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On the persistence of decorrelation in the theory of wave turbulence

Anne-Sophie de Suzzoni

À propos de la persitance des décorrélations dans la théorie de la wave turbulence

Résumé

On étudie les propriétés statistiques des solutions des équations de Kadomstev-Petviashvili (KP-I et KP-II) sur le tore lorsque la condition initiale est une variable aléatoire. On se donne une variable aléatoire u_0 à valeurs dans un espace de Sobolev de régularité suffisamment importante telle que ses coefficients de Fourier soient indépendants. On suppose également que les lois de ces coefficients sont invariantes par multiplication par $e^{i\theta}$ pour tout $\theta \in \mathbb{R}$. On s'intéresse alors à la persistance des décorrélations des coefficients de Fourier $(u_n(t))_n$ des solutions de KP-I et KP-II ayant pour condition initiale u_0 au sens où l'on estime l'espérance $E(u_n\overline{u_m})$ en fonction du temps et de la taille ε de la donnée initiale. Ces estimées sont sensibles à la présence ou l'absence de résonance au sein des interactions à trois ondes, c'est-à-dire, en notant ω_k la relation de dispersion de KP-I ou KP-II, à si $\omega_k + \omega_l - \omega_{k+l}$ s'anulle (modèle résonnant, KP-I) ou non (modèle non résonnant, KP-II). Dand le cas de l'équation résonante, les espérances $E(u_n\overline{u_m})$ restent petites jusqu'aux temps d'ordres $o(\varepsilon^{-1})$ alors que dans le cas de l'équation non-résonante, elles le restent jusqu'aux temps d'ordre $o(\varepsilon^{-5/3})$. Les techniques sont différentes en fonction du cas considéré, on utilise le lemme de Gronwall et des estimées de large déviation gaussiennes dans le cas résonant, et la structure de forme normale de KP-II dans l'autre.

MSC 2000: 35Q35, 35Q53.
Keywords: Wave turbulence, statistical equilibrium, random initial datum.

Abstract

We study the statistical properties of the solutions of the Kadomstev-Petviashvili equations (KP-I and KP-II) on the torus when the initial datum is a random variable. We give ourselves a random variable u_0 with values in the Sobolev space H^s with s big enough such that its Fourier coefficients are independent from each other. We assume that the laws of these Fourier coefficients are invariant under multiplication by $e^{i\theta}$ for all $\theta \in \mathbb{R}$. We investigate about the persistence of the decorrelation between the Fourier coefficients $(u_n(t))_n$ of the solutions of KP-I or KP-II with initial datum u_0 in the sense that we estimate the expectations $E(u_n\overline{u_m})$ in function of time and the size ε of the initial datum. These estimates are sensitive to the presence or not of resonances in the three waves interaction, that is, denoting ω_k the dispersion relation, whether $\omega_k + \omega_l - \omega_{k+l}$ can be null (resonant model, KP-I) or not (non-resonant model, KP-II). In the case of a resonant equation, the expectations $E(u_n\overline{u_m})$ remain small up to times of order $o(\varepsilon^{-1})$ whereas in the case of a non-resonant equation, they do up to times of order $o(\varepsilon^{-5/3})$. The techniques used are different depending on the cases, we use Gronwall lemma and Gaussian large deviation estimates for the resonant case, and the normal form structure of KP-II in the other one.

1. Introduction

We present here the results of [1] and [2] about the persistence of decorrelation between Fourier coefficients of the solution of a Hamiltonian equation. We call

$$u(t) = \sum_{n=(n_x,n_y)} u_n(t) e^{i(n_x x + n_y y)}$$

the solution of the Kadomstev-Petviashvili equation on the torus of dimension 2 given in terms of its Fourier coefficients. Assuming that initially, these Fourier coefficients u_n are independent random variables, we are interested in the evolution in time of the expectation $E(u_n\overline{u_m})$.

We start by giving the motivation of these studies. It comes from the theory of wave turbulence and more precisely from the project initiated by Zakharov and Filonenko in [8]. We refer to [5, 9] for further reading the subject.

1.1. Capillary waves

In this paper, [8], Zakharov and Filonenko introduce the notion of statistical equilibrium in the framework of capillary waves. They consider a fluid contained in a domain Ω_t depending on time, whose bottom is fixed at $z = -h$ (h may be ∞) and that has a free surface $\eta(t)$, that is

$$\Omega_t = \{(x,y,z) \in \mathbb{R}^3 \mid -h \le z \le \eta(t)\} .$$

The velocity field v of the fluid is supposed to be irrotational, incompressible and with constant density. Hence, v is the gradient of a potential ϕ that satisfies inside

Ω_t

$$\triangle\phi = 0 \ .$$

The conservation of matter on the free surface gives the following kinematic equation on η :

$$\partial_t\eta(x,y) = (\partial_z\phi)(x,y,\eta(x,y)) - (\partial_x\phi)(x,y,\eta(x,y))\partial_x\eta(x,y) \\ - (\partial_y\phi)(x,y,\eta(x,y))\partial_y\eta(x,y) \ .$$

On the free surface, they assume that the fluid is only submitted to surface tension (they neglect gravity and viscosity), which gives

$$(\partial_t\phi)(x,y,\eta(x,y)) - \frac{1}{2}(\nabla\phi)^2(x,y,\eta(x,y)) = \nabla.\frac{\nabla\eta(x,y)}{\sqrt{1+|\eta(x,y)|^2}} \ .$$

What is more, they assume that the velocity is null at the bottom, that is

$$\nabla\phi(x,y,-h) = 0.$$

This system of equation can be written as a 2D Hamiltonian equation on the variable η and $\varphi(x,y) = \phi(x,y,\eta(x,y))$. With a change of unknown in the Fourier variable :

$$\eta_n = |n|^{-1/4}(\beta_n + \overline{\beta_{-n}}) \ , \ \varphi_n = -i|n|^{-1/4}(\beta_n - \overline{\beta_n})$$

and approaching the non-linearity by a quadratic one, they get a complex Hamiltonian equation whose non linearity takes the form of a three waves interaction $k, l \to k+l$, or $k \to k-l, l$, or $0 \to k, l, -(k+l)$:

$$\dot{\beta}_n - i\omega_n\beta_n = \int \Big(V_n^{k,l}\delta(k+l-n)\beta_k\beta_l \\ + 2W_n^{k,l}\delta(k-l-n)\beta_k\overline{beta_l} + X_n^{k,l}\delta(n+k+l)\overline{\beta_k\beta_l}\Big) dkdl \ ,$$

where ω_n is the dispersion relation of their equation.

1.2. Statistical equilibrium

They consider the solution of this equation when initially, the β_n are independent random variables whose laws are invariant under multiplication by $e^{i\theta}$. They compute statistical equilibrium of the system, that is a solution for the β_n such that the expectations $E(|\beta_n|^2)$ do not depend in time. For that, they derive what is called a kinetic equation, which means that they compute the time derivative $\partial_t E(|\beta_n|^2)$ in function of the map $k \mapsto E(|\beta_k|^2)$ admitting some approximations.

Calling $n_k = E(|\beta_k|^2)$, they find that it behaves according to

$$\partial_t n_k = S_t(n,n) - 2\nu k^2 n_k \ , \tag{1.1}$$

where n is the map $k \mapsto n_k$ and S_t is bilinear (and depends on k). The second term $-2\nu k^2 n_k$ is a damping term added a posteriori to ensure the convergence of the energy and which they forget to compute the statistical equilibrium. The first one is written

$$S_t(n,n) = 4\pi \int |V_k^{k_1,k_2}|^2(n_{k_1}n_{k_2} - n_kn_{k_1} - n_kn_{k_2}\delta(k-k_1-k_2)\delta_{\omega_k-\omega_{k_1}-\omega_{k_2}}dk_1dk_2 + \\ 8\pi \int |V_{k_1}^{k,k_2}|^2(n_{k_1}n_{k_2} + n_kn_{k_1} - n_kn_{k_2}\delta(k-k_1+k_2)\delta_{\omega_k-\omega_{k_1}+\omega_{k_2}}dk_1dk_2 \ .$$

The first δ is a Dirac δ, the second one means that they keep only the resonant three waves interaction, that is the ones whose frequencies $\omega_k-\omega_{k_1}-\omega_{k_2}$ and $\omega_k-\omega_{k_1}+\omega_{k_2}$ are equal to 0.

They then compute particular solutions of $S_t(n,n)=0$ and find $n_k=C|k|^{-1/2}$, which corresponds to an invariant measure that uses the conservation of the matter. Another solution is $n_k=C|k|^{-18/4}$ which is interpreted as an exchange of energy that is not seen when the expectation is taken. These solutions are called Kolmogorov-Zakharov (or KZ) spectra.

1.3. Approximations

In order to compute the kinetic equation (1.1), they do several approximations.

The major one is that they assume that not only initially the $\beta_n(t=0)$ are independent, but also that the $\beta_n(t)$ are at all times t independent. They justify this approximation in the same way as they justify the fact that they keep only the quadratic terms in the non linearity : they assume that the initial datum is small. Hence if the solution is of size ε, the non linearity should be of size ε^2, and $E(|\beta_n|^2)-|E(\beta_n)|^2$ should be of the same size as its initial value, that is ε^2 whereas when $n\neq m$, as the initial value of $E(\beta_n\overline{\beta}_m)-E(\beta_n)E(\overline{\beta}_m)$ is 0, $E(\beta_n\overline{\beta}_m)-E(\beta_n)E(\overline{\beta}_m)$ should be of size at most ε^3 because of the non linearity. Indeed, in first approximation, it can be considered as behaving as

$$E(\beta_n\overline{\beta}_m)\sim\int\int\Big(V_n^{k,l}\delta(k+l-n)E(\beta_k\beta_l\overline{\beta}_m)+ \\ 2W_n^{k,l}\delta(k-l-n)E(\beta_k\overline{\beta_l\beta}_m)+X_n^{k,l}\delta(n+k+l)E(\overline{\beta_k\beta_l\beta}_m)\Big)dkdl\ .$$

As we will see in the sequel, thanks to the assumptions on the initial datum and cancellations in the computations, it is actually of size at most ε^5 or ε^6 depending on the properties of the initial datum.

Another approximation is that they keep only resonant three waves interaction, i.e. the ones such that the frequency of this interaction $\omega_{k+l}-\omega_k-\omega_l$ is null.

We now propose to understand these approximations by doing the same computations (though on another framework) but without assuming the independence at all times and without forgetting the non-resonant terms. This will provide a "kinetic equation" of the form :

$$\partial_t n_k=S_t(n(t=0),n(t=0))+\text{remainder}$$

where the default in independence appears in the remainder and the role of resonance appears in the bilinear form S_t. Notice that the bilinear form S_t is taken on the initial datum, but it did not seem such a restriction considering that we are looking for solutions of $\partial_t n_k=0$.

To make sure we do the difference between resonant and non-resonant cases, we treat two close asymptotic models of water waves, the Kadomstev-Petviashvili equations, one presenting resonances within the three waves interaction and the other not.

2. Description of the problem and framework

2.1. The KP-equations

The KP-equations models water waves (in which both gravity and surface tension are considered) in the approximation of small amplitudes and long wavelengths. They have been built by Kadomstev and Petviashvili in [4] as an extension of the Korteweg de Vries equation in dimension 2, in the case of transverse perturbation, and they are given by

$$\partial_x(\partial_t u + \alpha\partial_x^3 u + \frac{\varepsilon}{2}\partial_x u^2) + \kappa\partial_y^2 u = 0 \; . \tag{2.1}$$

We consider them on the torus of dimension 2 and with real-valued solutions.

First, if the mean value of u along x

$$\int_{\mathbb{T}} u(x,y)dx$$

is initially equal to 0, then it remains so as u evolves in time. Hence we consider the equation in the space

$$H^s = \Big\{ \sum_{n_x,n_y} u_{n_x,n_y} e^{i(n_x x + n_y y)} \mid \forall n_x, n_y \in \mathbb{Z}\times\mathbb{Z} \;,\; \Big(u_{0,n_y} = 0 \text{ and } u_{-n_x,-n_y} = \overline{u_{n_x,n_y}} \Big) \text{ and } \sum_{n_x\neq 0} |n|^{2s}|u_{n_x,n_y}|^2 < \infty \Big\}$$

normed by

$$\|u\|_{H^s} = \sqrt{\sum_{n_x\neq 0} |n|^{2s}|u_{n_x,n_y}|^2} \; .$$

Thus, we can rewrite the equation as

$$\partial_t u + \alpha\partial_x^3 u + \frac{\varepsilon}{2}\partial_x u^2 + \kappa\partial_x^{-1}\partial_y^2 u = 0 \; .$$

The parameter α is positive due to Physics considerations. The parameter ε is small compared to 1 and represents the former smallness of the initial datum. Indeed, because of its quadratic non linearity, it is equivalent to consider KP, with no constant in front of the non linearity and an initial datum of size ε or with an ε in front of the non linearity and an initial datum of size 1. Remark that with this formulation, the fact that $E(\beta_n\overline{\beta}_m)$ is as small as ε^5 in the context of Zakharov and Filonenko's paper corresponds to $E(u_n\overline{u}_m)$ being as small as ε^3 in our context. Both conventions have been used in the Physics literature. Depending on whether the surface tension predominates over gravity (KP-I) or not (KP-II), κ is equal to -1 or $+1$.

2.2. Dispersion and resonance within the three waves interaction

The dispersion relation of KP equation is given by, with $n = (n_x, n_y) \in \mathbb{Z}^* \times \mathbb{Z}$

$$\omega_n = \alpha n_x^3 - \kappa\frac{n_y^2}{n_x}$$

indeed, $-i\omega_n$ is the eigenvalue of $\alpha\partial_x^3 + \kappa\partial_x^{-1}\partial_y^2$ associated to $e^{i(n_x x + n_y y)}$.

In terms of Fourier coefficients, the equation is written :

$$\dot{u}_n - i\omega_n u_n = -\frac{\varepsilon}{2} i n_x \sum_{k+l=n} u_k u_l \ .$$

Its non-linearity is thus of the form of a three waves interaction.

The frequency of the three waves interaction $k, l \to k+l$ is

$$\Delta_n^{k,l} := \omega_k + \omega_l - \omega_n = 3\alpha n_x k_x l_x + \kappa \frac{(k_y l_x - k_x l_y)^2}{n_x l_x k_x}$$

when $n = k + l$.

In the case of KP-I, we have $\kappa = -1$, the two terms of $\Delta_n^{k,l}$ are of opposite signs, hence $\Delta_n^{k,l}$ can be zero, the three waves interaction is resonant. In the case of KP-II, $\kappa = +1$, hence the two terms are of the same sign and we have $|\Delta_n^{k,l}| \geq 3|n_x k_x l_x| > 0$, the three waves interaction is not resonant.

2.3. Initial datum

We want to inquire about the statistical properties of a solution of KP equations thus we have to describe the random initial datum.

Let $(g_n)_{n\in\mathbb{N}^*\times\mathbb{Z}}$ a sequence of independent identically distributed random variables. We assume that the law of the g_n is invariant under the multiplications by $e^{i\theta}$, $\theta \in \mathbb{R}$. These g_n contain the statistical information of the initial datum.

Let $(\lambda_n)_{n\in\mathbb{N}^*\times\mathbb{Z}}$ a sequence of complex number. The λ_n represent the size of the Fourier coefficients of the initial datum.

As we consider real-valued solutions of KP, we define what is left of the sequences by

$$\lambda_{-n} = \overline{\lambda_n} \ , \ g_{-n} = \overline{g_n} \ .$$

The immediate consequence of these independence and invariance assumptions is that the expectation of a product of an odd number of g is always 0, that is, for all $p \in \mathbb{N}$, $(n_1, \dots, n_{2p+1}) \in (\mathbb{Z}^* \times \mathbb{Z})^{2p+1}$, we have

$$E\Big(\prod_{i=1}^{2p+1} g_{n_i}\Big) = 0 \ .$$

Besides, for the product of even numbers of g, we have the formula :

$$E\Big(\prod_{i=1}^{2p} g_{n_i}\Big) = \prod_{n\in A} \delta_{m_1(n)}^{m_2(n)} E(|g_n|^{2m_1(n)})$$

where $A = \{|n_i| \mid i = 1, \dots, 2p\}$, $m_1(n)$ is the cardinal of the set $\{i \mid n_i = n\}$, $m_2(n)$ the cardinal of the set $\{i \mid n_i = -n\}$, and δ is the Kronecker symbol. In particular, this yields, for $p = 2$,

$$E\Big(\prod_{i=1}^{4} g_{n_i}\Big) = \begin{cases} E(|g_n|^4) & \text{if } \exists\sigma \in S^4 \text{ such that } n_{\sigma(1)} = n_{\sigma(2)} = -n_{\sigma(3)} = -n_{\sigma(4)} \\ E(|g_n|^2)^2 & \text{if } \exists\sigma \in S^4 \text{ such that } n_{\sigma(1)} = -n_{\sigma(3)} \\ & \quad\text{and } n_{\sigma(2)} = -n_{\sigma(4)} \text{ and } |n_{\sigma(2)}| \neq |n_{\sigma(1)}| \\ 0 & \text{otherwise,} \end{cases}$$

where S^4 is the set of permutations of $\{1, 2, 3, 4\}$.

The initial datum is given by

$$u(t=0) = \sum_{n\in\mathbb{Z}^*\times\mathbb{Z}} g_n\lambda_n e^{i(n_x x + n_y y)} \ .$$

3. Results

In the first subsection, we describe the results of [1]. We assume there usual integration properties on the sequence $(g_n)_n$, that the g_n satisfy large Gaussian deviation estimates, which only allows to expand $E(u_n\overline{u}_m)$ in ε up to the second order.

In the second subsection, we describe the result of [2]. We assume more integrability on g_n, that they are L^∞, in order to perform the expansion up to order ε^3.

3.1. Expansion to order 2

We present here the results of [1]. We assume that g_n satisfies large Gaussian deviation estimates, which means that

$$E(e^{\gamma \mathrm{Re} g_n}),\ E(e^{\gamma \mathrm{Im} g_n}) \leq Ce^{c\gamma^2}$$

for all $\gamma\in\mathbb{R}$. Remark that as the law of g_n is invariant under the multiplication by $e^{-i\pi}$, the law of $\mathrm{Re}(g_n)$ is the same as the one of $\mathrm{Im}(g_n) = \mathrm{Re}(e^{-i\pi}g_n)$.

On the λ_n, we assume that

$$\sum |n|^{2s}|\lambda_n|^2$$

is finite, for some $s > 3$.

This implies that for some c small enough, the mean value of $e^{c\|u_0\|^2_{H^s}}$ is finite, and thus that $u(t=0)$ belongs to H^s almost surely.

Theorem 3.1 ([1]). *Under these assumptions, and until times of order ε^{-1}, i.e. for $t\in[-T,T]$ with $1+T\leq C\varepsilon^{-1}$, we have*

$$E(\overline{u_m}(t)u_n(t)) = \delta_n^m E(|g_n|^2)|\lambda_n|^2 + \varepsilon^2\delta_n^m G_n(t) + \varepsilon^3 R_{n,m,\varepsilon}(t)$$

with

$$G_n(t)\sim t^2$$

and

$$|R_{n,m,\varepsilon}(t)| \leq C_{n,m}|t|^3 \ .$$

We have a similar result for KP-II. We make the same assumptions on the initial datum, except that the regularity is restricted to $s > 2$ instead of $s > 3$.

Theorem 3.2 ([1]). *Under the previous assumptions (but with $s > 2$) and until times of order ε^{-1}, we have*

$$E(\overline{u_m}(t)u_n(t)) = \delta_n^m E(|g_n|^2)|\lambda_n|^2 + \varepsilon^2\delta_n^m G_n(t) + \varepsilon^3 R_{n,m,\varepsilon}(t)$$

with $G_n(t)$ bounded uniformly in time and

$$|R_{n,m,\varepsilon}(t)| \leq C_{n,m}\ |t|(1+|t|) \ .$$

Remark 3.1. *We have an explicit formula for $G_n(t)$, the term of second order.*

$$G_n(t) = 2 \sum_{k+l=n} \left(\int_0^t \int_0^{t'} \cos(\Delta_n^{k,l}\tau) d\tau dt' \right) \times$$
$$\left(\aleph_{k,l} n_1 |\lambda_k|^2 |\lambda_l|^2 - \aleph_{l,-n} k_1 |\lambda_l|^2 |\lambda_n|^2 - \aleph_{-n,k} l_1 |\lambda_n|^2 |\lambda_k|^2 \right) \ ,$$

with

$$\aleph_{k,l} = \begin{cases} E(|g_n|^4) & \text{if } k = l \\ E(|g_n|^2)^2 & \text{otherwise.} \end{cases}$$

The mean value $E(\overline{u_m} u_n)$ is null up to order 2 in ε when $n \neq m$.

For KP-I, it is an expansion in εt, valid if $t = o(\varepsilon^{-1})$. For KP-II, it is an expansion for $|t| = o(\varepsilon^{-1/2})$.

The term of order 3 shall be null but we need more assumptions on the initial datum to prove it.

For KP-I, $\Delta_n^{k,l}$ can be null, thus $\left(\int_0^t \int_0^{t'} \cos(\Delta_n^{k,l}\tau) d\tau dt' \right)$ can be $t^2/2$. Besides, given a suitable α, the sum $\sum_{k+L=n}$ involves enough couples (k,l) for one to be such that $\Delta_n^{k,l} = 0$, which explains the behaviour in time (t^2) of $G_n(t)$.

For KP-II, $\Delta_n^{k,l}$ can not be null, thus $\left(\int_0^t \int_0^{t'} \cos(\Delta_n^{k,l}\tau) d\tau dt' \right) = -\frac{\cos(\Delta_n^{k,l} t) - 1}{(\Delta_n^{k,l})^2}$, which explains the uniform bound in time.

The time scale ε^{-1} seems quite natural for KP-I but could be improved for KP-II, using the structure of normal form of the equation. Indeed, using this technique, the equation can be transformed into an equation with a cubic non linearity, making it of order ε^2 instead of ε. Besides, the computations performed to prove the result comfort one in the idea that the term of order 3, if it could be computed, would be null. However, the initial datum is not integrable enough (Gaussian estimates are not sufficient) to bound the remainder term in the case of an expansion of the expectation to order 3.

3.2. Expansion to order 3

In [2], we make other assumptions on the initial datum, losing in integrability in the probability space but gaining in regularity and in time in the result.

On g_n, we assume $g_n \in L^\infty_{\text{proba}}$, and on λ_n, we assume that

$$\sum |n|^{2s} |\lambda_n|^2$$

is finite, for some $s > 1$. This implies that the norm $\|u(t=0)\|_{L^\infty, H^s}$ is finite and in particular that $u(t=0)$ belongs almost surely to H^s.

Theorem 3.3. *Under the previous assumptions and until times of order $\varepsilon^{-5/3}$, we have*

$$E(\overline{u_m}(t) u_n(t)) = \delta_n^m E(|g_n|^2) |\lambda_n|^2 + \varepsilon^2 \delta_n^m G_n(t) + \varepsilon^4 R_{n,m,\varepsilon}(t)$$

with $G_n(t)$ bounded uniformly in time and

$$|R_{n,m,\varepsilon}(t)| \leq C_{n,m} \, |t| (1 + |t|)^{7/5} \ .$$

Remark 3.2. *The assumption on g_n allows us to do the expansion on ε up to order 3.*

The remainder $\varepsilon^4 R_{n,m,\varepsilon}(t)$ is small compared to 1 if $t = o(\varepsilon^{-5/3})$, but it is small compared to ε^2 only if $t = o(\varepsilon^{-5/6})$.

4. Sketch of the proof

The strategy consists in expanding the solution u in ε and then inputing this expansion in the computation of $\partial_t E(e^{i(\omega_m-\omega_n)t}u_n\overline{u}_m)$ which gives explicit formulae for the first terms in ε and a bound depending on the expansion of u of the remainder. We then bound the remainder of the expansion of u to estimate the one of $E(u_n\overline{u}_m)$.

4.1. Picard interactions

The general idea is to expand the solution u in its first Picard interactions and then input this expansion in the expression of

$$\partial_t E\Big(e^{-i\omega_n t}u_n e^{i\omega_m t}\overline{u_m}\Big) \ .$$

Depending on the different techniques we use, we expand the solution u to either order 2 :

$$u = a + \varepsilon b + \varepsilon^2 e(\varepsilon) \ ;$$

or 3 :

$$u = a + \varepsilon b + \varepsilon^2 c + \varepsilon^3 d(\varepsilon) \ .$$

We recall that u satisfies, in terms of Fourier coefficients :

$$\dot{u}_n - i\omega_n u_n = -\varepsilon i n_x \frac{1}{2}\sum_{k+l=n} u_k u_l \ .$$

Keeping only the terms of order 0 in ε, we get for a_n :

$$\dot{a}_n - i\omega_n a_n = 0 \ ,$$

and with the initial datum, we get

$$a_n(t) = e^{i\omega_n t}\lambda_n g_n \ .$$

Keeping only the terms of order 1, we get

$$\dot{b}_n - i\omega_n b_n = -i n_x \frac{1}{2}\sum_{k+l=n} a_k a_l \ ,$$

which gives, as $b_n(t=0) = 0$,

$$b_n(t) = -i n_x \frac{e^{i\omega_n t}}{2}\sum_{k+l=n}\int_0^t e^{i\Delta_n^{k,l}t'}dt'\lambda_k\lambda_l g_k g_l \ .$$

We recall that $\Delta_n^{k,l} = \omega_k + \omega_l - \omega_n$ is the three waves interaction frequency. As it can be 0 in the case of KP-I, the norm of $b_n(t)$ behaves like $|t|$. In the case of KP-II, as $\Delta_n^{k,l}$ can not be null, b_n is uniformly bounded in time.

This leaves the following equation on e :

$$\partial_t e + (\alpha\partial_x^3 + \kappa\partial_x^{-1}\partial_y^2)e = -\frac{1}{2}\partial_x\Big(\frac{u^2 - a^2}{\varepsilon}\Big) \ ,$$

with initial datum 0 and where u is the short cut for $a + \varepsilon b + \varepsilon^2 e$.

Keeping only the terms of order 2 in the equation on u, we get an equation for c and then an explicit formula for c_n which describes the behaviour on time of c. The norm of c behaves like $1+|t|$ in the case of KP-II (the three wave interaction is resonance free, but this is not the case of the four waves interaction, involved in c), and like $1+t^2$ in the case of KP-I. The second order is given by :

$$c_n(t) = -n_x e^{i\omega_n t} \sum_{j+k+l=n} \Big[\int_0^t \Big(e^{i\Delta_n^{j+k,l} t'} \int_0^{t'} e^{i\Delta_{j+k}^{j,k}\tau} d\tau \Big) dt' \Big] \frac{j_x+k_x}{2} \lambda_j\lambda_k\lambda_l g_j g_k g_l \; .$$

In the case of KP-II, the behaviour in time is driven by

$$\int_0^t \Big(e^{i\Delta_n^{j+k,l} t'} \int_0^{t'} e^{i\Delta_{j+k}^{j,k}\tau} d\tau \Big) dt' = \int_0^t \frac{e^{i(\omega_j+\omega_k+\omega_l-\omega_n)t'} - e^{i\Delta_n^{j+k,l}t'}}{i\Delta_{j+k}^{j,k}} dt' \; ,$$

this term involves the frequency of the four waves interaction $j,k,l \to n$, $\omega_j+\omega_k+\omega_l-\omega_n$, which can be 0.

This leaves the following equation on d :

$$\partial_t d + (\alpha\partial_x^3 + \kappa\partial_x^{-1}\partial_y^2) d = -\frac{1}{2}\partial_x\Big(\frac{u^2-a^2-2\varepsilon ab}{\varepsilon^2}\Big) \; ,$$

with initial datum 0 and where u is the short cut for $a+\varepsilon b+\varepsilon^2 c+\varepsilon^3 d$.

We now want to expand $\partial_t E\Big(e^{-i\omega_n t}u_n e^{i\omega_m t}\overline{u_m}\Big)$. First, we can write it as

$$\partial_t E\Big(e^{-i\omega_n t}u_n e^{i\omega_m t}\overline{u_m}\Big) = A_{n,m} + \overline{A_{m,n}}$$

with

$$\begin{aligned} A_{n,m} &= E\Big(\partial_t(e^{-i\omega_n t}u_n)e^{i\omega_m t}\overline{u_m}\Big) \\ &= -in_x e^{i(\omega_n-\omega_n)t}\varepsilon\frac{1}{2}\sum_{k+l=n} E(u_k u_l \overline{u_m}) \; . \end{aligned}$$

We then have to input the expansion of u in $A_{n,m}$.

4.2. Probabilistic cancellations

Depending on the order of the expansion of u, we can write $A_{n,m}$ either as

$$A_{n,m} = \varepsilon A_{n,m}^{(1)} + \varepsilon^2 A_{n,m}^{(2)} + \varepsilon^2 B_{n,m}(\varepsilon)$$

or

$$A_{n,m} = \varepsilon A_{n,m}^{(1)} + \varepsilon^2 A_{n,m}^{(2)} + \varepsilon^3 A_{n,m}^{(3)} + \varepsilon^4 C_{n,m}(\varepsilon)$$

where the $A_{n,m}^{(j)}$ do not depend on ε. Indeed, they can be written

$$\begin{aligned} A_{n,m}^{(1)} &= -in_x e^{i(\omega_n-\omega_n)t}\frac{1}{2}\sum_{k+l=n} E(a_k a_l \overline{a_m}) \; , \\ A_{n,m}^{(2)} &= -in_x e^{i(\omega_n-\omega_n)t}\frac{1}{2}\sum_{k+l=n} E(b_k a_l \overline{a_m} + a_k b_l \overline{a_m} + a_k a_l \overline{b_m}) \; , \\ A_{n,m}^{(3)} &= -in_x e^{i(\omega_n-\omega_n)t}\frac{1}{2}\sum_{k+l=n} E(c_k a_l \overline{a_m} + a_k c_l \overline{a_m} + a_k a_l \overline{c_m} + a_k b_l \overline{b_m} + b_k a_l \overline{b_m} + b_k b_l \overline{a_m}) \; . \end{aligned}$$

This is where the importance of the assumptions on the law of g_n appears. We recall that thanks to the independence of g_n and g_m when $|n| \neq |m|$, and to the invariance of g_n under the multiplication by $e^{i\theta}$, we have, for every $(n_1, \ldots, n_{2p+1}) \in (\mathbb{Z}^* \times \mathbb{Z})^{2p+1}$,

$$E\Big(\prod_{i=1}^{2p+1} g_{n_i}\Big) = 0$$

and for every $(n_1, \ldots, n_4) \in (\mathbb{Z}^* \times \mathbb{Z})^4$,

$$E\Big(\prod_{i=1}^{4} g_{n_i}\Big) = \begin{cases} E(|g_n|^4) & \text{if } \exists \sigma \in S^4 \text{ such that } n_{\sigma(1)} = n_{\sigma(2)} = -n_{\sigma(3)} = -n_{\sigma(4)} \\ E(|g_n|^2)^2 & \text{if } \exists \sigma \in S^4 \text{ such that } n_{\sigma(1)} = -n_{\sigma(3)} \\ & \qquad \text{and } n_{\sigma(2)} = -n_{\sigma(4)} \text{ and } |n_{\sigma(2)}| \neq |n_{\sigma(1)}| \\ 0 & \text{otherwise,} \end{cases}$$

where S^4 is the set of permutations of $\{1, 2, 3, 4\}$.

Since $a_k = e^{i\omega_k t}\lambda_k g_k$, we have that $A^{(1)}_{n,m}$ is a sum of mean values of products of 3 g. As the mean value of any product of an odd number of g is null, $A^{(1)}_{n,m}$ is null.

Then, b_n is a sum of products of 2 g, whereas c_n is a sum of products of 3 g, which makes $A^{(3)}_{n,m}$ a sum of mean values of products of 5 g. Again, thanks to the fact that the mean value of any product of an odd number of g is null, $A^{(3)}_{n,m} = 0$.

Finally, further computations give the structure of $A^{(2)}_{n,m}$, which leads to $A^{(2)}_{n,m} = 0$ if $n \neq m$, and

$$\begin{aligned} G_n(t) &= \int_0^t 2\Re(A^{(2)}_{n,n}(t'))dt' \\ &= 2\sum_{k+l=n} \Big(\int_0^t \int_0^{t'} \cos(\Delta_n^{k,l}\tau)d\tau dt'\Big) \\ &\quad \times \Big(\aleph_{k,l} n_1 |\lambda_k|^2|\lambda_l|^2 - \aleph_{l,-n} k_1 |\lambda_l|^2|\lambda_n|^2 - \aleph_{-n,k} l_1 |\lambda_n|^2|\lambda_k|^2\Big) \end{aligned}$$

with

$$\aleph_{k,l} = \begin{cases} E(|g_n|^4) & \text{if } k = l \\ E(|g_n|^2)^2 & \text{otherwise.} \end{cases}$$

To get an idea why $A^{(2)}_{n,m} = 0$ when $n \neq m$, let us input the expression of b_k in function of g and λ in the sum

$$e^{i(\omega_n - \omega_n)t} \sum_{k+l=n} E(b_k a l \overline{a}_m) \ .$$

We have

$$b_k(t) = i\frac{ik_x}{2} e^{i\omega_k t} \sum_{j+q=k} \int_0^t e^{i\Delta_k^{j,q}t'}dt' \lambda_j \lambda_q g_j g_q$$

such that the sum can be written

$$e^{i(\omega_n - \omega_n)t} \sum_{k+l=n} E(b_k a l \overline{a}_m) = -\frac{i}{2} \sum_{k+l=n, j+q=k} \Big[e^{i\Delta_n^{k,l}t} \int_0^t e^{i\Delta_k^{j,q}t'}dt'\Big] \times k_x \lambda_j \lambda_q \lambda_l \overline{\lambda}_m E(g_j g_q g_l \overline{g_m}) \ .$$

For the expectation $E(g_j g_q g_l \overline{g_m})$ not to be null, we have to pair the indexes. If we pair j with q and l with m (i.e. $j = -q$ and $l = m$), we get $k = j + q = (0, 0)$ which makes the term under the sum null. We have then to pair j with either l or m and q with the other one. If the pairing is $j = -l$ and $q = m$, we get $n = k + l = j + q + l = q = m$. Therefore, there are solutions (j, q, l) such that the term under the sum is not 0 only if $n = m$. A similar discussion can be done for the other sums describing $A^{(2)}_{n,m}$.

4.3. Gronwall lemma and probabilistic bounds

In the case of Gaussian large deviation estimates on the initial datum, we are constrained to expand the solution only to order 2 if we want to be able to bound the remainder. Indeed, differentiating the L^2 norm of e to the square for a fixed event of the probability space yields

$$\partial_t \|e\|_{L^2} \lesssim \|2ab + \varepsilon b^2\| + \|\varepsilon a + \varepsilon^2 b\| \; \|e\|_{L^2} \ ,$$

where we do not precise the norms except for e but we assumed enough regularity on the initial datum. Using Gronwall lemma provides the estimate

$$\|e(t)\|_{L^2} \lesssim \int_0^t \|2ab + \varepsilon b^2\| dt' e^{\int_0^t \|\varepsilon a + \varepsilon^2 b\|} \ .$$

In order to estimate the remainder in $E(u_n \overline{u_m})$, we have to estimate the expectation of the norm of e, and for that, we need to ensure that $e^{\varepsilon^2 \int_0^t \|b(t')\| dt'}$ is integrable (in probability). Since b is the first Picard interaction, and from its explicit formula, we can bound it for each event ω of the probability space Ω :

$$\|b(t,\omega)\| \lesssim |t| \; \|u_0(\omega)\|^2$$

and with a loss of regularity in the case of KP-I and

$$\|b(t,\omega)\| \lesssim \|u_0(\omega)\|^2$$

without loss of regularity in the case of KP-II. Since there exists c such that $e^{c\|u_0\|^2_{H^s}}$ is integrable, we get a bound for e for suitable times, and until time $|t|$ of order ε^{-1}, it is given by $E(\|e(t)\|^p_{L^2}) \lesssim t^2$ in the case of KP-I, and $E(\|e(t)\|^p_{L^2}) \lesssim (1 + |t|)$ in the case of KP-II, which correspond to a remainder of size $\varepsilon^3|t|^3$ in the case of KP-I and $\varepsilon^3|t|(1 + |t|)$ in the case of KP-II, once the integration over time is done.

If we look now at the equation on d, in order to inquire on an expansion to order 3 of the solution, we get, thanks to Gronwall lemma, that

$$\|d\|_{L^2} \lesssim \int_0^t \|(b^2 + 2ac) + \varepsilon 2bc\varepsilon^2 c^2\| dt' e^{\int_0^t \|\varepsilon a + \varepsilon^2 b + \varepsilon^3 c\|} \ ,$$

which is not integrable in probability, as the norm of c behaves like the norm of u_0 to the cube and since $e^{\|u_0\|^3}$ is not integrable in general with our assumptions on the initial datum.

To expand the expectation $E(u_n \overline{u_m})$, we have to use another technique.

4.4. Normal forms

We focus now on KP-II.

The main idea is to perform a contraction argument on d. However, the time scale for KP-I seems natural, but we could get a better time scale for KP-II, hence, instead of performing a direct contraction on d, we use normal forms. Indeed, the source term in the direct equation on d involves the product ac, which behaves in time like $1+|t|$, as d solves

$$\partial_t d + (\alpha\partial_x^3 + \partial_x^{-1}\partial_y^2)d = -\frac{1}{2}\partial_x \frac{u^2 - a^2 - 2\varepsilon ab}{\varepsilon^2}$$

and

$$\frac{u^2 - a^2 - 2\varepsilon ab}{\varepsilon^2} = b^2 + 2ac + 2\varepsilon bc + \varepsilon^2 c^2 + 2d(\varepsilon a + \varepsilon^2 b + \varepsilon^3 c) + \varepsilon^4 d^2 \ .$$

Performing the contraction argument directly would lead to bounds on d of size $(1+|t|)^2$ until times ε^{-1}. This technique does not provide a gain in time.

In [7], N. Tzvetkov introduces the multi linear maps S and F, which gives the normal form structure for the KP-II. For earlier references of this type of normal forms, we can mention [3, 6]. The map S is given, in terms of Fourier coefficients, by

$$S(u,v)_n = \frac{in_x}{2} \sum_{k+l=n} \frac{u_k v_l}{i\Delta_n^{k,l}}$$

and $F(u,v,w) = -S(w, \partial_x(uv))$. Both maps are continuous from H^s to itself if $s>1$.

If u is a solution of KP-II, then $v = u + \varepsilon S(u,u)$ is the solution of

$$\partial_t v + (\partial_x^3 + \partial_x^{-1}\partial_y^2)v = \varepsilon^2 F(u,u,u) \ .$$

Note that the non linearity is now cubic with ε^2 in front of it. This is the reason why we expect to obtain a better time scale using normal forms.

The expansion of v is given by

$$v = a + \varepsilon(b + S(a,a)) + \varepsilon^2(c + 2S(a,b)) + \varepsilon^3\Big(2S(a,c) + S(b+\varepsilon c, b+\varepsilon c) + d + 2S(\varepsilon a + \varepsilon^2 b + \varepsilon^3 c, d) + \varepsilon^4 S(d,d)\Big) \ .$$

We can check that $b + S(a,a)$ satisfies :

$$\partial_t(b + S(a,a)) + (\partial_x^3 + \partial_x^{-1}\partial_y^2)(b + S(a,a)) = 0$$

with initial datum $S(u_0,u_0)$ and that $c + 2S(a,b)$ satisfies

$$\partial_t(c + 2S(a,b)) + (\partial_x^3 + \partial_x^{-1}\partial_y^2)(c + 2S(a,b)) = F(a,a,a)$$

with initial datum 0. This yields to the fact that

$$w = 2S(a,c) + S(b+\varepsilon c, b+\varepsilon c) + d + 2S(\varepsilon a + \varepsilon^2 b + \varepsilon^3 c, d) + \varepsilon^4 S(d,d)\Big)$$

solves the equation

$$\partial_t w + (\partial_x^3 + \partial_x^{-1}\partial_y^2)w = \frac{F(u,u,u) - F(a,a,a)}{\varepsilon}$$

with initial datum 0.

As $\frac{F(u,u,u)-F(a,a,a)}{\varepsilon}$ factorizes in

$$\frac{F(u,u,u) - F(a,a,a)}{\varepsilon} = F(\frac{u-a}{\varepsilon}, u, u,) + F(a, \frac{u-a}{\varepsilon}, u) + F(a, a, \frac{u-a}{\varepsilon})$$

and since $u = a + \varepsilon b + \varepsilon^2 c + \varepsilon^3 d$ and $\frac{u-a}{\varepsilon} = b + \varepsilon c + \varepsilon^2 d$, each time that c appears in the equation, it is preceded by ε. Compared to the equation on d, we gain a ε in front of $|t|$ in the source term, which allows us to get a better time scale.

The map Λ that gives w in function of d is invertible on the ball of $L^\infty_{\text{proba}}, L^\infty([-T,T], H^s)$ of centre 0 and radius of order ε^{-4} for T of order ε^{-3}, hence it is not an obstacle to perform the contraction argument as

$$d(t) = \Lambda^{-1}\Big[\int_0^t e^{(t-t')(\alpha\partial_x^3 + \partial_x^{-1}\partial_y^2)}\Big(\frac{F(u(t'),u(t'),u(t')) - F(a(t'),a(t'),a(t'))}{\varepsilon}\Big)dt'\Big]\ .$$

We get that until times of size $\varepsilon^{-\beta}$, for $\beta \in [1,2]$,

$$\|d(t)\|_{L^\infty_{\text{proba}}, H^s} \le C(1+|t|)^{2-1/\beta}\ .$$

The most relevant β for the expansion of $E(u_n\overline{u_m})$ is $\beta = 5/3$.

Conclusion. Finally, we can say that, in both cases, smooth initial data that satisfy Gaussian large deviation estimates are enough to use Gronwall lemma on an expansion of the solution, and then expand $E(u_n\overline{u_m})$ to order 2 in ε at some fixed time less than ε^{-1}. In the resonant case, KP-I, it is even an expansion in εt, besides, the time scale is satisfying. What is more, ignoring the non resonant terms in the term of second order in ε of $E(|u_n|^2)$, $G_n(t)$, we retrieve the kinetic equation of wave turbulence.

In the non resonant case, KP-II, we perform a contraction argument on the expansion of the solution using normal forms, which enables us to expand the expectation to higher orders, and take rougher initial data, but enforces more integrability assumptions (L^∞ in probability instead of Gaussian estimates).

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