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ON THE HOMOGENIZATION OF THE STOKES EQUATIONS IN PERFORATED DOMAINS WITH APPLICATION TO FLUID/SOLID INTERACTION PROBLEMS

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Abstract. In these notes, we consider the Stokes equations on a perforated domain. The inclusions represent particles moving in a viscous fluid so that the partial differential equations in the bulk are completed with non-zero boundary conditions simulating the motion of the particles. We review recent results on the associated homogenization problem in the regime where the number of particles increases while their diameters converge to 0.

1. Introduction

Considering the motion of a small number of indeformable particles $(B_i(t))_{i=1,\dots,N}$ in a viscous fluid inside a $3D$ container Ω, one may study the system coupling incompressible Navier-Stokes equations with Newton laws:

$$(1)\qquad \left\{\begin{array}{rcl} \rho_f(\partial_t \bar{u} + \bar{u}\cdot\nabla\bar{u}) &=& \nu_f \Delta \bar{u} - \nabla \bar{p}\,, \\ \mathrm{div}\ \bar{u} &=& 0\,, \end{array}\right. \qquad \text{in } \Omega \setminus \bigcup_{i=1}^N \overline{B_i(t)}$$

$$(2)\qquad \left\{\begin{array}{rcll} \bar{u}(t,x) &=& \dot{\hbar}_i(t) + \omega_i \times [x - \hbar_i(t)]\,, & \text{on } \partial B_i(t)\,, \\ \bar{u}(t,x) &=& 0\,, & \text{on } \partial\Omega\,. \end{array}\right.$$

$$(3)\qquad \left\{\begin{array}{rcl} m\ddot{\hbar}_i &=& -\displaystyle\int_{\partial B_i(t)} (\nu_f(\nabla\bar{u} + \nabla\bar{u}^\top) - \bar{p})n\mathrm{d}\sigma, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}[J\bar{\omega}_i] &=& -\displaystyle\int_{\partial B_i(t)} [x - \hbar_i] \times [(2\nu(\nabla\bar{u} + \nabla\bar{u}^\top) - \bar{p})n]\mathrm{d}\sigma. \end{array}\right.$$

We denote here by $(\bar{u},\bar{p})$ the fluid velocity-field/pressure whose density ρ_f, viscosity ν_f are assumed to be constant. As for the particle unknowns, we introduce $\hbar_i \in \Omega$ their centers of mass and $\bar{\omega}_i \in \mathbb{R}^3$ their rotation velocities. The symbol m and J stand for their common masses and inertias. For instance, if the particles are identical spheres with the same density ρ_s and radius $\bar{R}$, we have:

$$m = \frac{4}{3}\pi\rho_s\bar{R}^3 \quad J = \frac{2\pi}{5}\rho_s\bar{R}^5\mathbb{I}_3.$$

When the number of particles increases, such a description one-by-one is irrelevant. An alternative approach is to introduce the particle density function $(t,x,v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^3 \mapsto f(t,x,v) \in [0,\infty)$ which measures the proportion of particles having velocity v in x at time t. Note that this description allows to measure the influence of translation velocity of particles only (no rotation effect). In the case of thin sprays, i.e., when the particle phase has negligible volume fraction, a similar system to the one depicted above consists in coupling a Vlasov equation for f with an incompressible Navier Stokes equations for the fluid unknowns (see [4] or [3] and the references therein):

$$(4) \qquad \begin{cases} \partial_t f + v\cdot\nabla_x f + 6\pi\nu\,\mathrm{div}_v[(\bar{u}-v)f] &= 0, & \text{on } \Omega\times\mathbb{R}^3\\ \rho_f(\partial_t\bar{u}+\bar{u}\cdot\nabla_x\bar{u}) - \nu\Delta_x u + \nabla_x p &= -6\pi\displaystyle\int_{\mathbb{R}^3} f(\bar{u}-v)\,\mathrm{d}v, & \text{on } \Omega\\ \mathrm{div}_x u &= 0, & \text{on } \Omega\end{cases}$$

On the right-hand side of the fluid momentum equation, a supplementary term appears balancing the force term in the Vlasov equation. Both terms take into account the exchange of momentum between the fluid and the particle phase.

In these notes, we report on the papers [10, 11, 13] in which we provide an analytical justification of the friction term that one needs to add in the Vlasov-Navier Stokes system in order to take into account the fluid viscous effects. The method is the following one. We consider the fluid viewpoint. We restrict to the Stokes equations:

$$(5) \qquad \begin{cases} -\Delta u^N + \nabla p^N &= 0\,,\\ \mathrm{div}\ u^N &= 0\,,\end{cases} \quad \text{in } \Omega\setminus\bigcup_{i=1}^N \overline{B_i^N}$$

with boundary conditions:

$$(6) \qquad \begin{cases} u^N(x) &= v_i^N + \omega_i^N\times(x-h_i^N) & \text{on } \partial B_i^N\,,\\ u^N(x) &= 0\,, & \text{on } \partial\Omega\,.\end{cases}$$

This configuration is reproduced in Figure 1. We assume that $N\to\infty$ and that the $(h_i^N, v_i^N, \omega_i^N, B_i^N)_{i=1,\dots,N}$ converge in a suitable given way. We relate to this convergence two applications $x\mapsto\mathbb{M}(x)\in\mathcal{M}_3(\mathbb{R})$ (symmetric and positive a.e.) and $x\mapsto j(x)\in\mathbb{R}^3$ such that the solutions (u^N,p^N) of the previous problems converge when $N\to\infty$ to the (unique) solution to:

$$\begin{cases} -\Delta u + \nabla p &= j - \mathbb{M}u, & \text{on } \Omega,\\ \mathrm{div} u &= 0, & \text{on } \Omega,\\ u &= 0 & \text{on } \partial\Omega.\end{cases}$$

We call this last system "Stokes-Brinkman" after H.C. Brinkman who illustrated at a formal level the existence of this friction term (see [2]). We remark that, in (4), the right-hand side has the same form as in this latter system when setting:

$$(7) \qquad j(x) = 6\pi\int_{\mathbb{R}^3} v f(x,v)\mathrm{d}v \qquad \mathbb{M}(x) = 6\pi\int_{\mathbb{R}^3} f(x,v)\mathrm{d}v\mathbb{I}_3.$$

So, we want to recover these fluxes j and dissipation $\mathbb{M}$ in the case of spheres. But, we want also to discuss

- the influence of shape and rotation velocity on the computation of $\mathbb{M}$ and j,
- the implicit assumptions leading to the simple form for j and $\mathbb{M}$ given in (7).

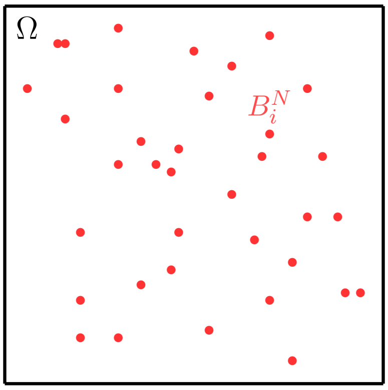

FIGURE 1. A N-obstacle configuration

Our computations reduce to a homogenization problem for the Stokes equations in perforated domains. First results on this problem are obtained in [1] in the periodic case. In this reference, the author assumes that the domain Ω is decomposed into cells of width $\varepsilon > 0$ and that each cell contains a hole of size $a_\varepsilon > 0$. He proves that there exists a threshold value of the capacitary parameter $\sigma_\varepsilon = \sqrt{\varepsilon^3/a_\varepsilon}$ such that, when $\varepsilon \to 0$, the solution $(u_\varepsilon, p_\varepsilon)$ to the Stokes problem on $\mathcal{F}^\varepsilon$ (the complement of the holes in the container Ω) with vanishing boundary conditions converges to a Stokes-Brinkman like problem. An interpretation of this result in terms of screening length is also provided in [7]. However, both results are resctricted to vanishing boundary conditions on the holes.

The very problem with non-zero constant boundary conditions that we consider herein is tackled in [5]. We give now a detailed description of this previous result. We consider the problem (5)-(6) when the B_i^N are copies of the same ball, and the rotation velocities ω_i^N vanish. We denote by $(h_i^N)_{i=1,\dots,N} \in \Omega^N$ the centers of the inclusions and we assume that their common radius scales like $1/N$. Without further restriction (since the size of Ω is not fixed), we assume that $B_i = B(h_i^N, 1/N)$ for all $i = 1, \dots, N$. Then, we denote $(v_i^N)_{i=1,\dots,N} \in [\mathbb{R}^3]^N$ the constant boundary conditions that are imposed on the respective ∂B_i^N. We assume that the inclusions are distributed in the same dilution regime (though non-periodic) as in [1]. With our notations, this reads as follows:

$$\min_{i=1,\dots,N} \left\{ \min_{j \neq i} |h_i^N - h_j^N|, \mathrm{dist}(h_i^N, \partial\Omega) \right\} \geqslant \frac{C_0}{N^{\frac{1}{3}}} \tag{8}$$

for some strictly positive constant C_0 independant of $N \in \mathbb{N}$. We also fix that we have a uniform bound on the kinetic energy of the particle phase:

$$\frac{1}{N} \sum_{i=1}^{N} |v_i^N|^2 \leqslant E_0 \tag{9}$$

for some strictly positive constant E_0 independant of $N \in \mathbb{N}$. We note here that, as we do not add any particle dynamics, assumptions on the particle masses are open. So, in equation (9) we include implicitly that we scaled the equations in time so that the particle Stokes number is $1/N$. One shortcoming of this approach is that we make the parameter N

play different roles: $1/N$ is the diameter of the particles and the Stokes number and N is the number of particles. Nevertheless, it is straightforward to extend these computations by including a radius R and a Stokes number St up to assume that they scale in a similar way with respect to the number N of particles.

In this framework, we have that, for N sufficiently large (depending solely on C_0), the B_i^N are disjoint subsets of Ω so that $\mathcal{F}^N = \Omega \backslash \cup_{i=1}^N \bar{B}_i^N$ has smooth boundaries. It is classical that there exists then a unique solution $(u^N, p^N) \in H^1(\mathcal{F}^N) \times L^2_0(\mathcal{F}^N)$ to (5)-(6). We abusively denote u^N the extension of this solution by setting $u^N = v_i^N$ in B_i^N for $i = 1, \ldots, N$. This yields a sequence $(u^N)_{N\in\mathbb{N}}$ of $H^1_0(\Omega)$ vector -fields. Our aim is to compute the asymptotics of this sequence. In that respect, and without the periodicity assumptions of [1], one has also to prescribe how the distribution of $(h_i^N)_{i=1,\ldots,N}$ and $(v_i^N)_{i=1,\ldots,N}$ behaves when $N \to \infty$. To this end, a classical tool of studies on many-particle systems is to define the empirical measures:

$$\rho^N := \frac{1}{N}\sum_{i=1}^N \delta_{h_i^N} \qquad j^N := \frac{1}{N}\sum_{i=1}^N v_i^N \delta_{h_i^N}. \tag{10}$$

These are both measures on $\overline{\Omega}$ so that we may assume convergence of both sequence of empirical measures in the dual space of $C(\overline{\Omega})$ and $C(\overline{\Omega};\mathbb{R}^3)$ respectively. With the assumptions that we detailed until now, the main result of [5] reads:

Theorem 1. *Assume that there exists $\rho \in C(\overline{\Omega})$ and $j \in C(\bar{\Omega};\mathbb{R}^3)$ such that*

$$\rho^N \rightharpoonup \rho, \qquad j \rightharpoonup j.$$

then, up to the extraction of a subsequence, $(u^N)_{N\in\mathbb{N}}$ converges weakly in $H^1_0(\Omega)$ to the unique vector-field $\bar{u} \in H^1_0(\Omega)$ for which there exists a pressure $\bar{p} \in L^2(\Omega)$ such that:

$$\left\{ \begin{array}{rcll} -\Delta\bar{u} + \nabla\bar{p} &=& 6\pi(j - \rho\bar{u}), & \text{in } \mathcal{D}'(\Omega), \\ div\bar{u} &=& 0, & \text{in } \mathcal{D}'(\Omega). \end{array} \right.$$

The proof of [5] is based on a compactness method. First, the authors prove that the u^N are uniformly bounded in $H^1_0(\Omega)$. Hence, up to the extraction of a subsequence, the u^N converge weakly in $H^1_0(\Omega)$ to some divergence-free $\bar{u}$. The second part of the proof is then to compute

$$\int_\Omega \nabla\bar{u} : \nabla w$$

for arbitrary divergence-free test-function $w \in C^\infty_c(\Omega;\mathbb{R}^3)$. To this end, one applies that:

$$\int_\Omega \nabla\bar{u} : \nabla w = \lim_{N\to\infty} \int_{\mathcal{F}^N} \nabla u^N : \nabla w$$

and aims at using w as a multiplier in the equation satisfied by u^N. Unfortunately this makes complicated boundary terms appear, so that, in [5], the authors introduce suitable correctors to lift the boundary values of u^N and w on the solid boundaries ∂B_i^N. The main difficulty lies then in computing the limit of the induced error terms. This difficulty is ruled out by an explicit choice of correctors that are computed on the basis of the resolution of the Stokes equations outside a ball.

The starting point of the results that we describe is to revisit the proof of [5] with variational arguments introduced in [1]. So, the outline of these notes is as follows. In the next section we give a brief reminder on the resolution of the Stokes problem with application to fluid/solid interaction problems (in the spirit of [6]). Then, we sketch an alternative proof to Theorem 1. In the following section, we show extensions including more general shapes of particles and weakening the restricting assumption (8). We end this last section by discussing the optimality of the more general assumptions that we introduce.

2. Revisiting the computation of [5]

2.1. **Reminders on the Stokes problem.** Let first consider the Stokes problem in $\mathbb{R}^3$ outside a unit ball with constant boundary condition V :

$$\left\{\begin{array}{rcl} -\Delta u + \nabla p &=& 0 \\ \mathrm{div} u &=& 0 \end{array}\right. \text{ on } \mathbb{R}^3 \setminus \overline{B(0,1)} \quad \left\{\begin{array}{rcl} u_{|\partial B(0,1)} &=& V \\ \lim_{|x|\to\infty} u(x) &=& 0. \end{array}\right. \tag{11}$$

It is well-known that this problem admits an exact solution (see [5, Section 6.2]):

$$u(x) = \nabla \times \left[\left(\frac{3}{2|x|} - \frac{1}{2|x|^3} \right) \frac{V \times x}{2} \right], \qquad p(x) = \frac{3V}{2} \cdot \nabla \left(\frac{1}{|x|} \right).$$

It can be proved that this solution is unique amongst the pairs (u,p) whose velocity-field gradient and pressure are $L^2(\mathbb{R}^3 \setminus \overline{B(0,1)})$. Such formulas entail that:

SB1: Denoting $\Sigma(u,p) = 2D(u) - p\mathbb{I}_3$ the fluid stress tensor, the force and torque applied by the sphere on the fluid satisfy:

$$\int_{\partial B(0,1)} \Sigma(u,p) n \mathrm{d}\sigma = 6\pi V, \qquad \int_{\partial B(0,1)} y \times \Sigma(u,p) n \mathrm{d}\sigma = 0.$$

The symbol n stands for the normal to $\partial B(0,1)$ directed toward the interior of $B(0,1)$.

SB2: The unique solution (u,p) to (11) obeys the decay estimates: given $\alpha \in \mathbb{N}^3$

$$|\partial_\alpha u(x)| \leqslant \frac{C_\alpha[V]}{|x|^{1+|\alpha|}}, \qquad |\partial_\alpha p(x)| \leqslant \frac{C[V]}{|x|^{2+|\alpha|}}, \qquad \forall\, |x| \gg 1.$$

The second property extends to any boundary data u_* outside any compact simply connected set B. More generally, we have:

Theorem 2. *Given a (lipschitz) simply connected obstacle B, $u_* \in H^{1/2}(\partial B)$, there exists a unique (weak or generalized) solution u to the Stokes problem on $\mathcal{F} := \mathbb{R}^3 \setminus \overline{B}$ with boundary data u_*.*

Furthermore, we have the two following properties:

S1: *The velocity-field u is characterized by the minimization problem:*

$$\int_{\mathcal{F}} |\nabla u|^2 = \inf \left\{ \int_{\mathcal{F}} |\nabla v|^2, v \in C_c^\infty(\mathbb{R}^3) \ s.t. \ \mathrm{div}\, v = 0 \ and \ v_{|\partial B} = u_* \right\} \tag{12}$$

S2: *Given $\alpha \in \mathbb{N}^3$ there exists $C_\alpha[u_*] < \infty$ for which:*

$$|\partial_\alpha u(x)| \leqslant \frac{C_\alpha[u_*]}{|x|^{1+|\alpha|}} \quad \forall\, |x| \gg 1.$$

We refer the reader to [6] for a proof of the first statement in this theorem while proofs for **S1** and **S2** can be found in [11]. The assumption that B is simply connected is made for convenience here. It is straightforward to extend the result to any B made of several smooth simply connected components when the flux of u_* through the boundary of any of these connected components vanishes.

We note that, when ∂B and u_* are smooth, the solution u and associated presure p are smooth on $\mathbb{R}^3 \setminus B$ so that the resulting force and torque are well defined. In particular, if $u_*(x)$ reads $V + \omega \times x$ and (u,p) is the corresponding solution, we define:

$$F[V,\omega] = \int_{\partial B} \Sigma(u,p) n \mathrm{d}\sigma, \quad T[V,\omega] = \int_{\partial B} y \times \Sigma(u,p) n \mathrm{d}\sigma.$$

Again, the symbol n stands for the normal to ∂B directed toward B. We note that the Stokes system can be rewritten $\mathrm{div}(\Sigma(u,p)) = 0$ which expresses the conservation of normal stress. We have then:

Proposition 3. *Given R s.t. $\overline{B} \subset B(0,R)$ we have:*

$$F[V,\omega] = \int_{\partial B(0,R)} \Sigma(u,p) n \mathrm{d}\sigma, \qquad T[V,\omega] = \int_{\partial B(0,R)} y \times \Sigma(u,p) n \mathrm{d}\sigma.$$

The symbol n stands for the normal directed toward the interior of $B(0,R)$.

For a given B, we can also let the boundary conditions (V,ω) vary and focus on the mapping $[F,T] : (V,\omega) \mapsto (F[V,\omega], T[V,\omega])$. The linearity and symmetries of the Stokes system entail that this mapping is linear symmetric, positive definite. It can be then reduced to a matrix $\mathbb{M}_B$ which characterizes the influence of the particle B on the flow. This matrix is known as the Stokes' resistance matrix (see [8, Section 3.5]). Simple scaling arguments induce also that:

Proposition 4. *Given $\varepsilon > 0$ let $B_\varepsilon = \varepsilon B$ and $(F_\varepsilon, T_\varepsilon)$ (resp. (F,T)) the force/torque corresponding to the solution to the Stokes equations outside the particle B_ε (resp. outside B). Given $(V,\omega) \in \mathbb{R}^3 \times \mathbb{R}^3$, there holds:*

$$F_\varepsilon[V,\omega] = \varepsilon F[V, \varepsilon\omega], \quad T_\varepsilon[V,\omega] = \varepsilon^2 T[V,\varepsilon\omega].$$

2.2. **Sketch of the proof of Theorem 1.** We proceed with the sketch of a proof for Theorem 1. So, we let (u^N, p^N) the solution to the problem (5)-(6) in the case of spheres with vanishing rotation velocities as in the assumptions of Theorem 1.

Uniform estimates and consequences. The first step of this proof is to obtain a uniform bound on the extended u^N by applying the variational characterization (12). This property derives directly from the uniform bound (9) under the dilution assumption (8). We can thus extract a weakly converging subsequence that we do not relabel for conciseness. The game is then to compute

$$I_w := \int_\Omega \nabla u : \nabla w \quad \text{for any divergence-free } w \in C_c^\infty(\Omega)$$

by remarking that:

$$I_w = \lim_{N\to\infty} I^N \text{ with } I^N := \int_\Omega \nabla u^N : \nabla w \quad \forall\, N \in \mathbb{N}.$$

So, in what follows, we fix w an arbitrary divergence-free test-function and we aim at computing the asymptotics of the corresponding integrals $(I^N)_{N\in\mathbb{N}}$.

Computation for fixed $N \in \mathbb{N}$. We first rewrite in a more suitable way I^N for fixed N. By applying that $u^N$is constant on the particle domains and solution to the Stokes equations on the fluid domain, we remark that:

$$I^N = \int_{\mathcal{F}^N} \nabla u^N : \nabla \widetilde{w} \tag{13}$$

for arbitrary divergence-free $\widetilde{w}$ matching the same boundary conditions as w on $\partial\mathcal{F}^N$. So, given $i \in \{1,\dots,N\}$ we introduce $\mathcal{C}_i^N := B(h_i^N, C_0/(2N^{1/3}))$ and (w_i, q_i) the unique solution to:

$$\left\{\begin{array}{rcll} -\Delta w_i + \nabla q_i &=& 0, & \text{on } \mathcal{C}_i^N \setminus \overline{B_i^N} \\ \operatorname{div} w_i &=& 0, & \text{on } \mathcal{C}_i^N \setminus \overline{B_i^N} \end{array}\right. \qquad \left\{\begin{array}{rcll} w_i(x) &=& w(x), & \text{on } \partial B_i^N \\ w_i(x) &=& 0, & \text{on } \partial \mathcal{C}_i^N. \end{array}\right.$$

Abusively, we still denote w_i its trivial extension to $\mathbb{R}^3$. We then combine these constructions in $\widetilde{w} = \sum_{i=1}^N w_i$. Because of assumption (8) we have that (13) holds true so that we can split I^N into:

$$I^N = \sum_{i=1}^N I_i^N \quad \text{with} \quad I_i^N = \int_{\mathcal{C}_i^N} \nabla u^N : \nabla w_i.$$

We work now on the "local" integral I_i^N for fixed i. We note that the radius of B_i^N is small with respect to the size of $\mathcal{C}_i^N$ so that, we may approximate w by a constant on ∂B_i^N. We set:

$$\left\{\begin{array}{rcll} -\Delta \widetilde{w}_i + \nabla \widetilde{q}_i &=& 0, & \text{on } \mathcal{C}_i^N \setminus \overline{B_i^N} \\ \operatorname{div} \widetilde{w}_i &=& 0, & \text{on } \mathcal{C}_i^N \setminus \overline{B_i^N} \end{array}\right. \qquad \left\{\begin{array}{rcll} \widetilde{w}_i &=& w(h_i^N), & \text{on } \partial B_i^N \\ \widetilde{w}_i &=& 0, & \text{on } \partial \mathcal{C}_i^N. \end{array}\right.$$

This yields:

$$I_i^N = \int_{\mathcal{C}_i^N} \nabla u^N : \nabla \widetilde{w}_i + \mathrm{Err}_{1,i}^N.$$

Applying then that $(\widetilde{w}_i, \widetilde{q}_i)$ is solution to a Stokes problem, we integrate by parts:

$$\begin{aligned}\int_{\mathcal{C}_i^N} \nabla u^N : \nabla \widetilde{w}_i &= F_i^N \cdot v_i^N + \int_{\partial \mathcal{C}_i^N} \Sigma(\widetilde{w}_i, \widetilde{q}_i) n \cdot u^N \, \mathrm{d}\sigma \\ &= F_i^N \cdot (v_i^N - \bar{u}_i^N) + \mathrm{Err}_{2,i}^N,\end{aligned}$$

where :

$$F_i^N = \int_{\partial B_i^N} \Sigma(\widetilde{w}_i, \widetilde{q}_i) n \, \mathrm{d}\sigma, \qquad \bar{u}_i^N = \frac{1}{|\mathcal{C}_i^N|} \int_{\mathcal{C}_i^N} u^N =: \fint_{\mathcal{C}_i^N} u^N.$$

We used here again that $\mathcal{C}_i^N$ has a small radius – so that we may approximate u^N by a constant that we chose to be $\bar{u}_i^N$, its mean on $\mathcal{C}_i^N$ – and the conservation property stated in Proposition 3. At this point, we note that $\widetilde{w}_i$ is nearly a solution outside a ball of radius $1/N$ so that, recalling **SB1**:

$$F_i^N = \frac{6\pi}{N} w(h_i^N) + \mathrm{Err}_{3,i}^N.$$

Combining the above computations, we obtain finally that:

$$I^N = \frac{1}{N} \sum_{i=1}^N 6\pi w(h_i^N) \cdot v_i^N - \frac{1}{N} \sum_{i=1}^N 6\pi w(h_i^N) \cdot \fint_{\mathcal{C}_i^N} u^N(x) \mathrm{d}x + \mathrm{Err}^N \tag{14}$$

where $\mathrm{Err}^N = \mathrm{Err}_1^N + \mathrm{Err}_2^N + \mathrm{Err}_3^N$ the three error terms being computed by aggregating the errors for each index i :

$$\mathrm{Err}_l^N = \sum_{i=1}^N \mathrm{Err}_{l,i}^N.$$

Asymptotics. One key-observation is that, because of assumptions (8)-(9), we may prove that the error term Err^N converges to 0 when $N \to \infty$. Consequently, the proof reduces to computing the asymptotics of the two remaing sums. By assumption on the convergence of the j^N we have at first:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N 6\pi w(h_i^N) \cdot v_i^N = 6\pi \lim_{N\to\infty} \langle j^N, w \rangle = 6\pi \int_\Omega j(x) \cdot w(x) \mathrm{d}x.$$

As for the other term, if $\bar{u}$ is smooth (say $C^1(\bar{\Omega})$) independant of $N \in \mathbb{N}$, we have

$$\frac{1}{N} \sum_{i=1}^N 6\pi w(h_i^N) \cdot \fint_{\mathcal{C}_i^N} \bar{u}(x) \mathrm{d}x = \frac{1}{N} \sum_{i=1}^N 6\pi w(h_i^N) \bar{u}(h_i^N) + \mathrm{Err}_4^N$$

where Err_4^N comes from the variations of $\bar{u}$ around h_i^N and is bounded by $1/N^{1/3}$ (the uniform diameter of the $\mathcal{C}_i^N$). Moreover, there holds:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N 6\pi w(h_i^N) \cdot \bar{u}(h_i^N) = 6\pi \lim_{N\to\infty} \langle \rho^N, w \cdot \bar{u} \rangle = 6\pi \int_\Omega w(x) \cdot \bar{u}(x) \rho(x) \mathrm{d}x.$$

We have thus weak-convergence (in the dual of $L^2(\Omega)$ for instance) to $x \mapsto 6\pi\rho(x)w(x)$ of the linear forms:

$$\langle L^N, \bar{u} \rangle = \frac{1}{N} \sum_{i=1}^N 6\pi w(h_i^N) \cdot \fint_{\mathcal{C}_i^N} \bar{u}(x) \mathrm{d}x.$$

In order to test the convergence of L^N against the sequence u^N we need to have continuity of L^N on a function space in which u^N converges strongly. For this, we remark that

$$\langle L^N, \bar{u} \rangle = 6\pi \int_\Omega \ell^N(x) \cdot \bar{u}(x) \mathrm{d}x \quad \text{with } \ell^N(x) = \sum_{i=1}^N \frac{\mathbb{1}_{\mathcal{C}_i^N}}{|\mathcal{C}_i^N| N} w(h_i^N).$$

It turns out that, with assumption (8), we have that $(\ell^N)_{N\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ so that we may multiply the weak convergence of $(\ell^N)_{N\in\mathbb{N}}$ to ρw in $L^2(\Omega)$ with the strong convergence of (a subsequence of) the $(u^N)_{N\in\mathbb{N}}$. We obtain finally that :

$$\int_\Omega \nabla u : \nabla w = 6\pi \int_\Omega (j(x) - \rho(x)u(x)) \cdot w(x) \mathrm{d}x.$$

This ends the proof.

We add that, similarly to the computations of [1], we may also quantify the convergence of the sequence u^N in $L^p(\Omega)$ with respect to the convergence of the empirical measures (ρ^N, j^N) (in dual spaces such as $(C^{0,\alpha}(\overline{\Omega}))^*$). We refer the reader to [13] for more details.

3. Improvements

In this section, we discuss improvements of Theorem 1 in several directions: more general shape/boundary conditions on the particles, more general dilution regimes.

3.1. **More general shapes and boundary conditions.** Firstly, we discuss the assumption that the particles are spheres that do not rotate. In this direction, a fundamental remark is that the convergence of the remainder term Err^N (see (14)) derives from the decay property **SB2.** satisfied by the solution to the Stokes problem outside a sphere. As this property is satisfied for any compact particle (see **S2** in Theorem 2), we expect that we can extend the result to particles of arbitrary shape. This is the problem that we tackle in [11].

In [11], we consider the asymptotics of the solutions (u^N, p^N) to (5)-(6) when the B_i^N are no longer spheres but particles of arbitrary shapes. Namely, we assume that

$$B_i^N = h_i^N + \frac{1}{N}\mathcal{B}_i^N \quad \text{where } \mathcal{B}_i^N \text{ is a smooth subset of } \mathbb{R}^3 \text{ satisfying } \mathcal{B}_i^N \subset B(0, R_0)$$

for some R_0 independant of i, N. As for boundary conditions we consider now translation and rotation velocities. Nevertheless, we restrict our computation to the favorable dilution regime (8) and the finite kinetic-energy case (for the particle phase). With the rotation velocities, this second condition reads:

$$\frac{1}{N} \sum_{i=1}^N \left[|v_i|^2 + |\omega_i/N|^2 \right] \leqslant E_0 \tag{15}$$

for some strictly positive constant E_0 independant of N. We recall here, that the parameter $1/N$ represents the diameter of $\mathcal{B}_i^N$. The quantities v_i and ω_i/N are then both linear velocities.

As for the empirical measures, we choose to assume that the computation of the force and torque for a given particle $\mathcal{B}_i^N$ is an independant problem. The associated resistance matrix $\mathbb{M}^{i,N}$ is supposed to be given and we focus on the summation process which yields the Brinkman term. To this end, we split the resistance matrix of a given particle $\mathcal{B}_i^N$ into:

$$\mathbb{M}^{i,N} = \begin{pmatrix} \mathbb{M}_{\mathrm{I}}^{i,N} & \mathbb{M}_{\mathrm{II}}^{i,N} \\ [\mathbb{M}_{\mathrm{II}}^{i,N}]^\top & \mathbb{M}_{\mathrm{III}}^{i,N} \end{pmatrix},$$

where all the blocks are 3×3 matrices. We denote:

$$\mathbb{M}^N := \frac{1}{N}\sum_{i=1}^N \mathbb{M}_I^{i,N}\delta_{h_i^N}, \qquad \mathbb{F}^N := \frac{1}{N}\sum_{i=1}^N \left[\mathbb{M}_I^{i,N} v_i^N + \frac{1}{N}\mathbb{M}_{II}^{i,N}\omega_i^N\right]\delta_{h_i^N}.$$

The $(\mathbb{M}^N)_{N\in\mathbb{N}}$ are matrix measures on $\overline{\Omega}$ while the $(\mathbb{F}^N)_{N\in\mathbb{N}}$ are vectorial measures on $\overline{\Omega}$. We again assume convergence of these measures when tested against arbitrary test-functions $x \mapsto w(x) \in C(\overline{\Omega};\mathbb{R}^3)$.

In the framework depicted above, our result reads:

Theorem 5. *Assume that there exists* $\mathbb{M} \in L^\infty(\Omega;\mathcal{M}_3(\mathbb{R}))$ *and* $\mathbb{F} \in H^{-1}(\Omega;\mathbb{R}^3)$ *such that*

$$\mathbb{M}^N \rightharpoonup \mathbb{M}, \qquad \mathbb{F}^N \rightharpoonup \mathbb{F}.$$

Then $(u^N)_{N\in\mathbb{N}}$ *converges weakly in* $H_0^1(\Omega)$ *to the unique vector-field* $\bar{u} \in H_0^1(\Omega)$ *for which there exists a pressure* $\bar{p} \in L^2(\Omega)$ *such that:*

$$\left\{ \begin{array}{rcll} -\Delta\bar{u} + \nabla\bar{p} & = & \mathbb{F} - \mathbb{M}\bar{u}, & \text{in } \mathcal{D}'(\Omega), \\ \mathrm{div}\bar{u} & = & 0, & \text{in } \mathcal{D}'(\Omega). \end{array} \right.$$

Several remarks are in order. Firstly, since $\mathbb{M}^{i,N}$ is positive definite for arbitrary i, N we have that the matrix $\mathbb{M}(x)$ is symmetric positive for a.e. x in Ω. Hence, the limit problem is well-posed in $H_0^1(\Omega)$. Secondly, the scheme of the proof is similar to the one in the case of spherical particles. However, to estimate remainder terms, we do not apply explicit formulas but the decay properties **S2.** One further difficulty here is that we need that the constant controlling these decays are independant of the shape $\mathcal{B}_i^N$. Finally, in the computation of the main term (see the paragraph *Asymptotics*), we treat the integrals on the two boundaries of $\mathcal{C}_i^N \setminus \overline{B_i^N}$ differently. Indeed, in the interior integral, we have (applying the scaling properties that we remarked in Proposition 4 and the symmetries of the matrices $\mathbb{M}^{i,N}$):

$$\begin{aligned}\int_{\partial B_i^N} \Sigma(\widetilde{w}_i, q_i)\cdot u^N \mathrm{d}\sigma &= \int_{\partial B_i^N} \Sigma(\widetilde{w}_i, q_i) n \cdot (v_i^N + \omega_i^N \times (x - h_i^N))\mathrm{d}\sigma \\ &= \int_{\partial B_i^N} \Sigma(\widetilde{w}_i, q_i) n \mathrm{d}\sigma \cdot v_i^N + \int_{\partial B_i^N} (x - h_i^N) \times \Sigma(\widetilde{w}_i, q_i) n \mathrm{d}\sigma \cdot \omega_i^N \\ &= \frac{1}{N} v_i^N \cdot \mathbb{M}_I^{i,N} w(h_i^N) + \frac{1}{N^2}\omega_i^N \cdot [\mathbb{M}_{II}^{i,N}]^\top w(h_i^N) \\ &= \frac{1}{N} w(h_i^N) \cdot \left[\mathbb{M}_I^{i,N} v_i^N + \frac{1}{N}\mathbb{M}_{II}^{i,N}\omega_i^N\right]\end{aligned}$$

and the vectorial measure $\mathbb{F}^N$ appears. While, in the outer integral, we have (by applying again the conservation properties of solutions to Stokes problems):

$$\begin{aligned}\int_{\partial \mathcal{C}_i^N} \Sigma(\widetilde{w}_i, q_i)\cdot \bar{u}_i^N \mathrm{d}\sigma &= -\int_{\partial B_i^N} \Sigma(\widetilde{w}_i, q_i) n \cdot \mathrm{d}\sigma \cdot \bar{u}_i^N \\ &= \frac{1}{N} w(h_i^N) \cdot \mathbb{M}_I^{i,N} \bar{u}_i^N.\end{aligned}$$

Remarking that $\bar{u}_i^N \sim u(h_i^N)$ this makes the vectorial measure $\mathbb{M}^N$ appear. We see here the reason why the rotation term in the flux $\mathbb{F}^N$ has no counterpart in the friction matrix $\mathbb{M}^N$.

3.2. **Toward a time-dependant model.** As we explained in the introduction, our motivation for studying homogenization of the Stokes problem in perforated domains is to contribute to an analytical derivation of the system (4). However, even in the case of spheres, it is unlikely that Theorem 1 is sufficient. Indeed, in the Navier Stokes/Newton system, the particle dynamics equations is of second order so that, for initial times, the particle motions are driven by their initial velocities. In particular, one has to expect that computations in the case where there is no fluid still hold and that a significant amount of particles become $1/\sqrt{N}$-close one-another in short time (in consistency with the Boltzman-Grad scaling, see the introduction of [9] for instance). Assumption (8) is then no longer relevant. A crucial difficulty is thus to obtain similar convergence results as Theorem 1 for more general configurations.

This issue is tackled in [10]. To describe the details obtained therein, we need further notations on the N-configuration. We go back to the case of the introduction where N spherical particles of radius $1/N$ translate inside a bounded fluid. We remind that the centers of the particles are denoted $(h_i^N)_{i=1,\dots,N}$ and that their respective translation velocities are $(v_i^N)_{i=1,\dots,N}$ (no rotation) . We introduce a parameter measuring the minimal distance between particles:

$$d_m^N := \min\left\{\mathrm{dist}(h_i^N, h_j^N), \mathrm{dist}(h_i^N, \partial\Omega)\right\}. \tag{16}$$

We also introduce a parameter measuring the concentration of obstacles. Given $\lambda \in (0,\infty)$, we set:

$$\rho_m^N(\lambda) := \max_{x\in\Omega} \sharp\{i \in \{1,\dots,N\}, \text{ s.t. } h_i^N \in B(x,\lambda)\} \tag{17}$$

(where the symbol $\sharp$ stands for the cardinal of a finite set). To compute the asymptotics of the solution (u^N, p^N) when $N \to \infty$ we assume again the uniform bound (9):

$$\frac{1}{N}\sum_{i=1}^{N} |v_i^N|^2 \leqslant E_0$$

and convergence of the empirical measures ρ^N, j^N:

$$\rho^N \rightharpoonup \rho \qquad j^N \rightharpoonup j,$$

where $\rho \in L^\infty(\Omega)$ and $j \in L^2(\Omega;\mathbb{R}^3)$. In this framework, our main result reads (see [10]):

Theorem 6. *Assume that there exists a sequence $(\lambda^N)_{N\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that:*

$$|\lambda^N|^3 \ll d_m^N, \qquad \rho_m^N(\lambda^N) \leqslant N|\lambda^N|^3, \qquad \frac{|\lambda^N|^5}{d_m^N} \leqslant \frac{1}{N^{\frac{2}{3}}}. \tag{18}$$

Then $(u^N)_{N\in\mathbb{N}}$ converges weakly in $H_0^1(\Omega)$ to the unique $\bar{u} \in H_0^1(\Omega)$ for which there exists a pressure $\bar{p} \in L^2(\Omega)$ such that:

$$\left\{\begin{array}{rcll} -\Delta\bar{u} + \nabla\bar{p} &=& 6\pi(j - \rho\bar{u}) & \text{in } \mathcal{D}'(\Omega)\,, \\ \mathrm{div}\bar{u} &=& 0, & \text{in } \mathcal{D}'(\Omega). \end{array}\right.$$

The main novelty of this result lies in the assumption (18). First, we note that $\rho_m^N(\lambda^N) \geqslant 1$ and $d_m^N \leqslant diam(\Omega)$. Consequently, assumption (18) implies:

$$d_m^N \gg \frac{1}{N} \qquad \lambda_N \geqslant \frac{c}{N^{1/3}}.$$

This assumption (18) is less restrictive than the previous (8). Indeed, if we assume $\lambda_N = 1/N^{1/3}$, the condition on d_m^N implies that we can reach a minimal distance larger than $1/N$ but arbitrary close to $1/N$ (which is less restrictive than the previous $1/N^{1/3}$). However, the counterpart is that the second condition in (18) requires that there is a finite number of particles in any ball of radius $1/N^{1/3}$ (the last condition is automatically satisfied for these λ^N). In the case $\lambda_N = 1/N^{1/3}$ the gain with respect to (8) is then to pass from "one particle in any ball of width $1/N^{1/3}$" to "at most a given finite number of particles in a ball of width $1/N^{1/3}$". However, this theorem encompasses more general sequences $(\lambda^N)_{N\in\mathbb{N}}$. In particular, without further informations, one may assume that the position/velocities of the particles are taken randomly according to a law $F^N(dx_1,\dots,dx_N,dv_1,\dots,dv_N)$. If we assume that F^N is asymptotically of the form $f^{\otimes N}(dx,dv)$ for a sufficiently smooth distribution f we may prove that assumption (18) contains "almost any" configuration.

The proof of Theorem 6 is a generalization of the one depicted in the previous section. With similar arguments, we obtain that the sequence $(u^N)_{N\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. So we may extract a subsequence that we do not relabel converging to some divergence-free $\bar{u} \in H_0^1(\Omega)$. The game is one more time to pass to the limit in

$$\int_\Omega \nabla u^N : \nabla w$$

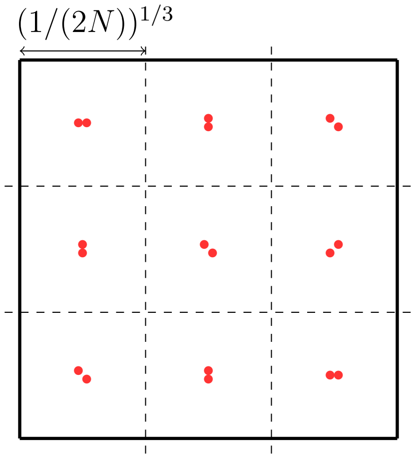

FIGURE 2. A N-obstacle configuration

for arbitrary divergence-free $w \in C_c^\infty(\Omega)$. We apply again the localization process by splitting Ω into cells $\mathcal{C}_i^N$. But, in this new approach, we do not choose the $\mathcal{C}_i^N$ with respect to the d_m^N but with respect to λ^N: the $(\mathcal{C}_i^N)_i$ is a covering of Ω with (semi-open) cubes of width λ^N. We introduce again w_i (resp. $\widetilde{w}_i$) the solutions to the Stokes problem inside $\mathcal{C}_i^N$ – outside the particles – having the same trace as w (resp. taking the values $w(h_l^N)$) on the particles B_l^N that are inside $\mathcal{C}_i^N$. We prove that $\widetilde{w}_i$ is almost the combination of solutions to the Stokes problem outside B_l^N with boundary condition $w(h_l^N)$. Hence, the computations of the limit leading term is similar to the previous case. We note that there is a further difficulty here because some particles may cross (or be simply too close to) the boundaries of the cells $\mathcal{C}_i^N$. By a measure-theory argument, we prove that we can choose the covering so that there are few such particles that we can delete in our computations. We refer to [10] for more details.

3.3. **Limiting cases.** Extending the dilution regime – for which we may prove that the Brinkman term has the form (7) – is a crucial step toward a rigorous justification of the transition between (1)-(2)-(3) and (4). However, much remains to be done for that purpose. In that respect, we prove also in [10] that, if one works with the discriminating parameters d_m^N and $\rho_m^N(\lambda)$ that we introduce above, one may not expect more general dilution regimes than (18) for which homogenization of the Stokes equations yield a Stokes-Brinkman problem (with the Brinkman term given by (7)). In particular, we discuss the two conditions:

$$\lim_{N\to\infty} \frac{d_m^N}{|\lambda^N|^3} = +\infty \tag{19}$$

and

$$\sup_{N\in\mathbb{N}} \frac{\rho_m^N(\lambda^N)}{N|\lambda^N|^3} < \infty. \tag{20}$$

Firstly, we focus on (19). For even N, we construct a configuration (see Figure 2) in which

- Ω is a unit cube splitted into $N/2$ sub-cubes of width $(2/N)^{1/3}$
- any sub-cube contains 2 particles, that we label with successive indices, each particle being at a distance h/N of the center of the sub-cube ($h > 0$ is a given parameter).

We then consider the solution (u^N, p^N) to (5)-(6) with vanishing rotation in the boundary conditions ($\omega_i^N = 0$) and such that two particles in the same sub-cube share the same translation velocity ($v_i^N = v_i^{N+1}$ for any odd integer i).

In this particular case, one can consider that the pair of particles in one sub-cube is one particle whose shape has two connected components. In this case, we may apply the computations of [11]. This yields that u^N will converge to a $\bar{u}$ satisfying:

$$\left\{ \begin{array}{rcl} -\Delta \bar{u} + \nabla \bar{p} & = & (\bar{\mathbb{F}} - \bar{\mathbb{M}}\bar{u})\,, \\ \mathrm{div}\bar{u} & = & 0 \end{array} \right. \quad \text{on } \Omega$$

but with $\bar{\mathbb{M}} \neq 6\pi \mathbb{I}_3$. In this first case, we keep the Brinkman term but we lose the simple form of this term.

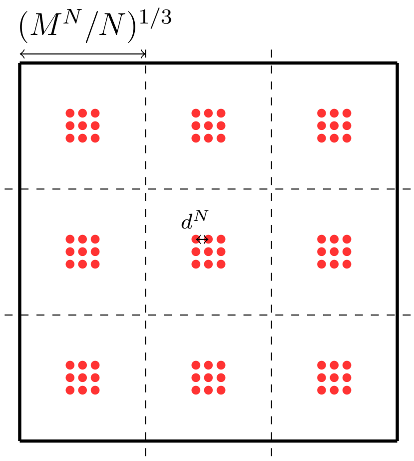

Figure 3. A N-obstacle configuration

We may also completly lose the Brinkman term when (20) is broken. Consider for instance the configuration depicted in Figure 3 i.e., consider that Ω is the unit cube in $\mathbb{R}^3$. Let $(M^N)_{N\in\mathbb{N}}$ be a diverging sequence of integers and split Ω into sub-cubes of width $(M^N/N)^{1/3}$. In each sub-cube, pack M^N particles distributed on an array of width d^N. Then, one may again interpret the group of particles inside each sub-cube as one particle with complicated shape with capacitary parameter $\sigma^N = |M^N|^{1/3}/\sqrt{N d^N}$. Taking $M^N = \ln(N)$ and $d^N = \sqrt{\ln(N)}/N$ we make σ^N converge to $+\infty$. Hence, as proven in [1], the influence of the holes disappears in the asymptotics $N \to \infty$. On the other hand, the empirical measures ρ^N converges to the uniform distribution in Ω, and the asymptotic problem do not contain a Brinkman friction term. This shows that whatever the chosen sequence λ^N one of the three conditions in (18) is not satisfied. If one chooses $\lambda^N = 1/N^{1/3}$ for instance, we do have that the first condition and last conditions are satisfied, but the second one is not: any of the clusters of particles is containted in a ball of radius $1/N^{1/3}$ so that $\rho_m^N(\lambda^N) = M^N$. Consequently

$$\lim_{N\to\infty} \frac{\rho_m^N(\lambda^N)}{N|\lambda^N|^3} = +\infty.$$

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