η small enough,

$$\|\nabla \phi_{f_0} - \nabla \phi_Q(\cdot - z_{\phi_{f_0}})\|_{L^2} + \|\phi_f(0) - \phi_Q(\cdot - z_{\phi_{f_0}})\|_{L^\infty} \leq \frac{\delta_0}{2}.$$

From (5.1), we first deduce that the corresponding solution $f(t)$ of (1.1) satisfies (5.4) for all $t \geq 0$. Hence, if we prove that

$$\forall t \geq 0, \qquad \|\nabla \phi_f(t) - \nabla \phi_Q(\cdot - z_{\phi_{f(t)}})\|_{L^2} + \|\phi_f(t) - \phi_Q(\cdot - z_{\phi_{f(t)}})\|_{L^\infty} < \delta_0, \tag{5.8}$$

then (5.6) holds true for all $t \geq 0$, which is nothing but (1.18). Now (5.8) follows for $\eta > 0$ small enough from a straightforward bootstrap argument using the continuity (5.2), (5.3) and an interpolation inequality. This ends the proof of Theorem 1.2. □

6. Some perspectives

There are many open questions after this work. We list some of them below:

(i) *Quantitative stability results.* A natural first question is to see whether is it possible or not to quantify all the estimates and the compactness arguments we have used in our proof of the nonlinear stability. We believe that this is possible to do and a work in this question is currently investigated. In particular, the compactness step described in section 4, may be completely reformulated in a quantified functional inequality. This inequality provides a quantitative control of the distance between f and the steady state Q by the difference between the two Hamiltonians $\mathcal{H}(f)$ and $\mathcal{H}(Q)$ and the distance between the two rearrangements f^* and Q^*.

(ii) *Extension to relativistic systems.* A second natural question is to explore the applicability of our strategy to relativistic Vlasov models. There are several kinds of relativistic models depending to which type of force fields the Vlasov equation is coupled: coupling with a wave equation on the potential, with the so-called Nördstrom equation or with the Einstein equations. While the most physically relevant model (Vlasov-Einstein) seems to be difficult to deal with at the present time, the two others are more reachable and the stability issues for these models may be directly explored using the present approach.

(iii) *Non monotonic steady states.* Of course, the monotonicity condition on the steady states is only a sufficient (but not necessary) to guarantee its stability. A natural question (which is an open problem) is therefore to analyze the stability or the instability of non monotonic steady states.

(iv) *Fluid mechanics.* The application of our strategy to the 2D Euler equation in fluid mechanics is also another interesting question. The problem in this context is to analyze the stability of the steady states of this system which are monotonic functions of the courant profiles. If one tries to follow the same algorithm as it is described in this paper, then one may reduce the stability analysis to the validity of certain Poincaré inequality. Unfortunately, the validity of such an inequality is still an open question.

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