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ASYMPTOTIC BEHAVIORS FOR NONLINEAR DISPERSIVE EQUATIONS WITH DAMPING OR DISSIPATIVE TERMS

ZE LI AND LIFENG ZHAO

ABSTRACT. In this note, we will review our recent work on the asymptotic behaviors of nonlinear Klein-Gordon equation with damping terms and Landau-Lifschitz flows from Euclidean spaces and hyperbolic spaces. By the method of concentration-compactness attractors, we prove that the global bounded solution will decouple into a finite number of equilibrium points with different shifts from the origin. For the Landau-Lifschitz flow from Euclidean spaces, we prove that the solution with energy below 4π will converge to some constant map in the energy space. While for the Landau-Lifschitz flow from two dimensional spaces, the solution will converge to some harmonic map.

1. DAMPED KLEIN-GORDON EQUATION

There are a lot of works devoted to the study of long time behaviour for nonlinear dispersive equations which are representative of Schrödinger equation, wave equations, Klein-Gordon equations and KdV equations, ect. It's widely bevelied that for "generic" large global solutions, the evolution asymptotically decouples into the superposition of divergent solitons, a free radiation term, and an error term tending to zero as time goes to infinity. This is the so called soliton resolution conjecture. For more expression and history, see A. Soffer [27].

The soliton resolution conjecture has been verified for one dimensional cubic Schrödinger equation and KdV equation because they are completely integrable. T. Duyckaerts, C. Kenig, and F. Merle [7] first made a breakthrough for non-integrable dispersive equations. For radial data to three dimensional focusing energy-critical wave equations, they proved the solution with bounded trajectory is in fact a superposition of a finite number of rescalings of the ground state plus a radiation term which is asymptotically a free wave. One of the key ingredient of their arguments is the novel tool, called "channels of energy" introduced by [7] [8]. The method developed by them has been applied to many other situations, such as [4] [5] [15] [17] [16] for wave maps, [6] [10] [13] for semilinear wave equations. By a weak version of outer energy inequality, the soliton resolution along a sequence of times was proved by R. Cote, C. Kenig, A. Lawrie and W. Schlag [6] for four dimensional critical wave equations in radial case, by R. Cote [3] for equivariant wave maps, and by H. Jia, and C. Kenig [14] for semilinear wave equations, wave maps. Since the soliton resolution is so hard for classical dispersive equations, it's natural to consider some simpler cases: dispersive equations with damping or dissipated terms.

Consider the damped Klein-Gordon equation

$$\begin{cases} u_{tt} + 2\alpha u_t - \Delta u + u - |u|^{p-1}u = 0, \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in \mathcal{H}, \end{cases} \quad (\text{DKG})$$

where $\mathcal{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, $\alpha \geq 0$ and $1 < p < \frac{d+2}{d-2}$. The energy is given by

$$E(u(t), \partial_t(u)) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla u(t)|^2 + \frac{1}{2} |u(t)|^2 + \frac{1}{2} |\partial_t(u)|^2 - \frac{1}{p+1} |u(t)|^{p+1} \right) dx.$$

There are some differences between damped Klein-Gordon equations (DKG) and the energy-critical wave equations. First, due to the damping term, the energy decrease

$$E(u(t_2), \partial_t u(t_2)) - E(u(t_1), \partial_t u(t_1)) = -2\alpha \int_{t_1}^{t_2} \|\partial_t u(t)\|_2^2 dt.$$

Second, since (DKG) is subcritical, it has an infinite number of radial equilibrium points except the ground state which is the radial positive stationary solution with the minimized energy among all the non-zero stationary solutions. Moreover, nothing is known about the uniqueness and the hyperbolicity of those nodal solutions. Finally, there are no type II finite time blow-up solutions. Hence it seems that subcritical problems need different techniques.

For $\alpha = 0$, namely nonlinear Klein-Gordon equation, T. Cazenave [2] gave the following dichotomy: solutions either blow up at finite time or are global forward in time and bounded in \mathcal{H} , provided $1 < p < \infty$, when $d = 1, 2$ and $1 < p < \frac{d}{d-2}$ if $d \geq 3$. For $\alpha > 0$, E. Feireisl [12] gave an independent proof of the boundedness of the trajectory to global solutions, for $1 < p < 1 + \min(\frac{d}{d-2}, \frac{4}{d})$ when $d \geq 3$, and in his paper [11], the case $d = 1$ is considered. N. Burq, G. Raugel, W. Schlag [1] studied the long time behaviors of solutions to nonlinear damped Klein-Gordon equations in radial case. By developing some dynamical methods especially invariant manifolds, they proved the ω -limit set of the trajectory is just one single point, hence they showed the dichotomy in forward time (1) the solution either blows up at finite time, (2) or converges to some equilibrium point.

Theorem 1 ([23]). *Let $\alpha > 0$, $1 \leq d \leq 4$, $1 < p < \infty$ for $d = 1, 2$ and $1 < p < 1 + \frac{4}{d-2}$ for $3 \leq d \leq 4$. For any data $(u_0, u_1) \in \mathcal{H}$. Then*

- (i) *either the solution of (DKG) blows up at finite positive time,*
- (ii) *or it is global forward in time with unbounded trajectory;*
- (iii) *or for any time sequence $t_n \rightarrow \infty$, up to a subsequence, there exist $0 \leq J < \infty$, $x_{j,n} \in \mathbb{R}^d$ for $j = 1, 2, \dots, J$ and equilibrium points $\{Q^j\}$ such that*

$$u(t_n) = \sum_{j=1}^J Q^j(x - x_{j,n}) + o_{H^1}(1),$$

and $\lim_{t \rightarrow \infty} \partial_t u(t) = 0$, in L^2 , where $\{x_{j,n}\}$ satisfies the separation property:

$$\lim_{n \rightarrow \infty} |x_{j,n} - x_{i,n}| = \infty, \text{ for } i \neq j.$$

Remark 2. It's conjectured that there are not global forward in time solutions with unbounded trajectory as in the radial case [1]. It's open to upgrade the sequential results to the one which holds for all time. It's also interesting to construct multi-soliton solutions with prescribed separation of spatial translation parameters.

The proof is based on the method of concentration-compactness attractors by T. Tao [28]. Given any $h \in \mathbb{R}^d$, let $\tau_h : \mathcal{H} \rightarrow \mathcal{H}$ be the shift operator $\tau_h f(x) = f(x - h)$, and we denote the translation group by $G = \{\tau_h : h \in \mathbb{R}^d\}$. Given any $K \subset \mathcal{H}$, we denote the orbit of K by $GK = \{gf : g \in G, f \in K\}$. Suppose $J \geq 1$ is an integer, we let

$$JK = \{f_1 + \dots + f_J : f_1, \dots, f_J \in K\}$$

denote the J -fold Minkowski sum of the set K , with the convention that $0K = \{o\}$. We say $E \subset \mathcal{H}$ is G -precompact with J components if $E \subset J(GK)$ for some compact $K \subset \mathcal{H}$ and $J \geq 1$. Thus it suffices to prove that

Proposition 3 (Concentration compact attractor). *There exists a G -precompact set \mathcal{K} with J components and \mathcal{K} consists solely of soliton solutions, such that for any forward global solution u , it holds that*

$$\lim_{t \rightarrow +\infty} \text{dist}_H(u(t), J(G\mathcal{K})) = 0.$$

Therefore, we need to prove the existence of the G -precompact set \mathcal{K} and show that \mathcal{K} consists solely of soliton solutions. The existence of \mathcal{K} can be reduced to the frequency localization and spatial localization due to the following proposition

Proposition 4 (Criterion for G -compact attractor, [28]). *Let \mathcal{U} be a collection of trajectories $u : \mathbb{R}^+ \rightarrow \mathcal{H}$, and let $J \geq 1$. Then the following are equivalent:*

- (i) *There exists a G -precompact set $\mathcal{K} \subset \mathcal{H}$ with J components such that $\lim_{t \rightarrow +\infty} \text{dist}_{\mathcal{H}}(u(t), G\mathcal{K}) = 0$ for all $u \in \mathcal{U}$.*
- (ii) *\mathcal{U} is asymptotically bounded such that for any $\mu_0 > 0$ there exists $\mu_1 > 0$ such that for every $u \in \mathcal{U}$ we have $x_1, \dots, x_J : \mathbb{R}^+ \rightarrow \mathbb{R}^d$ for which we have the asymptotic frequency localisation estimates*

$$\limsup_{t \rightarrow +\infty} \|P_{\geq 1/\mu_1} \bar{u}(t)\|_{\mathcal{H}} \leq \mu_0$$

and

$$\limsup_{t \rightarrow +\infty} \|P_{\leq \mu_1} \bar{u}(t)\|_{\mathcal{H}} \leq \mu_0$$

and the improved spatial localisation estimate

$$\limsup_{t \rightarrow +\infty} \int_{\inf_{1 \leq j \leq J} |x - x_j(t)| \geq 1/\mu_1} |u(t, x)|^2 + |\nabla u(t, x)|^2 + |\partial_t u(t, x)|^2 dx \leq \mu_0^2.$$

From the above proposition, we need to prove the localization of frequency first, namely

Proposition 5. *For any $\mu_0 > 0$ there exists $c(\mu_0) > 0$ depending on E such that*

$$\limsup_{t \rightarrow \infty} \|P_{\geq \frac{1}{c(\mu_0)}} u(t)\|_{H^1} \leq \mu_0, \quad (1)$$

$$\limsup_{t \rightarrow \infty} \|P_{\geq \frac{1}{c(\mu_0)}} \partial_t u(t)\|_{L^2} \leq \mu_0. \quad (2)$$

Proof. For illustration, we just consider the case $1 < p < \frac{d}{d-2}$ for $d \geq 3$ here. From Duhamel principle,

$$P_{\geq \frac{1}{\mu}} u(t) = S_{1,\alpha} P_{\geq \frac{1}{\mu}} u_0 + S_{2,\alpha} P_{\geq \frac{1}{\mu}} u_1 + \int_0^t S_{2,\alpha}(t-s) P_{\geq \frac{1}{\mu}} (|u|^{p-1} u)(s) ds.$$

Since

$$\|S_{1,\alpha} P_{\geq \frac{1}{\mu}} u_0\|_{H^1} \leq e^{-\alpha t} \|u_0\|_{H^1}, \quad \|S_{2,\alpha} P_{\geq \frac{1}{\mu}} u_1\|_{H^1} \leq e^{-\alpha t} \|u_1\|_{L^2},$$

for μ sufficiently small, we have

$$\|P_{\geq \frac{1}{\mu}} u(t)\|_{H^1} \leq C e^{-\alpha t} \|(u_0, u_1)\|_{\mathcal{H}} + \int_0^t e^{-\alpha(t-s)} \|P_{\geq \frac{1}{\mu}} (|u|^{p-1} u)(s)\|_2 ds.$$

Let $h(u) = |u|^{p-1} u$, split u into $u = P_{\leq \mu^{-1}} u + P_{\geq \mu^{-1}} u$, then

$$h(u) = h(P_{\leq \mu^{-1}} u) + P_{\geq \mu^{-1}} u O(|u|^{p-1}).$$

Bernstein's inequality and Hölder's inequality imply

$$\begin{aligned} \|P_{\geq \mu^{-1}} h(u)\|_2 &\leq \|P_{\geq \mu^{-1}} h(P_{\leq \mu^{-1}} u)\|_2 + \|P_{\geq \mu^{-1}} (P_{\geq \mu^{-1}} u O(|u|^{p-1}))\|_2 \\ &\leq \mu \|\nabla h(P_{\leq \mu^{-1}} u)\|_2 + \|P_{\geq \mu^{-1}} u O(|u|^{p-1})\|_2 \\ &\leq \mu \|\nabla P_{\leq \mu^{-1}} u\|_2 \|P_{\leq \mu^{-1}} u\|_2^{p-1} + \|P_{\geq \mu^{-1}} u\|_m \| |u|^{p-1} \|_{\frac{2^*}{p-1}}, \end{aligned}$$

where $\frac{1}{m} + \frac{p-1}{2^*} = \frac{1}{2}$. By Bernstein's inequality, we have

$$\begin{aligned} \mu \left\| \nabla P_{\leq \mu^{-1}} u \right\|_2 & \leq \mu \left\| \nabla u \right\|_2 \left\| P_{\leq \mu^{-1}} u \right\|_\infty^{p-1} \leq \mu^{-\frac{d(p-1)}{2^*}+1} \left\| \nabla u \right\|_2 \left\| P_{\leq \mu^{-1}} u \right\|_{2^*}^{p-1}. \end{aligned}$$

Since $1 < p < \frac{d}{d-2}$, we conclude for some $\kappa > 0$,

$$\left\| P_{\geq \mu^{-1}} h(P_{\leq \mu^{-1}} u) \right\|_2 \leq \mu^\kappa \|u\|_{H^1}. \quad (3)$$

Applying Bernstein's inequality, we have

$$\begin{aligned} \left\| P_{\geq \mu^{-1}} u \right\|_m & \leq \left(\sum_{N \geq \mu^{-1}}^\infty \|P_N u\|_m^2 \right)^{1/2} \leq \left(\sum_{N \geq \mu^{-1}}^\infty N^{2d(\frac{1}{2}-\frac{1}{m})-2} N^2 \|P_N u\|_2^2 \right)^{1/2} \\ & \leq \mu^{-d(\frac{1}{2}-\frac{1}{m})+1} \left(\sum_{N \geq \mu^{-1}}^\infty N^2 \|P_N u\|_2^2 \right)^{1/2}. \end{aligned}$$

which combined with (3) gives (1) by $1 < p < \frac{d}{d-2}$. Next, we bound $\partial_t u$. From Duhamel principle, we have

$$\begin{aligned} \partial_t u(t) & = -\alpha u(t) + e^{-\alpha t} \left[-\sqrt{-\Delta + 1 - \alpha^2} \sin \left(t \sqrt{-\Delta + 1 - \alpha^2} \right) \right. \\ & \quad \left. + \alpha \cos \left(t \sqrt{-\Delta + 1 - \alpha^2} \right) \right] u(t - \delta) \\ & \quad + e^{-\alpha t} \cos \left(t \sqrt{-\Delta + 1 - \alpha^2} \right) \partial_t u(t - \delta) \\ & \quad + \int_{t-\delta}^t \cos \left((t-s) \sqrt{-\Delta + 1 - \alpha^2} \right) e^{-\alpha(t-s)} \left(|u|^{p-1} u \right) (s) ds. \end{aligned}$$

For $\mu_1 \ll \mu_0$, (1) implies that there exist $\eta > 0$ and $T_0 > 0$ such that

$$\left\| P_{\geq \eta^{-1}} u(t) \right\|_{H^1} < \mu_1,$$

for $t > T_0$. Taking δ large such that $e^{-\alpha \delta} < \mu_1$, then for $t > T_0 + \delta$, it suffices to prove

$$\left\| P_{\geq \eta^{-1}} h(u(t)) \right\|_2 \leq \eta^\lambda,$$

for some $\lambda > 0$. The rest of the proof of (2) is the same as (1).

Next we need to prove the spatial localization

Proposition 6. *Let u be a global solution to (DKG) with \mathcal{H} norm at most $E > 0$. Then there exist $J = J(E)$ depending only on E , and functions $x_1(t), \dots, x_J(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^d$, such that for any $\mu > 0$ there exists $\eta = \eta(E, \mu) > 0$ such that*

$$\limsup_{t \rightarrow \infty} \int_{\text{dist}(x, \{x_1(t), \dots, x_J(t)\}) > \eta^{-1}} |\nabla u|^2 + |u|^2 + |\partial_t u|^2 \leq \mu.$$

In fact, the key ingredient of the proof is to prove a weaker version:

Proposition 7. *Let u be a global solution to (DKG) with \mathcal{H} norm at most $E > 0$. Then for $\mu_0 > 0$, there exists $J = J(E, \mu_0)$ and functions $\tilde{x}_1(t), \dots, \tilde{x}_J(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^d$, and $\eta = \eta(E, \mu_0) > 0$ such that*

$$\limsup_{t \rightarrow \infty} \int_{\text{dist}(x, \{\tilde{x}_1(t), \dots, \tilde{x}_J(t)\}) > \eta^{-1}} |u|^2 \leq \mu_0.$$

Proof. The proof is very technical, so we just give the skech of the proof. Fix $E > 0$ and μ_0 , choose parameters $\mu_0 \gg \mu_1 \gg \mu_2 \gg \mu_3 \gg \mu_4 > 0$. The whole proof is divided into five parts.

- i) *Selecting a “good” time sequence.* For any $t_0 > T_0$, There exits good time $t_* \in [t_0 - \mu_1^{-1}, t_0 + \mu_1^{-1}]$, such that

$$\|\partial_t u(t_*)\|_2 \leq \mu_2^2.$$

- ii) L_x^∞ *spatial localization at fixed time.* Fix $t > T_1$, there exist $\{x_1, \dots, x_J\}$ such that

$$|u_{<c(\mu_2)^{-1}}(t, x_j(t))| < \mu_3, \quad \text{whenever } \text{dist}(x, \{x_1, \dots, x_J\}) \geq 2\mu_3^{-1}.$$

- iii) L_x^∞ *spatial localization on an interval centered at good time.*

- iv) L^2 *localization of good times.* Define the distance function

$$D(x) = \text{dist}(x, \{x_1(t_*), x_2(t_*), \dots, x_J(t_*)\}),$$

then for T_1 sufficiently large, $t > T_1$,

$$\|1_{D > \mu_4^{-3}} u(t_*)\|_2 = O_{L^2}(\mu_1).$$

- v) L^2 *localization of all time.*

For $t \in (t_*, t_* + 4\mu_1^{-1})$.

$$\|1_{D \geq \mu_4^{-3}} u(t)\|_2 \leq \mu_1.$$

We are now in the position to prove the main theorem.

Proof of Theorem 1. We have for any $t_n \rightarrow \infty$, up to a subsequence there exits J_1, J_2, \dots, J_M and $w_m \in J_m(GK)$ such that

$$\begin{aligned} u(t_n) &= \sum_{m=1}^M \tau_{x_{m,n}} w_m + o_{H^1}(1) \\ \partial_t u(t_n) &= \sum_{m=1}^M \tau_{x_{m,n}} v_m + o_{L^2}(1), \end{aligned}$$

where $x_{m,n} \in \mathbb{R}^d$ and they satisfies $\lim_{n \rightarrow \infty} |x_{m,n} - x_{k,n}| = \infty$, for $k \neq m$.

By linear energy decoupling property, we have $\sup_m \|(w_m, v_m)\|_{\mathcal{H}} < C$, by the local theory, there exists $T > 0$ such that the solution W_j to (7) with initial data (w_j, v_j) is wellposed on $[0, T]$. From perturbation theorem and separation of $x_{m,n}$, we obtain

$$\partial_t u(t_n + t) = \sum_{j=1}^M \partial_t W_j(x - x_{j,n}, t) + o_{L^2}(1).$$

Since $\lim_{n \rightarrow \infty} \int_0^T \|\partial_t u(t_n + t)\|_2^2 dt = 0$, by the separation of linear energy, we conclude

$$\int_0^T \|\partial_t W_j(t)\|_2^2 dt = 0.$$

Therefore, W_j is an equilibrium, the same holds for w_j , thus we have proved there exists a finite number of equilibrium points Q_m such that for any sequence $t_n \rightarrow \infty$, there exists $x_{m,n}$ for which

$$u(t_n) = \sum_{m=1}^M Q_m(x - x_{m,n}) + o_{H^1}(1), \quad \partial_t u(t_n) = o_{L^2}(1).$$

By contradiction arguments, we can prove our theorem.

2. LANDAU-LIFSHITZ EQUATION

We first consider the two dimensional Landau-Lifshitz equation:

$$\begin{cases} u_t = \sum_{i=1}^2 \alpha \nabla_{x_i} \partial_{x_i} u - \beta J(\nabla_{x_i} \partial_{x_i} u) \\ u(0) = u_0, \end{cases} \quad (\text{LL})$$

where $u(x, t) : \mathbb{R}^2 \times [0, \infty) \rightarrow \mathcal{N}$, (\mathcal{N}, J, h) is a Kähler manifold, ∇ is the induced connection by u , $\partial_{x_i} u = u_* (\frac{\partial}{\partial x_i})$, $\partial_t u = u_* (\frac{\partial}{\partial t})$. $\alpha \geq 0$ is called the Gilbert constant. When $\alpha = 0$, (LL) is called the Schrödinger flow. When $\beta = 0$, $\alpha > 0$, it reduces to the heat flows of harmonic maps.

For general compact Kähler targets and general data, we have

Theorem 8 ([24]). *Let (\mathcal{N}, h, J) be a compact Kähler manifold, $\alpha > 0$, $\beta \in \mathbb{R}$. For any data $u_0 \in W^{1,2}(\mathbb{R}^2; \mathcal{N})$, there exists a weak solution in $L^\infty([0, \infty); W^{1,2}(\mathbb{R}^2; \mathcal{N}))$ to (LL), which is regular on $\mathbb{R}^2 \times (0, \infty)$ with the exception of finitely many points (x_l, T_l) , $1 \leq l \leq L$, characterized by*

$$\limsup_{t \rightarrow T_l} \int_{B(x_l, R)} |\nabla u(t, y)|^2 dy > \varepsilon_1, \quad \text{for all } R \in (0, 1],$$

where ε_1 is some positive constant depending only on \mathcal{N} . Furthermore, for any fixed pair (x_l, T_l) there exist sequences $t_m \rightarrow T_l$, $x_m \rightarrow x_l$, $R_m \rightarrow 0$ and a harmonic map $u_\infty \in C^\infty(\mathbb{R}^2; \mathcal{N})$ such that

$$u(t_m, R_m x + x_m) \rightarrow u_\infty \quad \text{locally in } W^{2,2}(\mathbb{R}^2; \mathcal{N}).$$

Theorem 8 can be proved by the Struwe's bubbling arguments on the heat flow. The new difficulty is the non-compactness of \mathbb{R}^2 and the second derivative term with the complex structure. The non-compactness will be overcome by an outer ball energy estimate. In order to avoid the obstacle to the energy arguments caused by the second derivative term with the complex structure, we fully exploit the skew-symmetry of the symplectic form to obtain some cancellation of the high derivative terms.

To state the next theorem, we define the critical energy as follows

$$E_* = \inf\{E : E = E(Q),$$

$$\text{where } Q(x) : \mathbb{R}^2 \rightarrow \mathcal{N} \text{ is a harmonic map with } E(Q) > 0\}.$$

We make the convention that $E_* = \infty$ if there is no non-trivial harmonic map from \mathbb{R}^2 to \mathcal{N} with finite energy. For compact Riemannian surfaces, we have

Theorem 9. *Let (\mathcal{N}, h, J) be a compact Riemannian surface, $\alpha > 0$, $\beta \in \mathbb{R}$. The LL flow with $u_0 \in W^{1,2}(\mathbb{R}^2; \mathcal{N})$ satisfying $E(u_0) < E_*$ admits a global solution $u \in L^\infty([0, \infty); W^{1,2}(\mathbb{R}^2; \mathcal{N}))$. Moreover, $u(t, x)$ converges to a constant map as $t \rightarrow \infty$ in the energy space, namely*

$$\lim_{t \rightarrow \infty} E(u(t)) = 0.$$

Remark 10. It is known that (Schoen and Yau [26]) $E_* = \infty$ if the sectional curvature of \mathcal{N} is non-positive. Therefore, Theorem 9 shows all the solutions of (7) with finite energy decay to zero if \mathcal{N} has a non-positive sectional curvature. Typical examples for compact Riemannian surfaces with non-positive curvature are Bolza surface, Klein quartic, Bring's surface and Macbeath surface. For general compact targets, E_* is always strictly positive and we have an explicit low bound for E_* by using the upper bound of the Riemannian curvature of \mathcal{N} . And it is known that $E_* = 4\pi$ when \mathcal{N} is a two-dimensional sphere. Considering that the S^2 target is of

special physical importance, we state the corresponding result of the S^2 target as a corollary below.

Corollary 11. *Let $\alpha > 0$, $\beta \in \mathbb{R}$. The Landau-Lifshitz-Gilbert equation (LL) with $u_0 \in W^{1,2}(\mathbb{R}^2; S^2)$ satisfying $E(u_0) < 4\pi$ admits a global solution and*

$$\lim_{t \rightarrow \infty} E(u(t)) = 0.$$

Before we proceed, we first rewrite the equation (LL) in Coulomb gauge. Assume that $u : [0, T] \times \mathbb{R}^2 \rightarrow \mathcal{N}$ is a solution of (LL). Choose an orthonormal frame $\{e_1, \dots, e_{2l}\}$ for $u^*T\mathcal{N}$ with respect to h , and $e_{l+1} = J(e_1), \dots, e_{2l} = J(e_l)$. Let the Latin indices take values in $\{1, 2, \dots, 2l\}$, the Roman indices in $\{1, 2\}$, and the Greek indices in $\{1, 2, \dots, l\}$. We make the convention that $f^{\bar{\gamma}} = f^{\gamma+l}$, $f^{\overline{\gamma+l}} = f^{\gamma}$ for vector-valued functions $(f^1, \dots, f^{2l})^t$, and $e_{\bar{\gamma}} = e_{\gamma+l}$, $e_{\overline{\gamma+l}} = -e_{\gamma}$. Expand $\nabla_{x,t}u$ in the frame $\{e_j\}$ as follows:

$$\partial_{x_i}u = \sum_{a=1}^{2l} h(\partial_{x_i}u, e_a)e_a \equiv \psi_i^a e_a, \quad \partial_t u = \sum_{a=1}^{2l} h(\partial_t u, e_a)e_a \equiv b^a e_a.$$

Since J commutes with $\nabla_{x,t}$, rewriting (LL) by ψ_i^a, b^a gives

$$b^a e_a = \sum_{i=1}^2 \alpha (\partial_{x_i} \psi_i^a e_a + \psi_i^a \nabla_{x_i} e_a) - \beta (\partial_{x_i} \psi_i^a J(e_a) - \psi_i^a \nabla_{x_i} J(e_a)).$$

Denote the space of C^l -valued field defined in $[0, T] \times \mathbb{R}^2$ by \mathfrak{X} , then ∇ induces a covariant derivative on \mathfrak{X} defined by

$$\begin{aligned} D_i(v)^\gamma &= \partial_{x_i} v^\gamma + ([A_i]_\theta^\gamma + \sqrt{-1}[A_i]_\theta^{\bar{\gamma}})v^\theta, \\ D_t(v)^\gamma &= \partial_t v^\gamma + ([A_t]_\theta^\gamma + \sqrt{-1}[A_t]_\theta^{\bar{\gamma}})v^\theta, \end{aligned}$$

where the corresponding connection coefficients matrices are given by

$$[A_i]_a^b = \langle \nabla_{x_i} e_a, e_b \rangle.$$

Considering the complexification of $\psi_{i,t}, A_{i,t}$ defined by

$$\begin{aligned} [\tilde{A}_t]_\theta^\gamma &= [A_t]_\theta^\gamma + \sqrt{-1}[A_t]_\theta^{\bar{\gamma}}, [\tilde{A}_i]_\theta^\gamma = [A_i]_\theta^\gamma + \sqrt{-1}[A_i]_\theta^{\bar{\gamma}}, \\ \phi_t &= (b^1 + \sqrt{-1}b^{\bar{1}}, \dots, b^l + \sqrt{-1}b^{\bar{l}})^t, \\ \phi_i &= (\psi_i^1 + \sqrt{-1}\psi_i^{\bar{1}}, \dots, \psi_i^l + \sqrt{-1}\psi_i^{\bar{l}})^t. \end{aligned}$$

If \mathcal{N} is a Riemannian surface, we can choose the frame $\{e_1, e_2\}$ to be a Coulomb gauge, namely $\partial_i \tilde{A}_i = 0$. In this case, for $i \in \{1, 2\}$, $[\tilde{A}_i] = a_i$, $[\tilde{A}_t] = a_t$ where $a_{i,t}$ is some pure-imaginary valued function defined on $[0, T] \times \mathbb{R}^2$. Moreover, (LL) can be rewritten as

$$\begin{cases} \partial_t \phi_j - z \Delta \phi_j = a_t \phi_j + \sum_{i=1}^2 (z a_i \partial_i \phi_j + z \partial_i a_i \phi_j + z a_i a_i \phi_j + z \mathcal{R}(\phi_i, \phi_j) \phi_i) \\ \Delta a_j = \partial_k (\mathbf{i} \kappa(u) \langle \phi_k, \mathbf{i} \phi_j \rangle) \\ \Delta a_t = -\partial_k (\mathbf{i} \kappa(u) (\partial_j \langle \phi_k, \phi_j \rangle - \frac{1}{2} \partial_k |\phi_j|^2)) \\ \partial_k a_k = 0 \end{cases} \quad (\text{CGL})$$

where $z = \alpha - \sqrt{-1}\beta$, $\mathbf{i} = \sqrt{-1}$, $\langle z, w \rangle = \text{Re}(z\bar{w})$, κ is the Gauss curvature.

From the Coulomb gauge representation, the estimate of ∇u can be reduced to some estimates of ϕ_j . Moreover, for (CGL) we have the standard Strichartz estimates:

Lemma 12. *Let z be a complex number with $\Re z > 0$. Then for an admissible pair (p, q) satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $2 \leq p, q \leq \infty$, $p \neq 2$, and any pair (r, s) satisfying $\frac{1}{r} + \frac{1}{s} = \frac{1}{2}$, $1 \leq r, s \leq 2$, $r \neq 2$, we have*

$$\|e^{zt\Delta} f\|_{L_t^p L_x^q} \|f\|_{L^2}, \left\| \int_{t_0}^t e^{z(t-\tau)\Delta} g(\tau, x) d\tau \right\|_{L_t^p L_x^q([t_0, t_1] \times \mathbb{R}^2)} \|g\|_{L_t^r L_x^s([t_0, t_1] \times \mathbb{R}^2)}.$$

We will prove Theorem 8 by the method of induction on energy. The classical line for the induction on energy argument involves three main ingredients: the scattering for small data; the existence of the critical elements; ruling out the critical elements. In fact, the problem can be reduced to

$$\int_0^\infty \int_{\mathbb{R}^2} |\nabla u(t, x)|^4 dx dt \leq C,$$

for some $C > 0$ and for any initial data $u_0 \in W^{1,2}$ satisfying $E(u_0) < E_*$. We will illustrate our main ideas by the small data result.

Lemma 13 (Small data). *Let $\varepsilon > 0$ be sufficiently small. For any initial data $u_0 \in W^{1,2}$ satisfying $\|\nabla u_0\|_{L_x^2} < \varepsilon$, (LL) has a unique global solution in $\mathcal{H}(\mathbb{R}^+ \times \mathbb{R}^2)$, furthermore we have*

$$\int_0^\infty \int_{\mathbb{R}^2} |\nabla u(t, x)|^4 dx dt \leq C, \quad (4)$$

for some $C > 0$.

Proof. Let $\varepsilon^2 < 2E_*$, the global well-posedness is a corollary of Theorem 8. Then we prove (4) by a bootstrap argument. Define

$$\mathcal{A} = \{T > 0, \|\nabla u(t, x)\|_{L_{t,x}^4([0, T] \times \mathbb{R}^2)} \leq C^* \varepsilon\},$$

where $C^* > 0$ will be determined later. The non-empty and closed-ness of \mathcal{A} follows from the energy inequality which implies

$$\|\nabla u(t, x)\|_{L^4([s, s'] \times \mathbb{R}^2)} \leq E(u_0) \|\nabla^2 u(t, x)\|_{L^2([s, s'] \times \mathbb{R}^2)},$$

and the fact that $u \in \mathcal{H}$. It remains to prove the openness of \mathcal{A} . Assume that $T \in \mathcal{A}$, it suffices to show

$$\|\nabla u(t, x)\|_{L^4([0, T] \times \mathbb{R}^2)} \leq \frac{1}{100} C^* \varepsilon. \quad (5)$$

By energy estimates we have

$$\alpha \int_0^T \|\partial_s u\|_{L_x^2}^2 ds + \alpha \int_0^T \|\Sigma_{i=1}^2 \nabla_i \partial_i u\|_{L_x^2}^2 ds \leq \left(\|\nabla u(0)\|_{L_x^2}^2 - \|\nabla u(T)\|_{L_x^2}^2 \right),$$

which together with

$$\int_{\mathbb{R}^2} |\nabla^2 u|^2 dx \leq R_N \int_{\mathbb{R}^2} |\nabla u|^4 dx + \int_{\mathbb{R}^2} \left| \sum_{i=1}^2 \nabla_i \partial_i u \right|^2 dx$$

and the bootstrap assumption $T \in \mathcal{A}$ implies

$$\|\nabla^2 u\|_{L_{t,x}^2([0, T] \times \mathbb{R}^2)}^2 \leq (C^* \varepsilon)^4 + \varepsilon^2. \quad (6)$$

Next it suffices to estimate $\|\phi_j\|_{L_t^4 L_x^4}$. For $2 < p < \infty$, we define p^* by $\frac{1}{p^*} = \frac{1}{p} + \frac{1}{2}$. For $m, n \in [1, \infty)$, we define (m, n) by $\frac{1}{(m, n)} = \frac{1}{m} - \frac{1}{n}$. The dual Strichartz

exponent \hat{r} for $r \in (1, 2)$ is define by $\frac{1}{\hat{r}} = \frac{3}{2} - \frac{1}{r}$. Consider (CGL), Strichartz estimates in Lemma 12 yield for some $n = 2^{-}$

$$\begin{aligned} & \|\phi_j\|_{L_t^4 L_x^4} \|\phi_j(0)\|_{L^2} + \|a_t \phi_j\|_{L_t^{\hat{r}} L_x^n} \\ & \quad + \sum_{i=1}^2 \|a_i \partial_i \phi_j\|_{L_t^{\hat{r}} L_x^n} + \|a_i a_i \phi_j\|_{L_t^{\hat{r}} L_x^n} + \|\phi_i \phi_j \phi_i\|_{L_t^4 L_x^2}, \end{aligned}$$

where the integration domains of the norms $L_t^p L_x^q$ are $[0, T] \times \mathbb{R}^2$. For illustration, we only bound $\|a_t \phi_j\|_{L_t^{\hat{r}} L_x^n}$. From (CGL), for any $p \in (2, \infty)$, we have

$$\begin{aligned} \|a_j\|_{L_x^p} & \sum_{k=1}^2 \|\phi_k \phi_j\|_{L_x^{p^*}} \\ \|a_t\|_{L_x^p} & \sum_{k=1}^2 \|\nabla^2 u\| |\nabla u| + |\nabla u|^2 |a_k| \|_{L_x^{p^*}}, \end{aligned}$$

where $\frac{1}{p^*} + \frac{1}{2} = 1 + \frac{1}{p}$. Therefore,

$$\begin{aligned} \|a_t \phi_j\|_{L_x^n} & \leq \|a_t\|_{L_x^p} \|\phi_j\|_{L_x^{(n,p)}} \\ & \leq \|\nabla^2 u\|_{L_x^2} \|\nabla u\|_{L_x^{(p^*,2)}} \|\nabla u\|_{L_x^{(n,p)}} + \|\nabla u\|_{L_x^4}^2 \|\phi_j \phi_k\|_{L_x^{(p^*,2)^*}} \|\nabla u\|_{L_x^{(n,p)}} \\ & \leq \left(\int_0^T \|\nabla^2 u\|_{L_x^2}^{\left(3 - \frac{2}{p} - \frac{2}{(n,p)}\right)\hat{n}} ds \right)^{1/\hat{n}}. \end{aligned}$$

Since $\left(3 - \frac{2}{p} - \frac{2}{(n,p)}\right)\hat{n} = 2$, (6) yields the acceptable bound for $a_t \phi_j$

$$\|a_t \phi_j\|_{L_t^{\hat{r}} L_x^n([0, T] \times \mathbb{R}^2)} \leq [\varepsilon^2 + (C^* \varepsilon^2)^4]^{\frac{1}{\hat{n}}}.$$

Other terms can be estimates similarly. Thus we have

$$\|\phi_j\|_{L_t^\infty L_x^2} \leq \varepsilon + \varepsilon^2 + (C^* \varepsilon)^4 + (\varepsilon^2 + (C^* \varepsilon)^4)^{\frac{1}{\hat{n}}}.$$

Then first choosing C^* sufficiently large, then taking ε sufficiently small, we obtain (5). Thus Lemma 13 follows.

Our next aim is to consider the Landau-Lifshitz flow from curved spaces. Usually the dynamics for flows defined on the Euclidean space and curved space are typically different and of independent interest. For the heat flow, $u : [0, T] \times \mathbb{H}^n \rightarrow \mathbb{H}^n$ is of particular interest because it is closely related to the Schoen-Li-Wang conjecture, namely any quasiconformal boundary map gives rise to a harmonic map of hyperbolic space. (see for instance Lemm, Markovic [22]) The wave map dynamics on curved space especially hyperbolic space were studied in the sequel works of Lawrie, Oh, Shahshahani [18, 19, 21].

Let (M, h) be a Riemannian manifold and (N, J, g) be a Kähler manifold, the Landau-Lifshitz flow is a map $u(x, t) : M \times [0, \infty) \rightarrow N$ satisfying

$$\begin{cases} u_t = \alpha \tau(u) - \beta J(u) \tau(u) \\ u|_{t=0} = u_0(x), \end{cases} \quad (7)$$

where $\alpha \geq 0$, $\beta \in \mathbb{R}$. In the local coordinates (x_1, x_2) for M and (y_1, y_2) for N , $\tau(u)$ is given by

$$\tau(u) = \left(\Delta_{\mathbb{H}^2} u^l + h^{ij} \bar{\Gamma}_{m,n}^l(u) \frac{\partial u^m}{\partial x^i} \frac{\partial u^n}{\partial x^j} \right) \frac{\partial}{\partial y^l},$$

where $h_{ij}dx^i dx^j$ is the metric tensor for M , (h^{ij}) is its inverse, $\bar{\Gamma}_{m,n}^l(u)$ is the Christoffel symbol at u . In the following, we consider the case when $M = \mathbb{H}^2$ and $N = \mathbb{H}^2$.

In order to state our main theorem, we introduce some notions.

Definition 14. We say $Q : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ is an admissible harmonic map if $\overline{Q(\mathbb{H}^2)}$ is compact and $\nabla^k Q \in L^2(\mathbb{H}^2)$ for $k \in \{1, 2, 3\}$. For this Q , the space \mathcal{H}_Q^3 is the extrinsic Sobolev space. Denote the union of \mathcal{H}_Q^3 with Q ranging over all the admissible harmonic maps by \mathbb{H}^3 .

Remark 15. We remark that any map which coincides with Q outside of a compact subset of \mathbb{H}^2 is a member of \mathcal{H}_Q^3 . Any analytic function $f : \mathbb{C} \rightarrow \mathbb{C}$ with $f(\mathbb{D}) \subset \mathbb{D}$ where \mathbb{D} is the Poincare disk is an admissible harmonic map. The harmonic map studied in Lawrie, Oh, Shahshahani [20] is in fact $f(z) = \lambda z$ with $\lambda \in (0, 1)$.

Theorem 16 ([25]). *Let $\alpha > 0$, $\beta \in \mathbb{R}$. For any initial data $u_0 \in \mathbb{H}^3$, there exists a global solution to (7) and as $t \rightarrow \infty$, $u(t, x)$ converges to some harmonic map $Q_\infty : \mathbb{H}^2 \rightarrow \mathbb{H}^2$, namely*

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{H}^2} d_{\mathbb{H}^2}(u(t, x), Q_\infty(x)) = 0.$$

Remark 17. The limit harmonic map Q_∞ in Theorem 1.1 is in fact Q if $u_0 \in \mathcal{H}_Q^3$. This is related to the uniqueness of harmonic maps with prescribed boundary harmonic map.

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