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Weak/BV Stability and a-contraction in fluid dynamics

Alexis Vasseur

Le principe BV/faible et la théorie de a-contraction pour la dynamique des fluides

Résumé

Cet article présente la théorie de a-contraction pour la stabilité en norme L^2 des solutions discontinues des lois de conservation et leurs limites asymptotiques. Elle peut être vue comme une extension du principe de stabilité fort/faible, introduit par Dafermos et DiPerna, à la stabilité des solutions discontinues–par exemple bornées en BV– au sein de la classe des solutions entropiques faibles générales. Par exemple, des solutions à petite variation totale (BV) des équations d'Euler barotropes peuvent être obtenues comme limites de viscosité évanescente de solutions des équations de Navier–Stokes compressibles associées. Dans le cadre barotrope, cela étend le célèbre travail de Bianchini et Bressan aux modèles avec viscosité physique.

Abstract

This paper introduces the a-contraction theory for the L^2 stability of discontinuous solutions to conservation laws and their inviscid limits. It can be viewed as an extension of the Weak–Strong stability principle, first introduced by Dafermos and DiPerna, to the stability of discontinuous solutions–e.g. bounded in BV–among general weak entropic solutions. For example, small BV solutions to barotropic Euler equations can be obtained as inviscid limits of solutions to the associated compressible Navier–Stokes equations. In the barotropic setting, this extends the celebrated work of Bianchini and Bressan to models with physical viscosity.

1. Introduction

Hyperbolic conservation laws can be cast, at a formal level, as systems of the type:

$$\begin{aligned} &\partial_t U + \sum_{i=1}^{d} \partial_{x_i} f_i(U) = 0 \qquad t>0,\ x\in\mathbb{R}^d, \\ &U(0,\cdot) = U^0, \end{aligned} \tag{1.1}$$

where $(t,x)\in\mathbb{R}^+\times\mathbb{R}^d$ are time and space, $U=(U_1,\cdot,U_n)\in\mathcal{V}_0\subseteq\mathbb{R}^n$ is the unknown, and U^0 is the given initial value. The equation is encoded in the flux functionals $f_i:\mathcal{V}_0\to\mathbb{R}^n$, which are assumed to be smooth in the interior $\mathring{\mathcal{V}}_0$. The hyperbolic structure means that for all states U in the interior of $\mathcal{V}_0$, and all direction $\xi\in\mathbb{R}^d_*$, the $n\times n$ matrix $\sum_{i=1}^d f_i'(U)\cdot\xi_i$ is diagonalizable with all distinct real eigenvalues.

These systems cover cornerstone models in fluid mechanics. Typically, solutions generate discontinuities in finite time, which makes rigorous analysis delicate–even in one spatial dimension. A paradigmatic case is the Euler system (see (2.1)) among the earliest PDEs formulated.

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Entropies and Uniqueness

An entropy is a scalar functional $\eta : \mathcal{V}_0 \to \mathbb{R}$ for which there exist associated fluxes $q_i : \mathcal{V}_0 \to \mathbb{R}$ satisfying, in $\mathring{\mathcal{V}}_0$,

$$q_i' = \eta' \cdot f_i' \qquad \text{for all } i = 1, \dots, d.$$

If η is convex, a weak solution to (1.1) is called entropic (relative to η) when it obeys–in the distribution sense–the inequatlity

$$\partial_t \eta(U) + \sum_{i=1}^{d} \partial_{x_i} q_i(U) \le 0 \qquad t > 0,\ x \in \mathbb{R}^d. \tag{1.2}$$

Entropy admissibility is necessary for uniqueness in the presence of discontinuities, but it is not sufficient in general: convex-integration counterexamples in [19] show nonuniqueness for the isentropic Euler system (where $n = 2$) in dimensions $d \ge 2$. Therefore, the uniqueness question is a complex problem that depends intrinsically on the structure of the considered solutions. Thus, uniqueness is tightly linked to the specific structural class of solutions under consideration. In particular, beyond the small-BV regime, the uniqueness of general entropic weak solutions for systems ($n \ge 2$) remains open even in one dimension.

Let us distinguish the set $\mathcal{S}_p$ of target patterns (solutions under study) from a broader class $\mathcal{S}_{\text{wild}}$ of less regular, *wild* solutions. In [22], Dafermos established the *weak/strong* uniqueness principle: *strong patterns*, namely solutions in

$$\mathcal{S}_p^{\text{Daf}} := \left\{ \text{Lipschitz solutions } \overline{U} \text{ to } (1.1) \text{ with values in } \mathring{\mathcal{V}}_0 \right\}$$

are unique within bounded entropic *weak solutions*; for any $K > 0$, define

$$\mathcal{S}_{\text{wild}}^K := \{ \text{Weak solutions } U \text{ to } (1.1)\ (1.2) \text{ such that } \|U\|_{L^\infty} \le K \}. \tag{1.3}$$

The same method yields L^2 stability of patterns $\overline{U}$ in $\mathcal{S}_p^{\text{Daf}}$ under perturbations of initial data by *wild* solutions U_ε in $\mathcal{S}_{\text{wild}}^K$:

$$\text{If } U_\varepsilon^0 \text{ converges in } L^2(\mathbb{R}^d) \text{ to } \overline{U}^0, \text{ then for every } t > 0,\ U_\varepsilon(t, \cdot) \text{ converges in } L^2(\mathbb{R}^d) \text{ to } \overline{U}. \tag{1.4}$$

For overviews, see [22], or [45]. Recent progress extends the framework to patterns with vacuum states (with values in $\mathcal{V}_0$) [26] and non-shock patterns with relaxed regularity [25].

Entropic solutions versus inviscid limits of Navier–Stokes

The a-contraction theory aims to extend the Dafermos weak/strong theory to study the uniqueness and stability of possibly *discontinuous* patterns. A surprising feature of the theory is that the results can be straightened by introducing more physics into the *wild solutions.* Most of the conservation laws can be derived, at least formally, from a (possibly degenerate) parabolic approximation. A typical case is the Euler system which can be seen as the inviscid limit of the Navier–Stokes system when the viscosity μ converges to 0. It is then convenient to replace the set of *wild perturbations* $\mathcal{S}_{\text{wild}}^K$ with the set of weak inviscid limits of solutions to the associated Navier–Stokes system (or other parabolic approximation):

$$\mathcal{S}_{\text{wild}}^{NS} := \{ \text{weak inviscid limits } U \text{ of solutions to the associated Navier–Stokes system} \}.$$

The stability result (1.4) can be reformulated in this context as an inviscid double limit of solutions to Navier–Stokes. Consider a pattern U solution to the inviscid conservation law (1.1). Then U is stable with respect to perturbations on the initial value among *wild* solutions in $\mathcal{S}_{\text{wild}}^{NS}$ if the following inviscid double limit holds. Consider a family of smooth initial conditions with values in $\mathring{\mathcal{V}}_0$. For $\varepsilon > 0, \nu > 0$, denote $U_{\varepsilon,\nu}$ the solutions to the Navier–Stokes system (or another parabolic approximation of the conservation laws) with initial value U_ε^0.

$$\begin{aligned} &\text{If } U_\varepsilon^0 \text{ converges strongly to } \overline{U}^0, \text{then for all } t > 0, \\ &U_{\varepsilon,\nu}(t, \cdot) \text{ converges to } \overline{U} \text{ when } \textit{both } \varepsilon, \nu \text{ converge to } 0. \end{aligned} \tag{1.5}$$

This is a double limit with respect to the viscosity coefficient ν, and the initial value U_ε^0. If we can first take the limit ν converges to 0, the inviscid double limit (1.5) implies the stability (1.4) among *wild* perturbations in $\mathcal{S}_{\text{wild}}^{NS}$. This kind of stability results is obtained at the level of the

Navier–Stokes systems, *uniformly* with respect to the viscosity. The theory of contraction with shifts is the first tool providing such uniform stability for discontinuous pattern limits. Note that these are very hard problems since small diffusions generically destabilize discontinuous patterns.

In [33], the stability of discontinuous planar entropy waves (or contact discontinuities) was shown to hold within the class of inviscid limits of solutions to Navier–Stokes–Fourier. This result holds in the multi-dimensional setting. However, we will stick to the mono-dimensional setting for this review.

2. Inviscid limits of the barotropic Navier–Stokes equations and small BV solutions to Euler

In one dimension, Euler equations take the form

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & t > 0,\ x \in \mathbb{R}, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, & t > 0,\ x \in \mathbb{R}, \\ (\rho(0,x), \rho u(0,x)) = (\rho^0, \rho^0 u^0), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $\rho : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ denotes the gas density and $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ its velocity field. In the barotropic case, the pressure is given by

$$p(\rho) = \rho^\gamma,$$

with adiabatic exponent $\gamma > 1$. The system is locally well-posed in the class of Lipschitz functions (see, for instance, [22]), although such solutions may develop singularities in finite time [43].

The entropy inequality can be written as:

$$\partial_t \eta(\rho, \rho u) + \partial_x(q(\rho, \rho u)) \leq 0, \qquad t > 0,\ x \in \mathbb{R}, \tag{2.2}$$

where the entropy (the total energy) and its flux are

$$\begin{aligned} \eta(\rho, \rho u) &= \rho \frac{|u|^2}{2} + \frac{\rho^\gamma}{\gamma - 1}, \\ q(\rho, \rho u) &= (\eta(\rho, \rho u) + \rho^\gamma) u. \end{aligned} \tag{2.3}$$

Solutions of (2.1) satisfying (2.2) are called *entropic solutions*. Lipschitz solutions automatically satisfy (2.2) as an equality.

For $\nu > 0$, consider now the compressible Navier–Stokes system

$$\begin{cases} \rho^\nu_t + (\rho^\nu u^\nu)_x = 0 \\ (\rho^\nu u^\nu)_t + (\rho^\nu |u^\nu|^2 + p(\rho^\nu))_x = \nu \big(\bar{\mu}(\rho^\nu) u^\nu_x\big)_x. \end{cases} \tag{2.4}$$

where $\bar{\mu}$ denotes the viscosity coefficient. We say that $U = (\rho, \rho u)$ is a *vanishing viscosity solution* of (2.1) if there exists a family of initial data $U^\nu_0 = (\rho^\nu_0, \rho^\nu_0 u^\nu_0) \to U^0 = (\rho^0, \rho^0 u^0)$ such that the corresponding solutions U^ν to (2.4) converge, up to a subsequence, to U as $\nu \to 0$.

Solutions of (2.4) can be constructed verifying the energy inequality (see [3, 24, 37])

$$\partial_t \eta(\rho^\nu, \rho^\nu u^\nu) + \partial_x(q(\rho^\nu, \rho^\nu u^\nu)) + \nu\, \bar{\mu}(\rho^\nu) |\partial_x u^\nu|^2 - \nu \partial_x(\mu(\rho^\nu) u^\nu \partial_x u^\nu) \leq 0, \tag{2.5}$$

which provides a natural justification for (2.2) in the limit $\nu \to 0$.

In the framework of small BV data for hyperbolic conservation laws, Glimm [27] established global existence for $n \times n$ systems (including (2.1)–(2.2) in one dimension) when the initial data are sufficiently small in $BV(\mathbb{R})$. Later, constructive proofs were obtained via front-tracking schemes [4], whose L^1 stability was shown in [5, 6, 13, 38]. Uniqueness was proved by Bressan and Goatin under the *Tame Oscillation Condition* [7], refining earlier work of Bressan and LeFloch [10], and further extended under the *Bounded Variation along space-like curves* condition [12]. These additional assumptions were later removed in the 2×2 case in [16, 28], which established a weak/BV stability principle for entropic solutions with strong traces. More recently, the general case was treated in [8, 11], though without the weak/BV structure.

In the celebrated work of Bianchini and Bressan [1], small BV solutions were shown to arise as vanishing viscosity limits for *artificial* viscosities (see also [9, 14] for convergence estimates). Yet, this result has not been extended to physical viscosity as in Navier–Stokes, the main difficulty being the absence of viscosity-uniform BV estimates (see [39] for fixed-viscosity estimates).

Using compensated compactness [44], Chen and Perepelitsa [18] proved in 2010 the existence of viscosity limits to (2.1) in one dimension for $\bar{\mu} = 1$, for general finite-energy initial data. However, even in this one-dimensional setting, it remained open whether small BV initial data yield viscosity limits that stay in BV, and whether such limits coincide with the unique small BV solution of (2.1).

In the paper [15], we establish two fundamental results concerning vanishing viscosity limits and weak/BV stability:

1. Any small BV solution of (2.1) is a vanishing viscosity limit; that is, it can be obtained as a double limit of Navier–Stokes solutions (2.4).

2. Small BV solutions are stable under perturbations of the initial data within the class of vanishing viscosity limits.

Importantly, the second statement requires neither strong trace assumptions nor a priori L^∞ bounds, in contrast with [16]. These results build on the a-contraction framework up to shift introduced in [29], and extend the relative entropy theory of Dafermos and DiPerna [21, 23] to the BV setting.

3. Precise statement of the results

We consider viscosity functionals satisfying the following property:

$$\begin{aligned}&\text{Assume that } \gamma > 1, \text{ and that there exist } 0 < C_1 < C_2 \text{ and } \alpha > 0 \text{ with}\\ &\gamma - 1 \le \alpha \le \gamma, \text{ and } C_1(1+\rho^\alpha) \le \tfrac{\bar{\mu}(\rho)}{\gamma} \le C_2(1+\rho^\alpha), \quad \forall\, \rho > 0.\\ &\text{If } \gamma \le \tfrac{5}{3}, \text{ we fix the value } \alpha = \gamma - 1.\end{aligned} \tag{3.1}$$

This assumption encompasses a broad class of viscosity functionals, while imposing a polynomial growth at high densities. Such a growth is consistent, for instance, with the shallow water case $\gamma = 2$, where $\alpha = 1$.

As discussed above, the analysis relies on the notion of relative entropy. For $U = (\rho, \rho u)$ and $\overline{U} = (\bar{\rho}, \bar{\rho}\bar{u})$ with $\bar{\rho} > 0$, the relative entropy of U with respect to $\overline{U}$ is defined by

$$\eta(U|\overline{U}) = \rho\frac{|u-\bar{u}|^2}{2} + \frac{p(\rho|\bar{\rho})}{\gamma - 1}, \qquad p(\rho|\bar{\rho}) = \rho^\gamma - \bar{\rho}^\gamma - \gamma\bar{\rho}^{\gamma-1}(\rho - \bar{\rho}) \ge 0.$$

For any fixed state $U_* = (\rho_*, \rho_* u_*) \in \mathbb{R}^+ \times \mathbb{R}$ defining the asymptotic states on the left and right, we introduce the following set of Euler initial data:

$$E^0 := \left\{ \begin{matrix} U^0 = (\rho^0, \rho^0 u^0) \\ \quad \in (L^1_{\mathrm{loc}} \cap L^\infty)(\mathbb{R}) \end{matrix} \;\middle|\; \inf(\rho^0) > 0,\ \int_{\mathbb{R}} \big(\rho^0|u^0 - u_*| + p(\rho^0|\rho_*)\big)\,\mathrm{d}x < \infty \right\}. \tag{3.2}$$

For any $U^0 \in E^0$, it follows that

$$\int_{\mathbb{R}} \eta(U^0(x)|U_*)\,\mathrm{d}x < \infty. \tag{3.3}$$

As in [18], we consider slightly regularized initial data at the Navier–Stokes level. Given $U^0 \in E^0$, we say that a smooth function $U_0^\nu := (\rho_0^\nu, \rho_0^\nu u_0^\nu)$ is an *adapted family of initial values* for the Navier–Stokes equations if it satisfies

$$\begin{aligned}&\text{There exist } \kappa_1^\nu, \kappa_2^\nu > 0 \text{ such that for all } x \in \mathbb{R},\\ &\kappa_1^\nu \le \rho_0^\nu(x) \le \kappa_2^\nu, \quad \text{and} \quad \lim_{\nu \to 0}\left(\int_{\mathbb{R}} \eta(U_0^\nu|U^0)\,\mathrm{d}x + \nu^2\int_{\mathbb{R}} |\partial_x\phi(\rho_0^\nu)|^2\,\mathrm{d}x\right) = 0,\end{aligned} \tag{3.4}$$

where ϕ is defined by $\phi'(\rho) = \frac{\bar{\mu}(\rho)}{\rho^{3/2}}$. It is showed in [15] that such adapted family of initial values exists for any initial value in E^0.

Our first result reads as follows.

Theorem 1. *Assume* (3.1)*. For any $U_* = (\rho_*, \rho_* u_*) \in \mathbb{R}^+ \times \mathbb{R}$, there exists $\varepsilon > 0$ such that the following holds. Let $U^0 \in E^0$ with $\|U^0\|_{BV(\mathbb{R})} \leq \varepsilon$, and let U_0^ν be any adapted family of initial values satisfying* (3.4)*. If U^ν denotes the solution to the Navier–Stokes system* (2.4) *with initial data U_0^ν, then*

$$U^\nu \longrightarrow U \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}),$$

where U is the small BV solution of (2.1) *associated with the initial data U^0, as $\nu \to 0$.*

Under the assumptions (3.4), (3.2), and (3.1), the existence and uniqueness of the solution to the Navier–Stokes system (2.4) follow from [40]. More precisely, [40, Theorem 2.1] ensures that for any $T > 0$, there exist $\kappa_2^\nu(T) > \kappa_1^\nu(T) > 0$ such that

$$\begin{aligned} &\rho^\nu - \rho_* \in L^\infty(0,T;H^1(\mathbb{R})), \quad \kappa_1^\nu(T) \leq \rho^\nu(t,x) \leq \kappa_2^\nu(T), \quad \forall\, x \in \mathbb{R}, t \leq T, \\ &u^\nu - u_* \in L^\infty(0,T;H^1(\mathbb{R})) \cap L^2(0,T;H^2(\mathbb{R})). \end{aligned} \tag{3.5}$$

For any $U^0 \in E^0$, and for any solution U^ν to (2.4)–(2.5) with initial values satisfying (3.4), the entropy inequality (2.5) implies

$$\limsup_{t>0,\nu>0} \int_{\mathbb{R}} \eta(U^\nu(t,x)|U_*)\,\mathrm{d}x \leq \int_{\mathbb{R}} \eta(U^0(x)|U_*)\,\mathrm{d}x. \tag{3.6}$$

Using the strict convexity of η and weak compactness, there exists $U \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ and a subsequence $\nu \to 0$ such that U^ν converges weakly in L^1_{loc} to U. We call any such U an *inviscid limit of Navier–Stokes* associated with the initial data U^0.

Our second theorem is as follows.

Theorem 2. *Assume* (3.1)*. For any $U_* = (\rho_*, \rho_* u_*) \in \mathbb{R}^+ \times \mathbb{R}$, there exists $\varepsilon > 0$ such that the following holds. Let $U^0 \in E^0$ with $\|U^0\|_{BV(\mathbb{R})} \leq \varepsilon$, and let $(U_n^0)_n$ be a sequence in E^0 such that*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \eta(U_n^0(x)|U^0(x))\,\mathrm{d}x = 0.$$

Then, for any sequence $(U_n)_n$ of inviscid limits of Navier–Stokes associated with the data U_n^0, one has

$$U_n \longrightarrow U \quad \text{strongly in } [L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})]^2,$$

where U is the small BV solution of (2.1) *corresponding to the initial value U^0.*

Note that only U is assumed to be of bounded variation; the perturbations U_n need not belong to BV. Hence, the theorem establishes a Weak/BV stability principle (in the spirit of the weak/strong principle of Dafermos and DiPerna): if U^0 is small in BV, then any inviscid limit of Navier–Stokes associated with U^0 coincides with the BV solution of the Euler system.

Theorem 1 shows that, in one space dimension, any small BV solution to (2.1) arises as a vanishing viscosity limit. Theorem 2 establishes the stability of these solutions under perturbations of the initial data within the class of vanishing viscosity solutions.

4. The a-contraction theory with shifts for Euler

In this section, we focus on the a-contraction theory with shifts as applied to the barotropic Euler and Navier–Stokes equations. The framework, however, has already been developed in several directions: for extremal shocks in general systems [29], for the stability of general solutions to 2×2 systems [16], and for the Riemann problem in the full Euler system and its associated Navier–Stokes–Fourier model [33, 42]. A multidimensional stability result for contact discontinuities in this setting was obtained in [32].

The inviscid limit results presented in Theorems 1 and 2 rely on earlier work concerning weak/BV stability at the level of the Euler equations themselves. Yet, these previous results are typically weaker than Theorem 2: uniform boundedness and strong trace properties were required for weak solutions in [16], whereas these conditional assumptions can be removed by exploiting the inviscid limit. For a general overview of recent progress in the a-contraction framework, see [46].

We begin by recalling the main ideas that apply directly to the hyperbolic equation before turning to the additional challenges that arise in the inviscid limit analysis.

4.1. The relative entropy method

For a fixed state $U_* \in \mathbb{R}^+ \times \mathbb{R}$, the map $U \mapsto \eta(U|U_*)$ defines an entropy associated with the corresponding entropy flux $q(U;U_*)$. From (2.1) and (2.2), any entropic solution satisfies, in the sense of distributions,

$$\partial_t \eta(U|U_*) + \partial_x q(U;U_*) \le 0. \tag{4.1}$$

The strict convexity of η implies that, as long as U and U_* take values in a compact set $\mathcal{C}$ away from vacuum,

$$\frac{1}{C}|U-U_*|^2 \le \eta(U|U_*) \le C|U-U_*|^2, \tag{4.2}$$

where C depends only on $\mathcal{C}$. Integrating the entropy inequality in x yields an L^2-type stability estimate (and contraction with respect to the relative entropy) for bounded entropic weak solutions relative to a constant state:

$$\int_{\mathbb{R}} |U(t,x)-U_*|^2 \,\mathrm{d}x \le C^2 \int_{\mathbb{R}} |U^0(x)-U_*|^2 \,\mathrm{d}x.$$

The relative entropy method, introduced by DiPerna and Dafermos [21, 23], consists in modulating the constant U_* by a Lipschitz solution V. Using finite propagation speed, the method establishes the uniqueness and L^2 stability of Lipschitz solutions within a broader class of bounded entropic weak solutions. It is also remarkably effective in asymptotic analyses when the approximating models are entropic—such as Navier–Stokes—and the limit solution is Lipschitz. In essence, Lipschitz solutions are stable under L^2 perturbations of the initial data and under entropic perturbations of the equation itself.

The situation becomes considerably more delicate when the reference pattern V is discontinuous. DiPerna already extended the approach to handle shock uniqueness in [23], and later developments extended it to Riemann solutions [17]. The aim of the a-contraction with shifts is to extend this L^2-based framework to the stability and asymptotic analysis of general discontinuous solutions.

4.2. A Kružkov-like theory for systems

For scalar conservation laws with flux $U \mapsto f(U) \in \mathbb{R}$, Kružkov developed a well-posedness theory for bounded solutions [35], based on the family of entropies

$$\eta_K(U|U_*) = |U-U_*|.$$

Any weak entropic solution U satisfies (4.1) for these entropies and their associated entropy flux

$$q_K(U;U_*) = \mathrm{sgn}(U-U_*)(f(U)-f(U_*)).$$

By modulating the constant through the doubling-of-variables method, Kružkov showed that two such weak entropic solutions U, V satisfy

$$\partial_t \eta_K(U|V) + \partial_x q_K(U;V) \le 0$$

in the sense of distributions. This yields the L^1 contraction property

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \eta_K(U(t,x)|V(t,x)) \,\mathrm{d}x \le 0, \tag{4.3}$$

from which well-posedness readily follows.

This elegant theory does not extend directly to systems, since the set of convex entropies is too restrictive and does not contain the Kružkov family. Nevertheless, from a single convex entropy, one can derive an associated Kružkov-like family of relative entropies satisfying (4.1). In general, rarefactions preserve the contraction property measured by relative entropy, but this is no longer true for arbitrary solutions. The obstruction arises because the relative entropy method is essentially L^2-based, while Kružkov's framework is L^1-based.

For shocks, the Rankine–Hugoniot condition implies that L^2 perturbations of order ε can induce a spatial shift producing an error of order $\sqrt{t\varepsilon}$ at time t. Hence, shocks cannot satisfy a strict L^2 contraction property. However, as shown in [29], a similar property can be recovered after introducing a weight $a > 0$ and compensating for the shift. This forms the foundation of the a-contraction theory with shifts.

Specifically, for any shock (U_L, U_R, σ_{LR}), that is, a discontinuous wave

$$S(t,x) = U_L \mathbf{1}_{\{x<\sigma_{LR}t\}} + U_R \mathbf{1}_{\{x>\sigma_{LR}t\}}$$

that is an entropic solution of (2.1), there exists a weight $a > 0$ such that, for any bounded entropic weak solution with strong traces, there exists a Lipschitz shift function $h(t)$ satisfying, for all $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{ a\int_{-\infty}^{\sigma_{LR}t+h(t)} \eta(U(t,x)|U_L)\,\mathrm{d}x + \int_{\sigma_{LR}t+h(t)}^{\infty} \eta(U(t,x)|U_R)\,\mathrm{d}x \right\} \le 0. \tag{4.4}$$

In the scalar case (see Léger [36]), one may take $a = 1$, recovering a contraction property akin to the L^1 theory (4.3), but expressed in terms of the L^2-based relative entropy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \eta(U(t,x)|S(t,x+h(t)))\,\mathrm{d}x \le 0.$$

For the Euler equations, however, the inequality does not hold without the weight (see [41]). Thus, (4.4) represents the closest L^2-analogue to Kružkov's L^1 contraction principle available for systems. At this stage, the inequalities are established only for elementary waves: shocks and rarefactions.

4.3. Weak/BV stability principle for Euler

Conservation laws possess finite propagation speed. Thanks to this property, one may hope to extend the contraction property from elementary waves to Riemann problems, and subsequently to small BV solutions. However, the presence of shifts makes it unclear how to define a consistent extension to general solutions. This program was first carried out in the scalar case in [34] and later adapted to 2×2 systems in [16].

Rather than seeking a pseudometric between weak solutions U_n and a small BV solution V, one uses the relative entropy to control the L^2 distance between U_n and the set of small BV functions. More precisely, let $U_* \in \mathbb{R}^+ \times \mathbb{R}$ and choose $\varepsilon > 0$ such that the small BV solution V belongs to

$$\mathcal{S} = \left\{ U \in L^\infty(\mathbb{R}) \;\middle|\; \|U - U_*\|_{L^\infty(\mathbb{R})} + \|U\|_{BV(\mathbb{R})} \le \varepsilon \right\},$$

where ε corresponds to the maximal BV norm admissible through the front-tracking construction. The a-contraction method with shift then yields, for all $t > 0$,

$$\inf_{W\in\mathcal{S}} \int_{\mathbb{R}} |U_n(t,x) - W(x)|^2\,\mathrm{d}x \le C\int_{\mathbb{R}} |U_n^0(x) - V^0(x)|^2\,\mathrm{d}x. \tag{4.5}$$

Combined with Bressan's small-BV uniqueness theory [4], this gives the weak/BV stability result: if a sequence of initial data U_n^0 converges in L^2 to the BV initial data V^0, then (4.5) implies that the limit U of U_n remains in $\mathcal{S}$ for all $t > 0$. Assuming uniform boundedness of the U_n, one obtains strong convergence, hence U is a small BV solution to Euler with initial value V^0. By uniqueness in this class, we conclude $U = V$.

To achieve (4.5), one constructs (via a modified front-tracking scheme) a function $\overline{U}_\delta$ taking values in $\mathcal{S}$ for all $t > 0$, together with a weight $a(t,x)$ satisfying $1/2 \le a(t,x) \le 2$, such that

$$\int_{\mathbb{R}} a(t,x)\,\eta(U_n(t,x)|\overline{U}_\delta(t,x))\,\mathrm{d}x \tag{4.6}$$

is nonincreasing in time, up to terms of order δ, the front-tracking parameter.

Front-tracking, a numerical approximation method involving only front dynamics, is a powerful tool for studying one-dimensional conservation laws (see [4]). The approximations are piecewise constant, with rarefactions treated in a suitable discrete form. To fit the relative entropy framework, the method is modified by introducing shifts on each front so that the a-weighted relative entropy decreases in time. The piecewise constant weight a is recomputed after each front interaction. Because these fronts do not exactly satisfy the Rankine–Hugoniot conditions (due to artificial shifts), $\overline{U}_\delta$ need not be an exact Euler approximation. Nevertheless, the front-tracking framework, based on the control of an interaction potential, remains valid and ensures that $\overline{U}_\delta$ stays within $\mathcal{S}$ for all t.

4.4. Towards the inviscid limit

The Navier–Stokes equations are compatible with the entropy structure, and thus with the relative entropy method. It is therefore natural to attempt a similar approach, seeking a control analogous to (4.6) (up to errors of order ν and δ) for solutions U^ν of Navier–Stokes. However, this extension is technically challenging for several reasons:

1. *Destabilization:* Navier–Stokes exhibits a destabilizing effect on shocks, with the formation of viscous layers (viscous shocks).

2. *Nonlocality:* The equations are no longer purely local—interactions between simple waves emerge.

3. *Lack of uniform bounds:* No uniform L^∞ bounds exist for solutions as viscosity varies, since only one entropy (the physical energy) is compatible with the Navier–Stokes system.

Consequently, the invariant regions ensuring L^∞ bounds for Euler are not preserved at the Navier–Stokes level—a major difference from artificial viscosity models [1]. As in [18], one must therefore contend with possible regions of high velocity or density, making the analysis significantly more intricate.

5. Main idea of the proof

Following [30, 31], the relative entropy method is applied to the Navier–Stokes equation in its Lagrangian form. The Lagrangian mass variables are the specific volume $v = 1/\rho$ and the velocity u, with the mass coordinate defined by $m(t,x) = \int_0^x \rho(t,z)\,\mathrm{d}z$. In these variables (where we keep the notation x for m), the Navier–Stokes system reads:

$$\begin{cases} v_t^\nu - u_x^\nu = 0, \\ u_t^\nu + p\,(v^\nu)_x = \nu\left(\dfrac{\bar{\mu}(v^\nu)}{v^\nu}u_x^\nu\right)_x, \end{cases} \tag{5.1}$$

with $p(v) = v^{-\gamma}$, $\gamma > 1$. The initial data (v_0^ν, u_0^ν) satisfy

$$\lim_{|x|\to\infty} (v_0^\nu, u_0^\nu) = (v_*, u_*),$$

so the total mass is infinite, though the density remains strictly positive (no vacuum). This property is preserved by the Navier–Stokes evolution, so (5.1) is equivalent to (2.4).

The Lagrangian framework is particularly suited to study the stability of viscous shock layers, which require a delicate balance between hyperbolic (Euler) and parabolic (viscous) effects. In particular, we need a robust stability estimate for these shock layers. Since they occur on the scale of ν, they must remain stable under perturbations of order δ, consistent with the front tracking approximation. This uniform stability with respect to viscosity was obtained in [30].

To achieve this, we introduce the effective velocity

$$h := u - \nu\frac{\bar{\mu}(v)}{v}v_x,$$

associated with the Bresch–Desjardins entropy [2]. In terms of (v^ν, h^ν), the system becomes

$$\begin{cases} v_t^\nu - h_x^\nu = -\nu\big(\mu(v^\nu)(v^\nu)^\gamma p\,(v^\nu)_x\big)_x, \\ h_t^\nu + p\,(v^\nu)_x = 0. \end{cases} \tag{5.2}$$

The relative entropy method is carried out on this system. Since the nonlinear term in the Euler equation is the pressure $p(v)$, depending only on v, the formulation (5.2) is advantageous: diffusion acts directly on v, providing control over this nonlinear term. After maximizing in h, stability reduces to a nonlinear Poincaré estimate in v.

Because no uniform L^∞ bound on $U^\nu = (v^\nu, h^\nu)$ holds with respect to ν, controlling extreme values of v^ν is delicate. This motivates the assumption (3.1) on the viscosity coefficient for large densities. Under this condition, one obtains a viscous analogue of (4.4) through the a-contraction with shift (see [31]): for any Euler shock (U_L, U_R, σ_{LR}), let $\widetilde{U}^\nu$ be the corresponding viscous layer, i.e. a solution to (5.2) with end states U_L, U_R. Then there exists a weight $a(x/\nu)$ (with limiting values determined by $a > 0$) such that for any solution U^ν to (5.2), there exists a Lipschitz shift $X(t)$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{-\infty}^{+\infty} a(x - X(t))\, \eta(U^\nu(t,x) \mid \widetilde{U}^\nu)\, \mathrm{d}x\right\} \le 0.$$

The use of the effective velocity imposes the condition on the initial data (3.4).

5.1. Construction of approximate solutions

As before, for any fixed viscosity $\nu > 0$, we want to construct an artificial small BV function $\overline{U}_{\nu,\delta}$ which stays close to U^ν for the relative entropy pseudometric. To construct this small BV function, the main idea is to construct an Euler-level front tracking approximation and to replace each wave by its viscous analogue, shifting them according to the a-contraction principle. Several difficulties arise, however. Let us fix the main parameters. The global BV strength $\varepsilon > 0$ is small but fixed throughout. The viscosity $\nu > 0$ tends to zero and is the smallest parameter. Two parameters belong to the front tracking scheme: δ, the discretization strength of rarefactions (vanishing slowest), and the interaction cutoff ρ_ν^{int}, used to select the Riemann solver and to bound below physical wave strengths. We choose $\rho_\nu^{\text{int}} = \nu^{1/3}$.

1. Since no uniform L^∞ bound holds for the general Navier–Stokes solution U^ν, the approximate solution $\overline{U}_{\nu,\delta}$ loses the finite speed of propagation (due to uncontrolled shifts). Moreover, for each t, $\eta(U^\nu(t,x) \mid \overline{U}_{\nu,\delta})$ is bounded only in L^1. Many error terms must therefore be handled using sign arguments. In particular, controlling pseudo-shock errors requires that the pseudo-shock velocity be monotone within the pseudo-shock layer.

2. Long-range interactions at the Navier–Stokes level pose another challenge. Superposing viscous waves is not an exact solution, and nonlinear interactions spread over wide regions: the width of a viscous shock of strength $|\sigma|$ is $\nu/|\sigma|$, large for small shocks. A sharp lower bound on shock strength is thus essential, leading to modified Riemann solvers in the front tracking scheme.

The standard algorithm [4] employs two solvers: an accurate solver (discretizing rarefactions) and a simplified one (used when $|\sigma_1\sigma_2| < \rho_\nu^{\text{int}}$) to prevent infinite cascades of waves. Simplified solvers merge physical waves and generate pseudo-shocks, all traveling at a fixed artificial speed to avoid mutual collisions. Here, due to the loss of uniform propagation speed, we instead set this pseudo-shock velocity to zero—no physical wave travels at this speed, even after shifts.

In our setting, the algorithm must satisfy three extra conditions: (i) all shock strengths remain bounded below proportionally to ρ_ν^{int}; (ii) the pseudo-shock velocity h increases with x; (iii) the total number of interactions up to time T grows at most like a power of $\log(1/\nu)$. To enforce these, we introduce three types of solvers: accurate, simplified, and adjusted. All are modified relative to their classical definitions to guarantee monotonicity of h. The adjusted solver handles overtaking interactions between shocks and rarefactions of the same family, ensuring outgoing shocks have strength $\mathcal{O}(\rho_\nu^{\text{int}})$. These adjustments, together with the barotropic structure, provide the desired $\log(1/\nu)$ control on total interactions.

Finally, interactions are resolved before waves become too close. The minimal distance d_j between them, gradually decreased after each interaction, is chosen as $d_j \sim (\rho_\nu^{\text{int}})^{1/4}$.

5.2. Construction of weight functions

At each time t with waves interactions, the a–contraction weight $a(\cdot, t)$ needs to be recomputed. To control the global a–contraction, it is then important to establish the decay of the weight $a(x,t)$ at these times:

$$\Delta a(x,t) = a(x,t^+) - a(x,t^-) \le 0.$$

for all x. The construction of $a(x,t)$ follows [20] for the Euler case, but unlike the inviscid setting, viscous fronts are diffuse, so the transition zones must be carefully repositioned. We must verify $\Delta a(x,t) \le 0$ for every x, not just the left and right interaction states. To achieve this, outgoing waves are shifted slightly in their propagation direction.

Among all interaction types, the overtaking of a shock by a rarefaction is the most delicate: here the total variation $L(t)$ decays by roughly twice the smaller incoming strength. Since the Glimm potential $Q(t)$ ensures decay of $L(t) + \kappa Q(t)$, we incorporate this combination into $a(x,t)$ to guarantee its decrease in accurate solvers. For the adjusted and simplified solvers, additional refined estimates are required.

5.3. Uniform estimates via a-contraction with shifts

The goal is to obtain uniform estimates of U^ν relative to the approximation $\overline{U}_{\nu,\delta}$. The argument relies on shifting the shocks in $\overline{U}_{\nu,\delta}$ to reconcile the Rankine–Hugoniot condition with the L^2 structure, and similarly shifting rarefaction fronts to control discretization errors. These shifts mitigate the L^1 errors induced by the lack of L^∞ control on U^ν.

5.4. Inviscid limit via compensated compactness

The combination of the weighted relative entropy and the front tracking with shifts yields an approximate solution $\overline{U}_{\nu,\delta}$ satisfying:

- its BV norm is uniformly bounded in ν and δ;
- $\eta(U^\nu \mid \overline{U}_{\nu,\delta}) \to 0$.

Hence any weak limit U_∞ (up to subsequence) of U^ν is uniformly of small BV. To show that U_∞ is an entropic Euler solution, strong convergence is required. However, due to uncontrolled shifts, time oscillations prevent strong convergence of $\overline{U}_{\nu,\delta}$. Following a variation of Chen and Perepelitsa [18], we instead apply compensated compactness to U^ν.

5.5. Conclusion

Since the inviscid limits are small-BV entropic Euler solutions, the uniqueness results of Bressan, and of Chen–Krupa–Vasseur, ensure that the limit is the unique small-BV solution. Consequently, the entire sequence U^ν converges. It is essential here that the uniqueness theorem applies without additional space-like BV conditions, which may fail for U_∞ due to residual shifts.

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