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Solution to the semilinear wave equation with a pyramid-shaped blow-up surface

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Abstract

We consider the semilinear wave equation with subconformal power nonlinearity in two space dimensions. We construct a finite-time blow-up solution with an isolated characteristic blow-up point at the origin, and a blow-up surface which is centered at the origin and has the shape of a stylized pyramid, whose edges follow the bisectrices of the axes in $\mathbb{R}^2$. The blow-up surface is differentiable outside the bisectrices. As for the asymptotic behavior in similarity variables, the solution converges to the classical one-dimensional soliton outside the bisectrices. On the bisectrices outside the origin, it converges (up to a subsequence) to a genuinely two-dimensional stationary solution, whose existence is a by-product of the proof. At the origin, it behaves like the sum of 4 solitons localized on the two axes, with opposite signs for neighbors.
This is the first example of a blow-up solution with a characteristic point in higher dimensions, showing a really two-dimensional behavior. Moreover, the points of the bisectrices outside the origin give us the first example of non-characteristic points where the blow-up surface is non-differentiable.
This note gives only the main ideas. For details, see [52].

MSC 2010 Classification: 35L05, 35L71, 35L67, 35B44, 35B40

Keywords: Semilinear wave equation, blow-up, higher dimensional case, characteristic point, multi-solitons.

1 Introduction and history of the problem

We consider the subconformal semilinear wave equation in 2 space dimensions:

$$\begin{cases} \partial_t^2 u = \Delta u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } \partial_t u(0) = u_1, \end{cases} \tag{1.1}$$

where $u(t) : x \in \mathbb{R}^2 \to u(x,t) \in \mathbb{R}$, $1 < p < 5$, $(u_0, u_1) \in H^1 \times L^2(\mathbb{R}^2)$. The Cauchy problem is locally wellposed, and we have the existence of blow-up solutions from Levine [28].

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Equation (1.1) can be considered as a lab model for blow-up in hyperbolic equations, because it captures features common to a whole range of blow-up problems arising in various nonlinear physical models, in particular in general relativity (see Donninger, Schlag and Soffer [16]), and also for self-focusing waves in nonlinear optics (see Bizoń, Chmaj and Szpak [4]).

In this note, we present our results proved in [52]. In that paper, our aim is to construct the first example of a blow-up solution with a truely two-dimensional behavior. In particular, our solution will be non-radial and will not depend only on a one-dimensional variable. In fact, we will construct a multi-soliton solution here, since we will have 4 decoupled solitons in some backward light cone centered at the origin. As a matter of fact, there has been many papers addressing the question of multi-solitons in the literature, for various PDEs: for the generalized KdV equation, see Martel [30, 31], Martel, Merle and Tsai [33]; for NLS, see Merle [36], Martel and Merle [32], Martel, Merle and Tsai [34], Côte, Martel and Merle [14] as well as Martel and Raphaël [62]; for water waves, see Ming, Rousset and Tzvetkov [55]; for the Yamabe flow, see Daskalopoulos, Del Pino and Sesum; for the subcritical wave equation, see Merle and Zaag [50] as well as Côte and Zaag [15]; for the critical wave equation, see Duyckaerts, Kenig and Merle [19].

More generally, constructing a solution to some PDE with a prescribed behavior (not necessarily multi-solitons solutions) is an important question. That question was solved for (gKdV) by Côte [12, 13], and also for parabolic equations exhibiting blow-up, like the semilinear heat equation by Bressan [7, 8] (with an exponential source), Merle [37], Bricmont and Kupiainen [9], Merle and Zaag in [40, 39], Schweyer [59] (in the critical case), Mahmoudi, Nouaili and Zaag [29] (in the periodic case), the complex Ginzburg-Landau equation by Zaag [63] and also by Masmoudi and Zaag in [35], a complex heat equation with no gradient structure by Nouaili and Zaag [57], a gradient perturbed heat equation in the subcritical case by Ebde and Zaag in [20], then by Tayachi and Zaag in the critical case in [60] (see also [61]), or a strongly perturbed complex-valued heat equation in Nguyen and Zaag [56]. Other examples are available for Schrödinger maps (see Merle, Raphaël and Rodnianski [38]), and also fo the Keller-Segel model (see Raphaël and Schweyer [58], and also Ghoul and Masmoudi [21]).

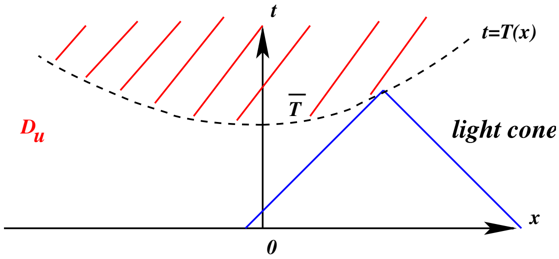

Figure 1: Domain of definition of the semilinear wave equation

If u is a blow-up solution of equation (1.1), we define (see for example Alinhac [1]) a 1-Lipschitz graph $x \mapsto T(x)$ such that the domain of definition of u is written as

$$D = \{(x,t) \mid x \in {}^2 \text{ and } 0 \le t < T(x)\}. \tag{1.2}$$

The graph of T is called the blow-up surface (or curve if $N = 1$) of u and will be denoted by Γ. A point $x_0 \in$ ^{2} is a non-characteristic point if there are

$$\delta_0 \in (0,1) \text{ and } t_0 < T(x_0) \text{ such that } u \text{ is defined on } \mathscr{C}_{x_0,T(x_0),\delta_0} \cap \{t \geq t_0\} \tag{1.3}$$

where

$$\mathscr{C}_{\bar{x},\bar{t},\bar{\delta}} = \{(x,t) \mid t < \bar{t} - \bar{\delta}|x - \bar{x}|\}. \tag{1.4}$$

If not, we say that x_0 is a characteristic point. We denote by $\mathscr{R} \subset$ ^{2} the set of non-characteristic points and by $\mathscr{S}$ the set of characteristic points.

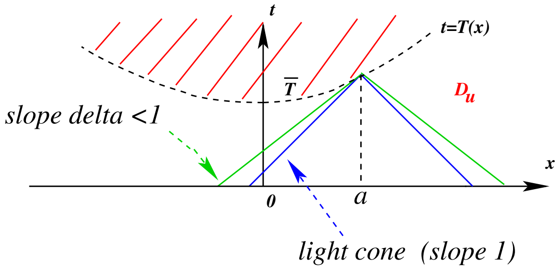

Figure 2: The point a is non-characteristic

The one-dimensional case in equation (1.1) has been understood completely, in a series of papers by the authors [44, 45, 47, 49, 50] and in Côte and Zaag [15] (see also the note [47]). This includes the first example of a solution with a characteristic point in [49] (in dimension one). For a general blow-up solution, we also proved that $\mathscr{S}$ is made of isolated points (see [50]), and that the blow-up curve is of class C^1 on $\mathscr{R}$ (see [49]). See Caffarelli and Friedman in [11, 10] for earlier results.

In higher dimensions $N \geq 2$, the situation is not as clear.
In fact, the blow-up rate is known (see [41], [43] and [42]; see also the extensions by Hamza and Zaag in [23] and [22] including the superconformal case in [25] also treated inKillip, Stoval and Vişan [26]).
For the asymptotic behavior and the regularity of the blow-up surface, the only known result is at non-characteristic points, where we show in [53] and [51] that Γ is C^1, under a reasonable assumption on the profile. The radial case outside the origin is also completely understood in [48], since it reduces to a perturbation of the one-dimensional case. Concerning the behavior of radial solutions at the origin, Donninger and Schörkhuber were able to prove the stability of the space-independent solution with respect to perturbations in initial data, in the Sobolev subcritical range [17] and also in the supercritical range in [18]. Some numerical results are available in a series of papers by Bizoń and co-authors (see [3], [5], [6]). See also Killip and Vişan [27].

In the note, we address the question of the existence of blow-up solutions to equation (1.1) with $\mathscr{S} \neq \emptyset$. As asserted above, the first example of such a solution was given in one space dimension in [49]. Later, Côte and Zaag [15] constructed other examples showing multi-solitons. Both approaches extend to the radial case and to perturbations of equation (1.1) with lower order terms (see Merle and Zaag [48], Hamza and Zaag [24]). Of course, all these one-dimensional examples can be considered as trivial 2-dimensional solutions, where $\mathscr{S}$ is either a line, or a circle. From the finite speed of propagation, we may have parallel lines or concentic circles, and the local blow-up behavior is always rigorously one dimensional. In particular, no example is known in higher dimensions, with $\mathscr{S}$ locally reduced to an isolated point. The aim of this paper is precisely to provide such an example. Moreover, we will give a sharp description of the blow-up behavior and the blow-up surface, locally near the characteristic point (this is related to an explicit description of the instabilities of the 4-soliton solution we construct at the origin).

2 Statement of the results

Before stating our result, let us introduce the following similarity variables, for any (x_0, T_0) such that $0 < T_0 \le T(x_0)$:

$$w_{x_0,T_0}(y,s) = (T_0-t)^{\frac{2}{p-1}}u(x,t), \quad y = \frac{x-x_0}{T_0-t}, \quad s = -\log(T_0-t). \tag{2.1}$$

If $T_0 = T(x_0)$, we write w_{x_0} for short. The function w_{x_0,T_0} (we write w for simplicity) satisfies the following equation for all $|y|<1$ and $s \ge -\log T_0$:

$$\partial_s^2 w - \mathscr{L}w + \frac{2(p+1)}{(p-1)^2}w - |w|^{p-1}w = -\frac{p+3}{p-1}\partial_s w - 2y\cdot\nabla\partial_s w \tag{2.2}$$

where

$$\mathscr{L}w = \frac{1}{\rho}\operatorname{div}\left(\rho\nabla w - \rho(y\cdot\nabla w)y\right), \quad \rho(y) = (1-|y|^2)^\alpha \text{ and } \alpha = \frac{5-p}{2(p-1)} > 0. \tag{2.3}$$

Equation (2.2) is studied in the energy space

$$\mathscr{H} = \mathscr{H}_0 \times L^2_\rho \text{ where } \|q_1\|^2_{\mathscr{H}_0} \equiv \int_{|y|<1}\left(q_1^2 + |\nabla q_1|^2 - |y\cdot\nabla q_1|^2\right)\rho dy. \tag{2.4}$$

We also introduce for all $|d|<1$ the following stationary solutions of (2.2) (or solitons) depending only on the one-dimensional variable $y\cdot d$ (if $d \ne 0$) and defined for all $|y|<1$ by

$$\kappa(d,y) = \kappa_0\frac{(1-|d|^2)^{\frac{1}{p-1}}}{(1+d\cdot y)^{\frac{2}{p-1}}} \text{ where } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}, \tag{2.5}$$

and

$$\bar{d}(s) = -\tanh\bar{\zeta}(s) \text{ where } \bar{\zeta}(s) = \left(\frac{p-1}{4}\right)\log s - \frac{(p-1)}{4}\log\left(\frac{p-1}{4\bar{c}}\right) \tag{2.6}$$

which is an explicit solution to the ODE

$$\frac{1}{\bar{c}}\frac{d\bar{\zeta}}{ds} = e^{-\frac{4}{p-1}\bar{\zeta}}$$

for some $\bar{c}(p) > 0$. Note that we have for some $C_0(p) > 0$,

$$1+\bar{d}(s) = C_0 s^{-\frac{p-1}{2}} + O(s^{-(p-1)}) \sim C_0 s^{-\frac{p-1}{2}} \text{ as } s\to\infty. \tag{2.7}$$

Let (e_1, e_2) be the canonical basis of $\mathbb{R}^2$. This is the statement of our result:

Theorem 1 (Existence of a blow-up solution with an isolated characteristic blow-up point and a blow-up surface which is a pyramid at the first order). *There exists $u(x,t)$ a solution to equation* (1.1)*, which is symmetric with respect to the axes and anti-symmetric with respect to bisectrices, with the following properties:*
(A) (Blow-up with an isolated characteristic point). *$u(x,t)$ blows up on some blow-up graph $\Gamma = \{(x, T(x)\}$, and for some $\delta > 0$, we have $\mathscr{S} \cap B(0,\delta) = \{0\}$.*
(B) (The blow-up surface is nearly a pyramid). *T is symmetric with respect to the axes and the bisectrices and C^1 outside the bisectrices. Moreover, when $0 \le x_2 < x_1 \le \delta$, we have for some $C_0 = C_0(p) > 0$:*

$$T(x) = T(0) - x_1(1 - C_0|\log x_1|^{-\frac{p-1}{2}}) + o(x_1|\log x_1|^{-\frac{p-1}{2}}) + o(x_2|\log x_1|^{-\frac{p-1}{4}}),$$
$$\nabla T(x) = -e_1(1 - C_0|\log x_1|^{-\frac{p-1}{2}}) + o(|\log x_1|^{-\frac{p-1}{2}})e_1 + o(|\log x_1|^{-\frac{p-1}{4}})e_2.$$

(C) (Blow-up behavior of the solution). *We have the following behavior for w_x for $0 \le x_2 \le x_1 \le \delta$ as $s \to \infty$:*
(i) if $x = 0$, then

$$\left\| w_0(y,s) - \left(\kappa(\bar{d}(s)e_1, y) + \kappa(-\bar{d}(s)e_1, y) - \kappa(\bar{d}(s)e_2, y) - \kappa(-\bar{d}(s)e_2, y)\right)\right\|_{\mathscr{H}} \to 0 \quad (2.8)$$

where $1 + \bar{d}(s) \sim C_0 s^{-\frac{p-1}{2}}$ as $s \to \infty$;
(ii) if $x_2 < x_1$, then $w_x(s)$ converges as $s \to +\infty$ to $\kappa(d(x)e_1)$, with

$$d(x) + 1 \sim C_0|\log x_1|^{-\frac{p-1}{2}} \text{ as } x_{\to}0.$$

(iii) if $x \neq 0$ with $x_1 = x_2$, then $w_x(s_n)$ converges to some stationary solution w_x^* for some sequence $s_n \to \infty$, where w_x^* is a genuinely two-dimensional stationary of equation (2.2)*.*

Let us remark that the existence of the new stationary solution of equation (2.2) just mentioned at the end of this theorem, follows from an indirect argument we use when x is on the bisectrices.

We would like to mention from the symmetries of the solution that we have $u(x,t) = 0$ on the bisectrices. In one space dimension, such a property implies that x is a characteristic point. Surprisingly, in our two-dimensional setting, only the origin is a characteristic point, and the other points on the bisecrtices are non-characteristic blow-up points, showing a genuinely two-dimensional behavior.

Let us also note that when $0 < x_2 = x_1 \le \delta$, the estimate on $T(x)$ in part (B) does hold, by continuity of T. Moreover, we can compute upper and lower left derivatives for T along any direction non parallel to the bisectrix $\{x_1 = x_2\}$, and the same holds from the right. In particular, if $|\omega| = 1$ and $\omega_2 - \omega_1 > 0$, then:
$\partial_{\omega,r,\pm}T(x) = (-1 + C_0|\log x_1|^{-\frac{p-1}{2}} + o(|\log x_1|^{-\frac{p-1}{2}}))\omega_1 + o(|\log x_1|^{-\frac{p-1}{4}})\omega_2$;
$\partial_{\omega,l,\pm}T(x) = (-1 + C_0|\log x_1|^{-\frac{p-1}{2}} + o(|\log x_1|^{-\frac{p-1}{2}}))\omega_2 + o(|\log x_1|^{-\frac{p-1}{4}})\omega_1$ as $x \to 0$,
where the subscript r and l stands for "right" and "left", whereas the subscript $\pm$ stands for "upper" or "lower".
At the origin, T has a right derivative with respect to x_1 whose value is $\partial_{x_1,r}T(0) = -1$, with similar statements from the left and in the direction x_2.

3 Generalization and extensions of the result

Our result can be generalized to other pyramids, with any regular polygon as a basis. In higher space dimensions $N \geq 3$, we naturally generalize our results to a pyramid with a hypercube as a basis. Moreover, using a Lorentz transform near some point of the bisectrices different from the origin, we can tilt the blow-up surface and obtain the existence of a blow-up solution of equation (1.1), with a tent-shaped (at the first order) blow-up surface, no characteristic point in some neighborhood, a slope approaching $\frac{\sqrt{2}}{2}$ and an upper edge depending on x_2. This tent is in fact new and different from the one obtained by considering a solution depending only on x_1 with a characteristic point at the origin. Indeed, in two space dimensions, this "naive" tent has a line a characteristic points on its upper edge, a slope approaching 1, and an upper edge that does not depend on x_2.

4 The strategy of the proof

Our proof relies on 3 main steps, which we present in the following subsections

4.1 Step 1: Construction of a solution for equation (1.1) showing 4 solitons in the backward light cone

In this step, we construct a blow-up solution to equation (1.1) defined only in the backward light cone with vertex $(0, T(0))$ and showing 4 solitons for w_0 at the origin as in (2.8). Then, using the finite speed of propagation, we derive from the latter a blow-up solution to the Cauchy problem of equation (1.1). Note that this construction step follows the classical scheme of a *"construction with a prescribed behavior"*, which proved to be efficient for various PDEs, as we already pointed-out in the introduction. Let us give some details for this method.

In fact, our goal is to construct a solution $w_0(y, s)$ for equation (2.2) defined for all $|y| < 1$ and $s \geq s_0$, for some large enough s_0 showing 4 solitons as in the following:

$$w_0(y,s) \sim \kappa(\bar{d}(s)e_1, y) + \kappa(-\bar{d}(s)e_1, y) - \kappa(\bar{d}(s)e_2, y) - \kappa(-\bar{d}(s)e_2, y) \text{ as } s \to \infty, \quad (4.1)$$

in accordance with our statement in Part (C) of Theorem 1, where the soliton $\kappa(d, y)$ is introduced in (2.5) and the paramester $\bar{d}(s)$ obeys the law given in (2.6) (here, $d \in {}^2$ and $y \in {}^2$), Applying the construction strategy in our setting, we linearize equation (2.2) around the intended behavior (4.1), and find three regions in the spectrum:
- *an infinite negative spectrum*, controlled thanks to a linearized version of the well-known Lyapunov functional associated to equation (2.2) ;
- $\lambda = 0$, which is controlled thanks to modulation in the parameter d in $\kappa(d, y)$ (2.5);
- $\lambda = 1$, which is controlled thanks to modulation in the parameter ν in the *generalized solitons*:

$$\kappa^*(d, \nu, y) = \kappa_0 \frac{(1 - |d|^2)^{\frac{1}{p-1}}}{(1 + \nu + d \cdot y)^{\frac{2}{p-1}}}. \quad (4.2)$$

Note that $\kappa^*(d, \mu e^s, s)$ is a solution of equation (2.2) for any $\mu \in$, which is obtained from $\kappa(d, y)$ through the use of the similarity variables' definition (2.1) back and forth, with a different scaling time.

We also would like to point-out that for this construction, we were inspired by the construction of multi-solitons in 1 space dimension, as performed by Côte and Zaag in [15].

4.2 Step 2: Instability of the 4-soliton solution when the center of the backward light cone leaves the origin

This step is the very heart of our argument. Here, we aim at understanding the instability of the 4-soliton solution of equation (2.2) we have for w_0 (4.1), when we move outside the origin to consider the behavior of w_{x_0} where $x_0 \neq 0$. We did that already in one space dimension in [50], but the situation is much more delicate here, mainly because of the dynamics at the bisectrices $\{x_{0,1} = \pm x_{0,2}\}$.

More precisely, we leave the origin and focus on the behavior of w_{x_0} for $x_0 \neq 0$. Using the decomposition into 4 solitons together with the upper blow-up bounds from [42], we first derive some rough estimates on the blow-up surface, showing in particular that it is under some pyramid, with a flatter slope line. Then, we find the behavior of $w_{x_0}(y,s)$ for large s, which turns to be different from the behavior of $w_0(y,s)$. This shows in particular that the 4-soliton solution is unstable. Let us give some details in the following.

Take $x_0 \neq 0$. We need to know the behavior of w_{x_0} for $|y| < 1$ and $s \geq -\log T(x_0)$. This is equivalent to knowing the behavior of $u(x,t)$ in the backward light cone $\mathscr{C}_{x_0,T(x_0),1}$ (1.4) with vertex $(x_0, T(x_0))$.
It happens that when x_0 is small and $t < \min(T(0), T(x_0))$, the sections of $\mathscr{C}_{x_0,T(x_0),1}$ and $\mathscr{C}_{0,T(0),1}$ are almost the same.
Moreover, we have the following relation between w_{x_0} and w_0, from the application of the similarity variables' transformation (2.1) back and forth:

$$w_{x_0}(y,s) = (1 - T(x_0)e^s)^{-\frac{2}{p-1}} w_0(Y,S), \quad Y = \frac{y + xe^s}{1 - T(x_0)e^s} \quad S = s - \log(1 - T(x_0)e^s).$$

Since w_0 shows 4 solitons, as in (4.1) and in Part (C) of Theorem 1:

$$w_0(y,s) \sim \kappa(\bar{d}(s)e_1, y) + \kappa(-\bar{d}(s)e_1, y) - \kappa(\bar{d}(s)e_2, y) - \kappa(-\bar{d}(s)e_2, y) \text{ as } s \to \infty,$$

the function w_{x_0} also shows 4 (generalized) solitons (see (4.2)), though with a *deformation*. Two cases then arise:

- **Case 1**: If x_0 is not on the bisectrices (say, $0 \leq x_{0,2} < x_{0,1}$ from the symmetries of the solution), only one soliton remains at some time $t^* = T(x_0) - e^{-s^*} = T(0) - e^{-S^*}$:

$$w_{x_0}(y,s^*) \sim \kappa(\bar{d}(S^*)e_1) \text{ with } S^* \sim -\log x_{0,1}.$$

Applying our trapping result near solitons (proved first in one space dimension in [45] then for higher dimensions in [54]), we see that if x_0 is non-characteristic, then

$$w_{x_0}(y,s) \to \kappa(\nabla T(x_0), y) \text{ as } s \to \infty \tag{4.3}$$

with

$$\nabla T(x_0) \sim \bar{d}(S^*)e_1 = (-1 + c_0 S^{*-\frac{p-1}{2}} + \ldots)e_1 = (-1 + c_0 |\log x_1|^{-\frac{p-1}{2}} + \ldots)e_1.$$

Remark. If x_0 is characteristic, we have no information; later, we will have to show that all points outside the bisectrices are non-characteristic.

Remark. Note the link between the asymptotic behavior of w_{x_0} and the regularity of $T(x_0)$ in (4.3).

- Case 2: If x_0 is on the bisectrices (say, $x_{0,2} = x_{0,1}$), then w_{x_0} is anti-symmetric with respect to the bisectrix, therefore, 2 solitons remain at some time $\tilde{t} = T(x_0) - e^{-\tilde{S}} = T(0) - e^{-\tilde{S}}$:

$$w_{x_0}(y, \tilde{s}) \sim \kappa(\bar{d}(\tilde{S})e_1) - \kappa(-\bar{d}(\tilde{S})e_2)$$

with

$$\tilde{S} \sim -\log x_{0,1}.$$

From the behavior of the neighbors outside the bisectrix, we derive that x_0 is non-characteristic. Therefore, from the existence of a Lyapunov functional in similarity variables (see Antonini and Merle [2]), we see that

$$\text{as } s \to \infty, w_{x_0}(y, s) \to w^*_{x_0}(y),$$

a stationary solution in similarity variables, with

$$w^*_{x_0}(y) \sim \kappa(\bar{d}(\tilde{S})e_1) - \kappa(-\bar{d}(\tilde{S})e_2).$$

Note that this is a *new* kind of stationary solutions, which are neither radial, nor 1d.

4.3 Step 3: The local behavior of $T(x)$ in connection with the dynamics of w_x

As usual with blow-up problems (heat, wave), the asymptotic behavior of the solution at blow-up and the regularity of the blow-up set are linked and advanced side by side in the proof (see [64], [65], [67] and [66] for the semilinear heat equation; see [44], [45], [49], [50], [47] and [46] for the semilinear wave equation).

The present situation is no exception. As a matter of fact, in this step, we make the link between dynamics of w_x from the previous section and the local behavior of $T(x)$. This is in fact the new feature of our paper which makes its originality. In particular, we use families of moving non-characteristic cones together with subtle elementary geometric methods to derive the pyramid shape of the blow-up surface. The delicate case is the case where x is on the bisectrices, since this is a new situation, not encountered in dimension 1. It is worth noticing that the moving cone technique simplifies the “moving plane” technique we use earlier in one space dimension in [45].

For details and proofs, see [52].

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