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SPATIAL BEHAVIOR FOR NLS AND APPLICATIONS TO SCATTERING

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ABSTRACT. We review recent results on the nonlinear Schrödinger equation

$$iu_t + \Delta u + \lambda|u|^\alpha u = 0$$

where $\lambda \in \mathbb{C}$ and $\alpha > 0$. In any space dimension $N \geq 1$ and for any $\alpha > 0$, we construct a class of (arbitrarily large) initial values for which there exists a local solution. Moreover, if $\alpha > 2/N$, we construct a class of (arbitrarily large) initial values for which there exists a global solution that scatters as $t \rightarrow \infty$. If $\alpha = 2/N$ and $\Im \lambda \leq 0$, we construct a class of (arbitrarily large) initial values for which there exists a global solution, of which we give a precise asymptotic expansion as $t \rightarrow \infty$ (of modified scattering type). These results rely on the construction of solutions that do not vanish, so as to avoid any issue related to the lack of regularity of the nonlinearity at $u = 0$. To study the asymptotic behavior, we apply the pseudo-conformal transformation. This yields the desired asymptotic behavior if $\alpha > 2/N$. In the case $\alpha = 2/N$, a further step is required, and we estimate the solutions by allowing a certain growth of the Sobolev norms, which depends on the order of regularity through a cascade of exponents.

In this note, we review recent results [6, 7] on the local Cauchy problem and the asymptotic behavior of solutions for the nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + \lambda|u|^\alpha u = 0 \\ u(0, x) = u_0 \end{cases} \quad (1)$$

on \mathbb{R}^N , where $\alpha > 0$ and $\lambda \in \mathbb{C}$.

The Cauchy problem (1) is locally well-posed in L^2 if $\alpha \leq 4/N$ (see [34, 9]), in H^1 if $\alpha \leq 4/(N-2)$ (see [13, 9]), and in H^2 if $\alpha \leq 4/(N-4)$ (see [19, 9, 5]). More generally, (1) is locally well-posed in H^s if

$$\text{either } s \geq \frac{N}{2}, \text{ or } 0 \leq s < \frac{N}{2} \text{ and } \alpha \leq \frac{4}{N-2s} \quad (2)$$

(see [8, 14, 20, 24]), but under the additional condition

$$\alpha > [s] \quad (3)$$

if $s > 1$ and α is not an even integer. (Here, $[s]$ the integer part of s .) Condition (3) appears because solutions are constructed by a fixed-point argument, for which one is led to estimate derivatives of order up to s of $|u|^\alpha u$. Indeed, even if u is smooth, $|u|^\alpha u$ need not be smooth if α is not an even integer. For instance in dimension 1, $u(x) = xe^{-x^2}$ belongs to $H^\infty(\mathbb{R})$, but if $0 < \alpha \leq 1/2$, then $|u|^\alpha u \notin H^2(\mathbb{R})$. Assumption (3) ensures that the map $u \mapsto |u|^\alpha u$ is sufficiently smooth for

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the appropriate estimates to hold. Condition (3) was improved in certain cases, see [27, 11], but not eliminated except for $s \leq 2$. We emphasize the fact that, even if condition (3) is not optimal, some regularity condition like (3) is necessary. For instance it follows from [4, Theorem 1.5] that if $0 < \alpha < 1$ and $s > 3 + \alpha + N/2$, then the Cauchy problem (1) is ill posed in $H^s(\mathbb{R}^N)$, even though condition (2) is clearly satisfied.

If α is sufficiently small ($\alpha \leq 4/(N-4)$, say), then one can apply the local L^2 , H^1 or H^2 theories, which do not require any regularity condition like (3). On the other hand, if α is sufficiently large (say, $\alpha > N/2$), then the local H^s theory applies for some $s > N/2$. However, in large dimensions there is a gap for intermediate values of α for which none of these theories apply. For instance, it seems that no available local theory applies if $N = 12$ and $\alpha = 1$. (Except for the case $\lambda \in \mathbb{R}$ and $\lambda < 0$, where the existence of a global weak solution for $u_0 \in H^1(\mathbb{R}^N) \cap L^{\alpha+2}(\mathbb{R}^N)$ follows from compactness arguments, see [30, 32].)

Our first goal is to establish a local existence result for (1) that applies in any dimension $N \geq 1$ and for any $\alpha > 0$, for an appropriate class of initial data u_0 . The following observation is crucial: Since the possible defect of smoothness of the nonlinearity $|u|^\alpha u$ is only at $u = 0$, there is no obstruction to regularity for a solution that does not vanish. This suggests to look for such solutions.

In order to determine an appropriate class of initial values, we consider the linear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = 0 \\ u(0, x) = u_0(x) \end{cases} \quad (4)$$

with an initial value $u_0 \in L^2(\mathbb{R}^N)$ such that¹ $\inf \langle x \rangle^n |u_0(x)| > 0$, where $n > N/2$ (so that $\langle x \rangle^{-n} \in L^2(\mathbb{R}^N)$). We want to estimate $\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u(t, x)|$ and we note that

$$u(t) = u_0 + i \int_0^t \Delta u(s) ds; \quad (5)$$

so that

$$\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u(t, x)| \geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u_0(x)| - t \|\langle x \rangle^n \Delta u\|_{L^\infty((0,t) \times \mathbb{R}^N)}. \quad (6)$$

We now must estimate the last term on the right-hand side of (6). We cannot simply use Sobolev's embedding $H^s \hookrightarrow L^\infty$ for $s > N/2$. Indeed, this would require in particular $\langle x \rangle^n \Delta u_0 \in L^2(\mathbb{R}^N)$. In the model case $u_0 = \langle \cdot \rangle^{-n}$, this means $\langle \cdot \rangle^{-2} \in L^2(\mathbb{R}^N)$, which fails if $N \geq 4$. On the other hand, still for $u_0 = \langle \cdot \rangle^{-n}$, we see that $|\langle x \rangle^n \Delta^{m+1} u_0| \leq C \langle x \rangle^{-2m-2}$, which belongs to $L^2(\mathbb{R}^N)$ if m is sufficiently large. Therefore, we estimate the last term on the right-hand side of (6) by applying Taylor's formula with integral remainder

$$u(t) = \sum_{j=0}^m \frac{(it)^j}{j!} \Delta^j u_0 + \frac{i^{m+1}}{m!} \int_0^t (t-s)^m \Delta^{m+1} u(s)$$

with m sufficiently large (for instance $m > N/2$). Applying the Laplacian, we obtain

$$\begin{aligned} \|\langle x \rangle^n \Delta u(t)\|_{L^\infty} &\leq \sum_{j=1}^{l+1} \frac{(it)^j}{j!} \|\langle x \rangle^n \Delta^j u_0\|_{L^\infty} \\ &\quad + \frac{i^{m+1}}{m!} \int_0^t (t-s)^m \|\langle x \rangle^n \Delta^{m+2} u(s)\|_{L^\infty}. \end{aligned} \quad (7)$$

¹We use the notation $\langle x \rangle = \sqrt{1 + |x|^2}$

We now can estimate $\|\langle x \rangle^n \Delta^{m+2} u(t)\|_{H^s}$ with $s > N/2$ by energy estimates. More precisely, we apply a derivative D^β of order $2m + 4$ to (4) and we multiply by $\langle x \rangle^{2n} D^\beta \bar{u}$. Integrating by parts yields

$$\frac{1}{2} \frac{d}{dt} \|\langle x \rangle^n D^\beta u\|_{L^2}^2 = -\Im \int D^\beta \bar{u} \nabla D^\beta u \cdot \nabla \langle x \rangle^{2n}.$$

Since $|\nabla \langle x \rangle^{2n}| \leq C \langle x \rangle^{2n-1}$, we deduce by applying Cauchy-Schwarz that

$$\frac{d}{dt} \|\langle x \rangle^n D^\beta u\|_{L^2} \leq C \|\langle x \rangle^{n-1} \nabla D^\beta u\|_{L^2}.$$

We now can iterate this estimate. Every such integration by parts will decrease by 1 the power of $\langle x \rangle$ which is involved in the estimate, but will at the same time increase by 1 the number of derivatives. At the last step, it only remains to estimate a derivative of u with no weight, and this is a standard energy estimate. Thus we see that we can obtain an estimate of $\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u(t, x)|$ for t sufficiently small by using (6), (7) and energy estimates. The requirements on the initial value u_0 are that $\langle x \rangle^n \partial^\beta u_0 \in L^\infty(\mathbb{R}^N)$ for $|\beta| \leq 2m$ for some sufficiently large m , that $\langle x \rangle^n \partial^\beta u_0 \in L^2(\mathbb{R}^N)$ for $2m + 1 \leq |\beta| \leq 2m + 2 + k$ with k sufficiently large, and then $\langle x \rangle^{n+2m+2+k-|\beta|} \partial^\beta u_0 \in L^2(\mathbb{R}^N)$ for $2m + 3 + k \leq |\beta| \leq 2m + 2 + k + n$.

The above calculations motivate the following definition. We fix $\alpha > 0$, we consider three integers k, m, n such that

$$k > \frac{N}{2}, \quad n > \max\left\{\frac{N}{2} + 1, \frac{N}{2\alpha}\right\}, \quad 2m \geq k + n + 1 \quad (8)$$

and we let

$$J = 2m + 2 + k + n. \quad (9)$$

We define the space \mathcal{X} by

$$\begin{aligned} \mathcal{X} = \{ & u \in H^J(\mathbb{R}^N); \langle x \rangle^n D^\beta u \in L^\infty(\mathbb{R}^N) \text{ for } 0 \leq |\beta| \leq 2m, \\ & \langle x \rangle^n D^\beta u \in L^2(\mathbb{R}^N) \text{ for } 2m + 1 \leq |\beta| \leq 2m + 2 + k, \\ & \langle x \rangle^{J-|\beta|} D^\beta u \in L^2(\mathbb{R}^N) \text{ for } 2m + 2 + k < |\beta| \leq J\} \end{aligned} \quad (10)$$

and we equip \mathcal{X} with the norm

$$\|u\|_{\mathcal{X}} = \sum_{j=0}^{2m} \sup_{|\beta|=j} \|\langle x \rangle^n D^\beta u\|_{L^\infty} + \sum_{\nu=0}^{k+1} \sum_{\mu=0}^n \sum_{|\beta|=\nu+\mu+2m+1} \|\langle x \rangle^{n-\mu} D^\beta u\|_{L^2} \quad (11)$$

so that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach space. Calculations similar to those sketched above show that, given $\psi \in \mathcal{X}$, the map² $t \mapsto e^{it\Delta} \psi$ is continuous $\mathbb{R} \rightarrow \mathcal{X}$. Moreover, there exists a constant C_1 such that

$$\|e^{it\Delta} \psi\|_{\mathcal{X}} \leq C_1 (1 + |t|)^{m+n+1} \|\psi\|_{\mathcal{X}} \quad (12)$$

for all $t \in \mathbb{R}$ and all $\psi \in \mathcal{X}$. Estimates (6) and (12) imply that if $u_0 \in \mathcal{X}$, then

$$\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |e^{it\Delta} u_0| \geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u_0(x)| - C_1 t (1 + |t|)^{m+n+1} \|u_0\|_{\mathcal{X}}. \quad (13)$$

In particular, if

$$\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u_0(x)| > 0 \quad (14)$$

then

$$\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |e^{it\Delta} u_0| > 0 \quad (15)$$

for all sufficiently small t . See [6, Proposition 1] for details and complete proofs of the above statements.

²We denote by $(e^{it\Delta})_{t \in \mathbb{R}}$ the Schrödinger group on \mathbb{R}^N

Estimate (15) indicates that \mathcal{X} might be a relevant space for solving the Schrödinger equation (1) by a perturbation argument. Indeed, if u_0 satisfies (14), then $u(t) = e^{it\Delta}u_0$ does not vanish for t small, so that there is no obstruction in differentiating $|u|^\alpha u$. In fact, one can prove that if $u \in \mathcal{X}$ satisfies (14) and if $\eta > 0$ is sufficiently large so that

$$\eta \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u(x)| \geq 1 \quad (16)$$

then $|u|^\alpha u \in \mathcal{X}$. Moreover, there exists a constant C_2 such that

$$\| |u|^\alpha u \|_{\mathcal{X}} \leq C_2 (1 + \eta \|u\|_{\mathcal{X}})^{2J} \|u\|_{\mathcal{X}}^{\alpha+1} \quad (17)$$

for all $u \in \mathcal{X}$ satisfying (16). In addition, if $u_1, u_2 \in \mathcal{X}$ both satisfy (16), then

$$\begin{aligned} & \| |u_1|^\alpha u_1 - |u_2|^\alpha u_2 \|_{\mathcal{X}} \\ & \leq C_2 ((1 + \eta(\|u_1\|_{\mathcal{X}} + \|u_2\|_{\mathcal{X}}))^{2J+1} (\|u_1\|_{\mathcal{X}} + \|u_2\|_{\mathcal{X}})^\alpha \|u_1 - u_2\|_{\mathcal{X}}). \end{aligned} \quad (18)$$

The proof of (17)-(18) follows from elementary (but tedious) calculations based on the fact that if $|\beta| \geq 2$, then the development of $D^\beta(|u|^\alpha u)$ contains on the one hand the term

$$A = \left(1 + \frac{\alpha}{2}\right) |u|^\alpha D^\beta u + \frac{\alpha}{2} |u|^{\alpha-2} u^2 D^\beta \bar{u}, \quad (19)$$

and on the other hand, terms of the form

$$B = |u|^{\alpha-2p} D^\rho u \prod_{j=1}^p D^{\gamma_{1,j}} u D^{\gamma_{2,j}} \bar{u} \quad (20)$$

where $\gamma + \rho = \beta$, $1 \leq p \leq |\gamma|$, $|\gamma_{1,j} + \gamma_{2,j}| \geq 1$, $\sum_{j=0}^p (\gamma_{1,j} + \gamma_{2,j}) = \gamma$, and $|\gamma_{i,j}| \leq |\beta| - 1$ for $i = 1, 2$. See [6, Proposition 2] for details.

A standard fixed-point argument, based on Duhamel's formula and on estimates (12), (17) and (18) yields the following local well-posedness result. (This is [6, Theorem 1], except for the blowup alternative (22).)

Theorem 1. *Let $N \geq 1$, $\alpha > 0$ and $\lambda \in \mathbb{C}$. Assume (8)-(9) and let \mathcal{X} be defined by (10)-(11). If $u_0 \in \mathcal{X}$ satisfies (14), then there exist $T > 0$ and a unique solution $u \in C([0, T], \mathcal{X})$ of (1) satisfying*

$$\inf_{0 \leq t \leq T} \inf_{x \in \mathbb{R}^N} (\langle x \rangle^n |u(t, x)|) > 0. \quad (21)$$

Moreover, u can be extended on a maximal existence interval $[0, T_{\max})$ with $0 < T_{\max} \leq \infty$ to a solution $u \in C([0, T_{\max}), \mathcal{X})$ satisfying (21) for all $0 < T < T_{\max}$; and if $T_{\max} < \infty$, then

$$\|u(t)\|_{\mathcal{X}} + \left(\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u(t, x)| \right)^{-1} \xrightarrow{t \uparrow T_{\max}} \infty. \quad (22)$$

Sketch of the proof of Theorem 1. We write equation (1) in the equivalent form

$$u(t) = e^{it\Delta} u_0 + i\lambda \int_0^t e^{i(t-s)\Delta} (|u|^\alpha u)(s) ds, \quad (23)$$

so we look for a fixed point of the map Φ defined by

$$\Phi(u)(t) = e^{it\Delta} u_0 + i\lambda \int_0^t e^{i(t-s)\Delta} (|u|^\alpha u)(s) ds$$

on some appropriate set. We let $T, \eta, M > 0$ and we define the set \mathcal{E} by

$$\mathcal{E} = \left\{ u \in C([0, T], \mathcal{X}); \sup_{0 \leq t \leq T} \|u(t)\|_{\mathcal{X}} \leq M \text{ and } \eta \inf_{\substack{x \in \mathbb{R}^N \\ 0 \leq t \leq T}} \langle x \rangle^n |u(t, x)| \geq 1 \right\}$$

It easily follows from (6), (12), (17) and (18) that

$$\begin{aligned} \|\Phi(u)\|_{L^\infty((0,T),\mathcal{X})} &\leq K_T\|u_0\|_{\mathcal{X}} + T|\lambda|K_T(1+\eta M)^{2J}M^{\alpha+1} \\ \|\Phi(u) - \Phi(v)\|_{L^\infty((0,T),\mathcal{X})} &\leq T|\lambda|K_T(1+2\eta M)^{2J+1}(2M)^\alpha\|u-v\|_{L^\infty((0,T),\mathcal{X})} \\ \inf_{\substack{x \in \mathbb{R}^N \\ 0 \leq t \leq T}} \langle x \rangle^n |\Phi(u)(t,x)| &\geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u_0(x)| - TK_T(\|u_0\|_{\mathcal{X}} + |\lambda|(1+\eta M)^{2J}M^{\alpha+1}) \end{aligned}$$

for every $u, v \in \mathcal{E}$, where

$$K_T = (1+C_1)(1+C_2)(1+T)^{m+n+1} \quad (24)$$

Given $T > 0$ and $u_0 \in \mathcal{X}$ such that (14) holds, we let

$$\eta = 2 \left(\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u_0(x)| \right)^{-1} \quad (25)$$

$$M = 2K_T\|u_0\|_{\mathcal{X}}. \quad (26)$$

In particular, if $u(t) \equiv u_0$, then $u \in \mathcal{E}$ so that $\mathcal{E} \neq \emptyset$. It follows easily from the above estimates that if

$$T \left(\eta + (1+\eta)|\lambda|K_T(1+2\eta M)^{2J+1}(2M)^\alpha(1+M) \right) \leq \frac{1}{2} \quad (27)$$

then Φ is a strict contraction $\mathcal{E} \rightarrow \mathcal{E}$. Thus Φ has a fixed point $u \in \mathcal{E}$, which is a solution of (23). Since $u \in \mathcal{E}$, u satisfies (21). Uniqueness easily follows from (18) and Gronwall's inequality. As a matter of fact, $\mathcal{X} \hookrightarrow L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, so uniqueness also follows from [20, Theorem 2.1]. Note that (27) is achieved if $T > 0$ is sufficiently small. The extension of the solution to a maximum interval and the blowup alternative follow from standard arguments, see [7, Proposition 4.1]. \square

Remark 2. Here are some comments on Theorem 1.

- (i) The space \mathcal{X} is determined by the parameters k, n, m , which can be chosen arbitrarily, as long as they are sufficiently large to satisfy assumption (8).
- (ii) Since $\mathcal{X} \hookrightarrow L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, it makes sense to say that $u \in C([0, T], \mathcal{X})$ is a solution of (1). See e.g. [20, Section 2].
- (iii) Theorem 1 shows the existence of solutions of (1) for $0 \leq t \leq T$. The existence of solutions for negative t also follows from Theorem 1. Indeed, $u(t)$ is a solution of (1) on $[0, T]$, if and only if $\bar{u}(-t)$ is a solution for $-T \leq t \leq 0$ of equation (1) with λ replaced by $\bar{\lambda}$. Note that changing u_0 to \bar{u}_0 does not change the \mathcal{X} -norm, nor the left-hand side of (14). Therefore, if $u_0 \in \mathcal{X}$ satisfies (14), then $\bar{u}_0 \in \mathcal{X}$ and \bar{u}_0 also satisfies (14). Therefore, Theorem 1 yields a solution of (1) for $-T \leq t \leq T$ for some $T > 0$.
- (iv) It is immediate that $\mathcal{S}(\mathbb{R}^N) \subset \mathcal{X}$. Furthermore, it is not difficult to show that $\langle x \rangle^{-p} \in \mathcal{X}$ if $p \geq n$. Therefore, if $u_0 = c(\langle \cdot \rangle^{-n} + \varphi)$ with $c \in \mathbb{C}$, $c \neq 0$, $\varphi \in \mathcal{S}(\mathbb{R}^N)$, and $\|\langle x \rangle^n \psi\|_{L^\infty} < 1$, then $u_0 \in \mathcal{X}$ and u_0 satisfies (14). In particular, Theorem 1 applies to such initial values.

We now discuss the low-energy scattering problem. It is a natural conjecture that if $\alpha > 2/N$, then small initial values (in an appropriate sense) give rise to global solutions of (1) that are asymptotically free, i.e. $u(t) \sim e^{it\Delta}u^+$ as $t \rightarrow \infty$ (in some norm) for some asymptotic state u^+ . This property is known in dimension $N = 1, 2, 3$, see [33, 10, 12, 25]. However, in larger dimension, the available methods leave a gap. This gap is not only due to the limitations discussed above, but also concerns values of α close to $2/N$, for which local existence is not an issue. The reason for this difficulty is better seen by using the pseudo-conformal transformation. Let

$u \in C([0, \infty), L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$, $b > 0$, $v \in C([0, 1/b), L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$, and suppose that

$$u(t, x) = (1 + bt)^{-N/2} e^{ib|x|^2/4(1+bt)} v\left(\frac{t}{1+bt}, \frac{x}{1+bt}\right) \quad (28)$$

for $t > 0$ and $x \in \mathbb{R}^N$. It follows that u is a solution of (1) for $t > 0$ if and only if v is a solution of the non-autonomous equation

$$\begin{cases} iv_t + \Delta v + \lambda(1 - bt)^{-(4-N\alpha)/2} |v|^\alpha v = 0 \\ v(0, x) = v_0(x) \end{cases} \quad (29)$$

or its equivalent integral formulation

$$v(t) = e^{it\Delta} v_0 + i\lambda \int_0^t (1 - bs)^{-(4-N\alpha)/2} e^{i(t-s)\Delta} |v(s)|^{2/N} v(s) ds \quad (30)$$

for $0 < t < 1/b$. In addition, $u \in C([0, \infty), \Sigma)$ if and only if $v \in C([0, 1/b), \Sigma)$, where

$$\Sigma = H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx) \quad (31)$$

and $e^{-t\Delta} u(t)$ has a limit in Σ as $t \rightarrow \infty$ if and only if $v(t)$ has a limit in Σ as $t \rightarrow 1/b$. Therefore, an asymptotically free solution u of (1) corresponds, via the pseudo-conformal transformation (28), to a solution v of (29) that exists up to $t = 1/b$. (See e.g. [10, Section 3].) The problem is then to solve (29) on $[0, 1/b)$. If $\alpha > 2/N$, then the non-autonomous factor $(1 - bt)^{-(4-N\alpha)/2}$ in (29) may be singular at $t = 1/b$ (if $\alpha < 4/N$), but is integrable. If we use the approach used for proving Theorem 1, we obtain the following result.

Theorem 3 ([6], Theorem 2). *Let $N \geq 1$, $\alpha > 2/N$ and $\lambda \in \mathbb{C}$. Assume (8)-(9), let \mathcal{X} be defined by (10)-(11) and Σ by (31). Let $\varphi \in \mathcal{X}$ satisfy (14), and let $u_0 = e^{ib|x|^2/4} \varphi$, where $b > 0$. If b is sufficiently large, then there exists a unique, global solution $u \in C([0, \infty), \Sigma) \cap L^\infty((0, \infty) \times \mathbb{R}^N)$ of (1). Moreover u scatters, i.e. there exists $u^+ \in \Sigma$ such that $e^{-it\Delta} u(t) \rightarrow u^+$ in Σ as $t \rightarrow \infty$. In addition, $\sup_{t \geq 0} (1+t)^{N/2} \|u(t)\|_{L^\infty} < \infty$.*

Sketch of the proof of Theorem 3. We solve equation (30) on $[0, 1/b]$ with $v_0 = \varepsilon \varphi$. This amounts to finding a fixed point, in some appropriate set, of the map Φ defined by

$$\Phi(v)(t) = e^{it\Delta} u_0 + i\lambda \int_0^t (1 - bs)^{-(4-N\alpha)/2} e^{i(t-s)\Delta} (|v|^\alpha v)(s) ds.$$

Arguing as in the proof of Theorem 1, we obtain a solution $v \in C([0, 1/b], \mathcal{X})$ provided

$$\frac{2}{b(N\alpha - 2)} \left(\eta + (1 + \eta) |\lambda| K_{1/b} (1 + 2\eta M)^{2J+1} (2M)^\alpha (1 + M) \right) \leq \frac{1}{2}. \quad (32)$$

(Indeed, $T = \int_0^T ds$ in (27) has to be replaced by $\int_0^{1/b} (1 - bs)^{-(4-N\alpha)/2} ds = 2/b(N\alpha - 2)$.) We see that (32) is satisfied if $b > 0$ is sufficiently large. Since $v \in C([0, 1/b], \mathcal{X})$, the corresponding u given by (28) is in $C([0, \infty), \Sigma) \cap L^\infty((0, \infty) \times \mathbb{R}^N)$ and scatters as $t \rightarrow \infty$. The property $\sup_{t \geq 0} (1+t)^{N/2} \|u(t)\|_{L^\infty} < \infty$ follows from the boundedness of v and formula (28). \square

Remark 4. Here are some comments on Theorem 3.

- (i) It follows from Remark 2 (iv) that Theorem 3 applies to the initial value $u_0 = z e^{ib|x|^2/4} (\langle x \rangle^{-n} + \psi)$ where $n > \max\{(N/2) + 1, N/2\alpha\}$, $\psi \in \mathcal{S}(\mathbb{R}^N)$ satisfies $\|\langle x \rangle^n \psi\|_{L^\infty} < 1$, $z \in \mathbb{C}$ and $b > 0$, provided b is sufficiently large.
- (ii) There are no restrictions on the size of the initial value in Theorem 3. Instead, b must be large.

- (iii) Theorem 3 does not say anything on what happens to the solution u for $t < 0$. In fact, one cannot in general expect that the initial values considered in Theorem 3 give rise to global solutions for negative times. See [6, Remark 1.4 (ix)].
- (iv) We can apply Theorem 3 to construct solutions of (1) that exist for all $t < 0$ and scatter as $t \rightarrow -\infty$. Indeed, it suffices to apply Theorem 3 to equation (1) with λ replaced by $\bar{\lambda}$. If u_0 satisfies the assumptions of Theorem 3 (for $\bar{\lambda}$) and u is the corresponding solution, then we see that $v(t) = \bar{u}(-t)$ is a solution of (1) (with λ) for $t < 0$, which scatters as $t \rightarrow -\infty$, and with initial value \bar{u}_0 . Of course, one cannot expect in general that v is global for positive times, since this would mean that u is global for negative times. (See (iii) above.)

If $\alpha \leq 2/N$, then scattering (including low energy scattering) cannot be expected, see Strauss [31], Theorem 3.2 and Example 3.3, p. 68. See also [1] for the one-dimensional case. Therefore,

$$\alpha = 2/N \quad (33)$$

is a limiting case, for which the relevant notion is modified scattering, i.e. standard scattering modulated by a phase. When $\Im\lambda = 0$, the existence of modified wave operators was established in [26] in dimension $N = 1$. More precisely, for all sufficiently small asymptotic state u^+ , there exists a solution of (1) which behaves as $t \rightarrow \infty$ like $e^{i\phi(t,\cdot)} e^{t\Delta} u^+$, where the phase ϕ is given explicitly in terms of u^+ . (See also [2]. See [18, 29] for extensions in dimension $N = 2$.) Conversely, for small initial values, it was proved in [15] that the asymptotic behavior of the corresponding solution has this form when $\Im\lambda = 0$, in dimensions $N = 1, 2, 3$. (See also [21].) If $\Im\lambda < 0$, then the nonlinearity has some dissipative effect, and an extra log decay (and also a log correction in the phase) appears in the description of the asymptotic behavior of the solutions. This was established in space dimensions $N = 1, 2, 3$ in [28]. (See also [16, 17] for related results.)

Our main results in the case (33) are the following.

Theorem 5 ([7], Theorem 1.1). *Let $N \geq 1$, $\alpha = 2/N$ and*

$$\lambda \in \mathbb{R}.$$

Assume (8), (9), let \mathcal{X} be defined by (10)-(11), and Σ by (31). Suppose that $u_0(x) = e^{ib|x|^2/4} v_0(x)$, where $b > 0$, and $v_0 \in \mathcal{X}$ satisfies (14). If b is sufficiently large, then there exists a unique, global solution u in the class $C([0, \infty), \Sigma) \cap L^\infty((0, \infty) \times \mathbb{R}^N) \cap L^\infty((0, \infty), H^1(\mathbb{R}^N))$ of (1). Moreover, there exist $\delta > 0$ and $w_0 \in L^\infty(\mathbb{R}^N)$ with $\langle \cdot \rangle^n w_0 \in L^\infty(\mathbb{R}^N)$ and $h \neq 0$ such that

$$\|u(t, \cdot) - z(t, \cdot)\|_{L^2} + (1+t)^{N/2} \|u(t, \cdot) - z(t, \cdot)\|_{L^\infty} \leq C(1+t)^{-\delta} \quad (34)$$

where

$$z(t, x) = (1+bt)^{-N/2} e^{i\Phi(t,\cdot)} w_0\left(\frac{\cdot}{1+bt}\right)$$

and

$$\Phi(t, x) = \frac{b|x|^2}{4(1+bt)} + \frac{\lambda}{b} \left| w_0\left(\frac{x}{1+bt}\right) \right|^{2/N} \log(1+bt).$$

In addition,

$$t^{N/2} \|u(t)\|_{L^\infty} \xrightarrow[t \rightarrow \infty]{} b^{-N/2} \|w_0\|_{L^\infty}. \quad (35)$$

Theorem 6 ([7], Theorem 1.2). *Let $N \geq 1$ and*

$$\lambda \in \mathbb{C} \text{ with } \Im\lambda > 0.$$

Assume (33), (8), (9), let \mathcal{X} be defined by (10)-(11), and Σ by (31). Suppose $u_0(x) = e^{ib|x|^2/4} v_0(x)$, where $b > 0$, and $v_0 \in \mathcal{X}$ satisfies (14). If b is sufficiently large, then there exists a unique, global solution $u \in C([0, \infty), \Sigma) \cap L^\infty((0, \infty) \times \mathbb{R}^N) \cap$

$L^\infty((0, \infty), H^1(\mathbb{R}^N))$ of (1). Moreover, there exist $\delta > 0$ and $f_0, w_0 \in L^\infty$, with f_0 real valued, $\|f_0\|_{L^\infty} \leq 1/2$, $w_0 \neq 0$ and $\langle \cdot \rangle^n w_0 \in L^\infty(\mathbb{R}^N)$ such that

$$\|u(t, \cdot) - z(t, \cdot)\|_{L^2} + (1+t)^{N/2} \|u(t, \cdot) - z(t, \cdot)\|_{L^\infty} \leq C(1+t)^{-\delta} \quad (36)$$

where

$$z(t, x) = (1+bt)^{-N/2} e^{i\Theta(t, \cdot)} \Psi\left(t, \frac{\cdot}{1+bt}\right) w_0\left(\frac{\cdot}{1+bt}\right)$$

with

$$\Theta(t, x) = \frac{b|x|^2}{4(1+bt)} - \frac{\Re\lambda}{\Im\lambda} \log\left(\Psi\left(t, \frac{x}{1+bt}\right)\right)$$

and

$$\Psi(t, y) = \left(\frac{1+f_0(y)}{1+f_0(y) + (2|\Im\lambda|/Nb)|v_0(y)|^{2/N} \log(1+bt)} \right)^{N/2}.$$

In addition,

$$(t \log t)^{N/2} \|u(t)\|_{L^\infty} \xrightarrow{t \rightarrow \infty} (\alpha |\Im\lambda|)^{-N/2}. \quad (37)$$

The proofs of Theorems 5 and 6 are unfortunately rather technical, so we only give a very brief sketch below. To prove Theorems 5 and 6, we first apply the same strategy we use for proving Theorem 3. We apply the pseudo-conformal transformation (28), which yields equation (30). In the present case (33), equation (30) takes the form

$$v(t) = e^{it\Delta} v_0 + i\lambda \int_0^t (1-bs)^{-1} e^{i(t-s)\Delta} |v(s)|^{2/N} v(s) ds \quad (38)$$

In the case $\alpha > 2/N$, a solution v of (30) can be constructed on the interval $[0, 1/b)$ by a fixed point argument. In the present case (33), this argument cannot be applied since $(1-bt)^{-1}$ is not integrable at $1/b$. We therefore have to modify the arguments used in the proof of Theorem 3.

We fix $v_0 \in \mathcal{X}$ satisfying (14), and we let $u_0(x) = e^{ib|x|^2/4} v_0(x)$, where $b > 0$. We note that it suffices to construct a solution $v \in C([0, 1/b), \mathcal{X})$ of (38) which is not too singular as $t \uparrow 1/b$. The asymptotic behavior of v as $t \uparrow 1/b$ (hence the behavior of u as $t \rightarrow \infty$, where u is given by (28)) is determined by standard arguments. See [7, Sections 5 and 6].

Arguing as in the proof of Theorem 1, we construct a solution of (38) defined on a maximal interval $[0, T_{\max})$ with $T_{\max} \leq 1/b$. Moreover, if $T_{\max} < 1/b$, then v satisfies (22). (See [7, Proposition 4.1].) We then need to show that if b is sufficiently large, then $T_{\max} = 1/b$ and $v(t)$ satisfies appropriate estimates as $t \uparrow T_{\max}$.

Crucial in our analysis is the elementary estimate

$$\int_0^t (1-bs)^{-1-\mu} ds = \frac{1}{b\mu} [(1-bt)^{-\mu} - 1] \leq \frac{1}{b\mu} (1-bt)^{-\mu} \quad (39)$$

for every $\mu > 0$ and $t < 1/b$. Inequality (39) implies that if a certain norm of $e^{i(t-s)\Delta} |v|^\alpha v$ is estimated by $C(1-bt)^{-\sigma}$, then the same norm of the integral term in (38) is estimated by $(C/b\sigma)(1-bt)^{-\sigma}$. Our strategy is to allow the norms $\|\langle x \rangle^n D^\beta u\|_{L^\infty}$, $\|\langle x \rangle^n D^\beta u\|_{L^2}$, $\|\langle x \rangle^{J-|\beta|} D^\beta u\|_{L^2}$ to have a growth like $(1-bt)^{-\sigma}$ as $t \rightarrow 1/b$, with σ depending on the norm under consideration. This requires a refinement of both the linear estimate (12) and the nonlinear estimates (17) (16). For the linear estimate, one shows that the solution of $iv_t + \Delta v = f$, $v(0) = v_0$ satisfies

$$\|\langle x \rangle^\nu D^\beta v(t)\| \leq \|v_0\|_{\mathcal{X}} + C \int_0^t (\|v(s)\|_{\mathcal{X}} + \|\langle x \rangle^\nu D^\beta f(s)\|) ds \quad (40)$$

where $\|\cdot\|$ is either the L^∞ or L^2 norm, and $\nu = n$ or $\nu = J - |\beta|$. See [7, Proposition 2.1]. For the nonlinear estimate, one observes that $D^\beta(|v|^\alpha v)$ contains on the one hand terms estimated by $|v|^\alpha |D^\beta v|$, and on the other hand terms that

contain products of derivatives of lower order. See [7, Formulas (3.11) and (3.12)]. It follows that if u satisfies (16) for a certain $\eta > 0$, then

$$\| \langle x \rangle^\nu D^\beta (|v|^\alpha v) \| \leq C \|v\|_{L^\infty}^\alpha \| \langle x \rangle^\nu D^\beta v \| + W \quad (41)$$

where $\| \cdot \|$ is either the L^∞ or L^2 norm, and $\nu = n$ or $\nu = J - |\beta|$; and W depends on η and on norms like the one on the left-hand side, but involving derivatives of (strictly) lower order. (See [7, Proposition 3.1].) Assuming $\| \langle x \rangle^\nu D^\beta v \| \leq C(1 - bt)^{-\sigma_{|\beta|}}$ and $\inf \langle x \rangle^n |v(t, x)| \geq C(1 - bt)^{\tilde{\sigma}}$, we deduce from (40) and (41) that

$$\| \langle x \rangle^\nu D^\beta v \| \leq C + \frac{C}{b} \|v\|_{L^\infty}^\alpha (1 - bt)^{-\sigma_{|\beta|}} + \frac{C}{b} (1 - bt)^{-\mu(|\beta|)} \quad (42)$$

where $\mu(|\beta|)$ is a combination of $\tilde{\sigma}$ and $\sigma_{|\gamma|}$ with $|\gamma| < |\beta|$. We then construct by induction $0 = \sigma_0 < \sigma_1 < \dots < \sigma_J$ such that $\sigma_J < 1$ and $\sigma_\ell \geq \mu_\ell$. The factor $1/b$ in the right-hand side of (42) is what allows us to obtain an estimate of $(1 - bt)^{\sigma_{|\beta|}} \| \langle x \rangle^\nu D^\beta v \|$ provided b is sufficiently large. Using this estimate, we deduce that $T_{\max} = 1/b$, hence the desired conclusion. See [7, Proposition 4.3] for details.

Remark 7. Here are some comments on the above Theorems 5 and 6.

- (i) Like Theorem 3, Theorems 5 and 6 do not provide any information on the behavior of the solution for $t < 0$.
- (ii) It follows from Remark 2 (iv) that Theorems 5 and 6 apply to the initial value $u_0 = ce^{ib|x|^2/4}(\langle x \rangle^{-n} + \psi)$ where $n > \max\{(N/2) + 1, N/2\alpha\}$, $\psi \in \mathcal{S}(\mathbb{R}^N)$ satisfies $\| \langle x \rangle^n \psi \|_{L^\infty} < 1$, $c \in \mathbb{C}$ and $b > 0$, provided b is sufficiently large.
- (iii) One can express formula (34) in the form of the standard modified scattering, see [7, Remark 1.3 (vi)].

Remark 8. Here are some open questions related to Theorems 5 and 6.

- (i) We do not know what happens if $\Im \lambda > 0$. Let us observe that if $\alpha < 2/N$ and $\Im \lambda > 0$, then it follows from [3, Theorem 1.1] that every nontrivial solution of (1) either blows up in finite time or else is global with unbounded H^1 norm. The proof in [3] apparently does not apply to the case $\alpha = 2/N$. See also [7, Remark 4.4].
- (ii) If $\alpha < 2/N$ and $\Im \lambda \leq 0$, it seems that no precise description of the asymptotic behavior of the solutions of (1) is available. When $\lambda \in \mathbb{R}$, $\lambda > 0$, it is proved in [35] that all H^1 solutions converge strongly to 0 in $L^p(\mathbb{R}^N)$, for $2 < p < 2N/(N - 2)$, but even the rate of decay of these norms seems to be unknown.

Remark 9. The strategy of constructing solutions of (1) that do not vanish was adapted to the derivative Schrödinger equations [23], and to generalized KdV equations [22].

REFERENCES

- [1] Barab J.E. Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation, *J. Math. Phys.* **25** (1984), no. 11, 3270–3273. (MR0761850) (doi: 10.1063/1.526074)
- [2] Carles R. Geometric optics and long range scattering for one-dimensional nonlinear Schrödinger equations. *Comm. Math. Phys.* **220** (2001), no. 1, 41–67. (MR1882399) (doi: 10.1007/s002200100438)
- [3] Cazenave T., Correia S., Dickstein F. and Weissler F.B. A Fujita-type blowup result and low energy scattering for a nonlinear Schrödinger equation. *São Paulo J. Math. Sci.* **9** (2015), no. 2, 146–161. (MR3457455) (doi: 10.1007/s40863-015-0020-6)
- [4] Cazenave T., Dickstein F. and Weissler F.B. Non-regularity in Hölder and Sobolev spaces of solutions to the semilinear heat and Schrödinger equations. *Nagoya Math. J.* **226** (2017), 44–70. (MR3650971) (doi: 10.1017/nmj.2016.35)

- [5] Cazenave T., Fang D. and Han Z. Local well-posedness for the H^2 -critical nonlinear Schrödinger equation. *Trans. Amer. Math. Soc.* **368** (2016), no. 11, 7911–7934. (MR3546788) (doi: 10.1090/tran6683)
- [6] Cazenave T. and Naumkin I. Local existence, global existence, and scattering for the nonlinear Schrödinger equation. *Commun. Contemp. Math.* **19** (2017), no. 2, 1650038, 20 pp. (MR3611666) (doi: 10.1142/S0219199716500383)
- [7] Cazenave T. and Naumkin I. Modified scattering for the critical nonlinear Schrödinger equation. *J. Funct. Anal.* **274** (2018), 402–432. (doi: 10.1016/j.jfa.2017.10.022)
- [8] Cazenave T. and Weissler F. B. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s , *Nonlinear Anal.* **14** (1990), no. 10, 807–836. (MR1055532) (doi: 10.1016/0362-546X(90)90023-A)
- [9] Cazenave T. and Weissler F. B. Some remarks on the nonlinear Schrödinger equation in the critical case, in *Nonlinear Semigroups, Partial Differential Equations, and Attractors*, T.L. Gill and W.W. Zachary (eds.) *Lecture Notes in Math.* **1394**, Springer, New York, 1989, 18–29. (1021011) (doi: 10.1007/BFb0086749)
- [10] Cazenave T. and Weissler F. B. Rapidly decaying solutions of the nonlinear Schrödinger equation, *Comm. Math. Phys.* **147** (1992), 75–100. (MR1171761) (link: <http://projecteuclid.org/euclid.cmp/1104250527>)
- [11] Fang D. and Han Z. On the well-posedness for NLS in H^s . *J. Funct. Anal.* **264** (2013), no. 6, 1438–1455. (MR3017270) (doi: 10.1016/j.jfa.2013.01.005)
- [12] Ginibre J., Ozawa T. and Velo G. On the existence of the wave operators for a class of nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré Phys. Théor.* **60** (1994), no. 2, 211–239. (MR1270296) (link: http://archive.numdam.org/article/AIHPA_1994__60_2_211_0.pdf)
- [13] Ginibre J. and Velo G. On a class of nonlinear Schrödinger equations. II. Scattering theory, general case, *J. Funct. Anal.* **32**, no. 1 (1979), 33–71. (MR0533219) (doi: 10.1016/0022-1236(79)90077-6)
- [14] Ginibre J. and Velo G. On the global Cauchy problem for some nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), no. 4, 309–323. (MR0778977) (link: http://www.numdam.org/item?id=AIHPC_1984__1_4_309_0)
- [15] Hayashi N. and Naumkin P.I. Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations. *Amer. J. Math.* **120** (1998), no. 2, 369–389. (MR1613646) (doi: 10.1353/ajm.1998.0011)
- [16] Hayashi N. and Naumkin P.I. Large time behavior for the cubic nonlinear Schrödinger equation. *Canad. J. Math.* **54** (2002), no. 5, 1065–1085. (MR1924713) (doi: 10.4153/CJM-2002-039-3)
- [17] Hayashi N. and Naumkin P.I. Logarithmic time decay for the cubic nonlinear Schrödinger equations. *Int. Math. Res. Not. IMRN* **2015**, no. 14, 5604–5643. (MR3384451) (doi: 10.1093/imrn/rnu102)
- [18] Hayashi N., Naumkin P.I., Shimomura A. and Tonegawa S. Modified wave operators for nonlinear Schrödinger equations in one and two dimensions. *Electron. J. Differential Equations* 2004, No. 62, 16 pp. (MR2047418) (link: <http://ejde.math.txstate.edu/Volumes/2004/62/hayashi.pdf>)
- [19] Kato T. Nonlinear Schrödinger equations, in *Schrödinger Operators (Sønderborg, 1988)*, *Lecture Notes in Phys.* **345**, Springer, Berlin, 1989, 218–263. (MR1037322) (doi: 10.1007/3-540-51783-9_22)
- [20] Kato T. On nonlinear Schrödinger equations. II. H^s -solutions and unconditional well-posedness. *J. Anal. Math.* **67** (1995), 281–306. (MR1383498) (doi: 10.1007/BF02787794)
- [21] Kita N. and Wada T. Sharp asymptotic behavior of solutions to nonlinear Schrödinger equations in one space dimension. *Funkcial. Ekvac.* **45** (2002), no. 1, 53–69. (MR1913680) (link: <http://fe.math.kobe-u.ac.jp/FE/FullPapers/vol45/fe45-1-3.pdf>)
- [22] Linares F., Miyazaki H., and Ponce G. On a class of solutions to the generalized KdV type equation. Preprint, 2018, arXiv:1802.07345 [math.AP]. (link: <http://arxiv.org/abs/1802.07345>)
- [23] Linares F., Ponce G., and Santos G.N. On a class of solutions to the generalized derivative Schrödinger equations. Preprint, 2017, arXiv:1712.00663 [math.AP]. (link: <http://arxiv.org/abs/1712.00663>)
- [24] Nakamura M. and Ozawa T. Nonlinear Schrödinger equations in the Sobolev space of critical order, *J. Funct. Anal.* **155** (1998), no. 2, 364–380. (MR1624557) (doi: 10.1006/jfan.1997.3236)
- [25] Nakanishi K. and Ozawa T. Remarks on scattering for nonlinear Schrödinger equations, *NoDEA Nonlinear Differential Equations Appl.* **9** (2002), no. 1, 45–68. (MR1891695) (doi: 10.1007/s00030-002-8118-9)

- [26] Ozawa T. Long range scattering for the nonlinear Schrödinger equation in one space dimension, *Comm. Math. Phys.* **139** (1991), no. 3, 479–493. (MR1121130) (link: <http://projecteuclid.org/euclid.cmp/1104203467>)
- [27] Pecher H. Solutions of semilinear Schrödinger equations in H^s , *Ann. Inst. H. Poincaré Phys. Théor.* **67** (1997), no. 3, 259–296. (MR1472820) (link: http://www.numdam.org/item?id=AIHPA_1997__67_3_259_0)
- [28] Shimomura A. Asymptotic behavior of solutions for Schrödinger equations with dissipative nonlinearities, *Comm. Partial Differential Equations* **31** (2006), no. 7-9, 1407–1423. (MR2254620) (doi: 10.1080/03605300600910316)
- [29] Shimomura A. and Tonegawa S. Long-range scattering for nonlinear Schrödinger equations in one and two space dimensions. *Differential Integral Equations* **17** (2004), no. 1-2, 127–150. (MR2035499) (link: <http://projecteuclid.org/euclid.die/1356060476>)
- [30] Strauss W.A. On weak solutions of semi-linear hyperbolic equations, *An. Acad. Brasil. Ciên.* **42** (1970), 645–651. (MR0306715)
- [31] Strauss W.A. Nonlinear scattering theory, in *Scattering Theory in Mathematical Physics*, NATO Advanced Study Institutes Series Volume **9**, 53–78. Springer Netherlands, 1974. (doi: 10.1007/978-94-010-2147-0_3)
- [32] Strauss W.A. Nonlinear invariant wave equations, in *Invariant wave equations (Proc. “Ettore Majorana” Internat. School of Math. Phys., Erice, 1977)*, Lecture Notes in Phys. **73**, Springer, 1978, 197–249. (MR0498955) (doi: 10.1007/BFb0032334)
- [33] Strauss W.A. Nonlinear scattering theory at low energy, *J. Funct. Anal.* **41** (1981), no. 1, 110–133.
- [34] Tsutsumi Y. L^2 -solutions for nonlinear Schrödinger equations and nonlinear groups, *Funkcial. Ekvac.* **30** (1987), no. 1, 115–125. (MR0915266) (link: <http://www.math.sci.kobe-u.ac.jp/~fe/>)
- [35] Visciglia N. On the decay of solutions to a class of defocusing NLS, *Math. Res. Lett.* **16** (2009), no. 5, 919–926. (MR2576708) (doi: 10.4310/MRL.2009.v16.n5.a14)

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