Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Forges-les-Eaux, 7 juin–11 juin 2004

V. Kamotski and G. Lebeau

On 2D Rayleigh-Taylor instabilities

J. É. D. P. (2004), Exposé n^{o} VII, 10 p.

<http://jedp.cedram.org/item?id=JEDP_2004____A7_0>

Journées *Équations aux dérivées partielles*
Forges-les-Eaux, 7 juin–11 juin 2004
GDR 2434 (CNRS)

On 2D Rayleigh-Taylor instabilities

V. Kamotski G. Lebeau

1. Introduction

Consider in the plane the flow in the gravity field of two ideal incompressible fluids of constant densities $\rho^{\pm} > 0$, with $\rho^{+} \neq \rho^{-}$. Velocity field satisfies the Euler equation

$$\frac{\partial \rho u}{\partial t} + u \cdot \nabla \rho u = -\nabla p + \rho g \tag{1}$$

$$\text{div } u = 0 \tag{2}$$

$$\rho_t + \text{div } (\rho u) = 0 \tag{3}$$

with initial data

$$u(x, 0) = u_0(x). \tag{4}$$

We suppose, that $u_0(x)$ satisfies the continuity equation (2), and the assumption that vorticity $\omega_0(x) = \text{rot } u_0$ is concentrated on some curve Σ_0 separating the two fluids, i.e. we suppose Σ_0 splits the plane into two domains Ω_+^0 and Ω_-^0, in which the fluids have constant densities ρ_+ and ρ_- respectively. So we have

$$\omega_0 = \Omega \delta_{\Sigma_0} \tag{5}$$

with $\Omega = [u_{||}]$ being the jump of the tangent to Σ_0 component of velocity. Applying the rot operator to (1) we find that vorticity $\omega = \partial_x u^y - \partial_y u^x$ is formally constant along the flow in each subdomain where ρ is constant :

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{6}$$

Thus for $t > 0$ one expects the vorticity to remain concentrated on some curve Σ_t, with a time dependant curve Σ_t separating the two fluids .

This problem is known as the Rayleigh-Taylor instability problem. It has been shown in [SS85] that in the case of a periodic interface close to a flat line and with analytic data Ω, Σ_0, this problem is locally in time well posed. This result of local analytical well-posedness can be extended to the case of arbitrary analytical initial data, not necessarily small.

Considering a related problem of Kelvin-Helmholtz instability, i.e in the case $\rho_+ = \rho_-, \quad g = 0$, in [Leb02] and [Wu], it has been proved that the evolution problem of the vortex sheet is strongly ill-posed in the sense that if Ω does not vanish, analyticity of the data is a necessary condition to get a local in time solution with a $C^{1+\alpha}$ interface Σ with $\alpha > 0$. On the other hand it is well known that 2D water wave problem (in a sense, it is the case $\rho_- = 0$) is well posed in Sobolev spaces. We shown in this paper, nonetheless, that results on Kelvin-Helmholtz instability apply to this more involved Rayleigh-Taylor case. The following result holds

Theorem 1.1. *Suppose* $\Sigma = \{(x,t) : \ x \in \Sigma_t, \ t \in (-T,T)\} \in C^{1+\alpha}$ *with some* $\alpha > 0$ *and* Σ_t *is a simple closed curve for all* $t \in (-T,T)$. *Suppose also, that* $u \in L^\infty_{\mathrm{loc}}((-T,T); L^2_{\mathrm{loc}}(\mathbb{R}^2))$ *is a weak solution of* (1)-(3), *satisfying*

$$\begin{cases} \mathrm{rot}(u) = \Omega(t,s)\delta_{\Sigma_t}; \\ \lim\limits_{|x,y|\to\infty} u(t,x,y) = 0 \end{cases} \tag{7}$$

and an assumption, that

$$\exists c_0 > 0 \quad \forall (t,s) \quad c_0 \le \Omega(t,s) \le 1/c_0, \tag{8}$$

then Σ *and* Ω *are* C^∞ *smooth . If moreover, the jump of the tangent to* Σ *impulse* $w = [\rho u_{||}] = \rho_+ u_{+,||} - \rho_- u_{-,||}$ *doesn't vanish, then for each* t, *the data* $\Sigma(t)$ *and* $\Omega(t)$ *are analytic in* s.

Thus we see, that the Cauchy problem associated with the Rayleigh-Taylor instability is strongly ill posed in the Hadamard sense. We deduce this result from a stronger local one:

Theorem 1.2. *Suppose* $\Sigma = \{(x,t) : t \in (-T,T), x = r(t,s)\}$ *with* $r(t,s) \in C^\nu((-T,T); C^{1+\alpha}(s))$ *for some* $\nu > 0, \alpha > 0$. *Let* $\tilde{U} = (-T,T) \times U \subset \mathbb{R}^3$ *be a vicinity of* $(t_0, r(t_0,s_0)) \in \Sigma$. *Suppose* $u \in L^\infty_{loc}((-T,T); L^2_{loc}(U)) \cap C^\nu((-T,T); \mathcal{D}'(U))$ *is a weak solution of* (1)-(3), *satisfying in* $\tilde{U}$ *for some* $\Omega \in C^\nu(t; C^\alpha(s))$

$$\mathrm{rot}\ u = \Omega(t,s)\delta_\Sigma \tag{9}$$

then if $\Omega(t_0,s_0) \neq 0$, *there exist* $\varepsilon > 0$ *such that one has* $r \in C^{1+\nu}(|t-t_0| < \varepsilon; C^\infty(s_0-\varepsilon, s_0+\varepsilon))$ *and* $\Omega(t,s) \in C^\nu(|t-t_0| < \varepsilon; C^\infty(s_0 - \varepsilon, s_0+\varepsilon))$.

We expect that local analyticity in s is true under the hypothesis of the theorem 1.2. The above result shows that there is no hope to solve the local Cauchy problem associated with the Rayleigh Taylor instability in reasonable spaces, no matter the sign of $\rho^+ - \rho^-$ and gravity is. This situation is thus different from the water-wave problem, i.e the evolution of a single fluid with free boundary under gravity, which corresponds to the limiting case $\rho^+ = 0$. One should note here that, the condition $\rho^\pm \neq 0$ is essential in the verification of ellipticity of the local problem. It will be of great interest to analyse the surface tension effect in the problem in order to get local solvability in Sobolev spaces.

In this paper we present mostly sketches of the proofs, referring to [LK04] for details. We first reduce the problem to an evolution equation on the vortex sheet, and we distinguish the global and local cases. We then prove the local theorem 1.2. The key point in the proof is to reduce the problem to an elliptic one, using paradifferential calculus. In order to perform this reduction, we introduce a suitable local parametrization of the vortex, taking advantage of the galilean invariance of the problem. We then deduce from the local theorem global smoothness and analyticity under the hypothesis of the theorem 1.1. In all the paper, we work with Hölder spaces $C^\nu(t; C^\alpha_s)$ with non integer values $\nu,\ \alpha$.

2. Reduction to the problem on the vortex sheet

Let u be a weak solution of the Euler equations satisfying the hypothesis of theorem 1.1. Let us introduce $(t,s) \mapsto (t, r(t,s)) \in \mathbb{R}^3$ a parametrization of the hypersurface Σ, with $r \in C^{1+\alpha}$ such that $\|\frac{\partial r}{\partial s}(t,s)\| \equiv 1$; the variable s is the arc-length parameter on the curve Σ_t; Let

$$\tau(t,s) = \frac{\partial r}{\partial s}(t,s)\ ,\ \nu(t,s) = R_{\pi/2}(\tau(t,s)) \tag{10}$$

be the unit tangent and perpendicular vectors to Σ_t, where $R_{\pi/2}$ is the $\pi/2$ rotation in the plane. The functions τ and ν belong to C^α.

The Biot-Savart law reads in this situation as follows:

$$u(t,Q) = R_{\pi/2}\frac{1}{2\pi}\int \frac{Q - r(t,s)}{\|Q - r(t,s)\|^2}\Omega(t,s)ds \quad (t,Q) \notin \Sigma. \tag{11}$$

Let $u_\pm$ be the restriction of u to $\Omega_\pm$. The vector fields $u_\pm$ are harmonic in (x,y). Classical potential theory, and $\Omega \in L^\infty$ imply that the traces $u_\pm|_\Sigma$ exist and belongs to $L^\infty_t(L^p_s)$ for any p. Let us denote

$$u_{\pm,s} = u_\pm|_\Sigma.\tau\ ;\ u_{\pm,n} = u_\pm|_\Sigma.\nu \tag{12}$$

The conservation laws div $(u) = 0$, $\rho_t +$ div $(\rho u) = 0$, and rot$(u) = \Omega(t,s)\delta_{\Sigma_t}$ imply

$$[u_n] = 0 \tag{13}$$
$$[\rho](\partial_t r \cdot \nu - u_n) = 0 \tag{14}$$
$$[u_s] = \Omega(t,s) \tag{15}$$

Here, as usual, we use square brackets to denote the jump of a value when crossing the interface, and angle brackets for the mean value: $[f](s) = f_+(s,0) - f_-(s,0)$, $\langle f\rangle(s) = (f_+(s,0) + f_-(s,0))/2$.

Applying the rot operator to the Euler equation, and evaluating the result in weak formulation we obtain:

$$\frac{\partial}{\partial t}[\rho u_s] - \partial_s\left([\rho u_s]\partial_t r\cdot\tau + \frac{1}{2}\left[\rho(u_n^2 - u_s^2)\right]\right) = [\rho]g\cdot\tau \tag{16}$$
$$\partial_t r\cdot\nu - u_n = 0 \tag{17}$$

We introduce the notations $v = \langle u \rangle, w = [\rho u_s]$. Note, that in complex representation $r = x + iy \in \mathbb{C}$ and $\overline{v}$ denoting the complex conjugate of v

$$wehave\overline{v}(t,s) = \frac{1}{2i\pi}\text{v.p.}\int \frac{\Omega(t,s')}{r(t,s) - r(t,s')}\,ds'. \tag{18}$$

In (18) the integral is understood in the sense of the principal value, but in fact it's only the normal component that is involved in the singular integral, indeed, for the tangent one we have the regular representation $(\tau = \partial_s r, \nu = i\tau)$

$$v_s(t,s) = \frac{1}{2\pi}\int Im(\frac{\partial_s r(t,s)\Omega(t,s')}{r(t,s) - r(t,s')})\,ds' \tag{19}$$

We may look at the right-hand side, as an integral operator (depending on r) applied to Ω, let us define

$$K\{r\}f = \frac{-1}{2\pi}\int Im(\frac{\partial_s r(s) f(s')}{r(s) - r(s')})\,ds'. \tag{20}$$

So (19) now reads as $v_s = -K\Omega$. Next, substituting the latter into identity

$$\Omega = \frac{1}{\langle \rho \rangle}(w - [\rho]v_s), \tag{21}$$

we obtain

$$\Omega - \frac{[\rho]}{\langle \rho \rangle}K\{r\}\Omega = \frac{1}{\langle \rho \rangle}w, \tag{22}$$

which is a solvable equation with respect to Ω. Indeed, one has the following

Lemma 2.1. *Suppose, that $r(s) \in C^{1+\alpha}$ with some positive $\alpha > 0$, then $K\{r\}$ as operator $C^\beta \to C^\beta$ is a compact one for $\beta \le \alpha$; moreover, its spectrum allows more detailed description: $\sigma K\{r\} \subset (-\frac{1}{2}, \frac{1}{2}]$.*

This result due to $|[\rho]| < 2\langle \rho \rangle$ implies that operator $\mathbb{I} - \frac{[\rho]}{\langle \rho \rangle}K\{r\}$ is an invertible one, and so

$$\Omega = \left(\mathbb{I} - \frac{[\rho]}{\langle \rho \rangle}K\{r\}\right)^{-1}\frac{1}{\langle \rho \rangle}w. \tag{23}$$

The core of the argument is the analysis of the singular integrals in above expressions. Let $J\{z,\Omega\}$ be the linear operator in Ω, depending on z, defined by

$$J\{z,\Omega\}(\lambda) = \text{v.p.}\int \frac{\Omega(\lambda')}{z(\lambda) - z(\lambda')}d\lambda'. \tag{24}$$

As in [Leb02], the main technical tool that we use is a paralinearization of J, including here its Ω dependance. To simplify notation, we state the next lemma without explicit dependance on t We will always assume that the derivative of z satisfies for some $c > 0$

$$|\partial_\lambda z| \ge c. \tag{25}$$

Following [Leb02] we obtain the following result [LK04]:

Lemma 2.2. *Let $\alpha > 0,\ 0 \leq \beta \leq \alpha$. Assume that $z(\lambda) = x(\lambda) + iy(\lambda) \in C^{1+\alpha}$ and $\Omega(\lambda) \in C^\beta$ for $\beta > 0$, $\Omega(\lambda) \in L^\infty$ for $\beta = 0$. Let $\theta(\lambda) \in C_0^\infty$ equal to 1 in the vicinity of λ_0. Then one has*

$$J\{z, \Omega\} = T^{(\delta)}_{-\frac{\pi\theta\Omega}{(z_\lambda)^2}} |D_\lambda| \theta z + T^{(\delta)}_{\frac{\theta}{z_\lambda}} \mathbb{H}\theta\Omega + R^{(\delta)}, \tag{26}$$

where $\mathbb{H}$ is the Hilbert operator:

$$\mathbb{H}v = \text{v.p.} \int \frac{v(\lambda')}{\lambda - \lambda'} d\lambda' \tag{27}$$

and $R^{(\delta)} \in C^\mu$ near λ_0 for all

$$\mu < \mu_0 = \min(\beta + \delta\alpha, \alpha + \min(1, \delta\beta)). \tag{28}$$

This lemma allows us to justify the reduction to the vortex sheet, we have

Lemma 2.3. *In assumption of theorem 1.1 the traces $u_\pm, v, w, \Omega$ belong to the space $C^\nu(t; C^\mu_s)$ for every ν, μ s.t $\nu + \mu < \alpha$ and satisfy the system*

$$w_t - \partial_s \left(w(r_t - v) \cdot \tau + [\rho] \left(\frac{1}{2}|v|^2 - \frac{1}{8}\Omega^2 \right) \right) = [\rho] g \cdot \tau, \tag{29}$$

$$r_t \cdot \nu = v_n, \tag{30}$$

$$\Omega - \frac{[\rho]}{\langle\rho\rangle} K\{r\}\Omega = \frac{1}{\langle\rho\rangle} w, \tag{31}$$

$$\overline{v}(t,s) = \frac{1}{2i\pi} v.p. \int \frac{\Omega(t,s')}{r(t,s) - r(t,s')} ds'. \tag{32}$$

As for the local problem, i.e. assuming conditions of the theorem 1.2 to be satisfied, we have, of course, the same equations of motion (29),(30), but expressions for v change. Let us introduce $\varphi \in C_0^\infty(U)$ which is equal to 1 in vicinity of $V \subset\subset U$. Due to div $(u) = 0$ we have $u = (-\partial_y \psi, \partial_x \psi)$ for some $\psi \in C^\nu((-T,T); \mathcal{D}')$; with $\tilde{u} = (-\partial_y \varphi\psi, \partial_x \varphi\psi)$ one has $\triangle\varphi\psi = \text{rot } (\tilde{u}) = \varphi\Omega\delta_\Sigma + \tilde{E}$, with $\tilde{E} \in C^\nu((-T,T); \mathcal{E}'(U \setminus V))$ so we get

$$u|_{\tilde{V}} = \frac{(-y, x)}{2\pi|(x,y)|^2} * (\varphi\Omega\delta_{\Sigma_t}) + (-\partial_y E, \partial_x E)|_{\tilde{V}}, \tag{33}$$

where the "error" E is C^ν in time with values in harmonic functions of (x, y). So we get

Lemma 2.4. *In assumption of theorem 1.2 the traces $u_\pm, v, w, \Omega$ belong, near t_0, s_0 to the space $C^\nu(t; C^\alpha_s)$ and satisfy the system*

$$w_t - \partial_s \left(w(r_t - v) \cdot \tau + [\rho] \left(\frac{1}{2}|v|^2 - \frac{1}{8}\Omega^2 \right) \right) = [\rho] g \cdot \tau, \tag{34}$$

$$r_t \cdot \nu = v_n, \tag{35}$$

$$\Omega = \frac{1}{\langle\rho\rangle}(w - [\rho] v_s) \tag{36}$$

$$\overline{v}(t,s) = \frac{1}{2i\pi} v.p. \int \frac{\varphi(r(t,s'))\Omega(t,s')}{r(t,s) - r(t,s')} ds' + E(t, r(t,s)). \tag{37}$$

Notice that Galilean transformations correspond to preserve t, s, Ω and to move w, v, r to $w - [\rho]V.\tau, v - V, r - tV$ where V is a constant real vector. The above system is invariant with respect to Galilean transformations, except that the error term E is replaced by $E - V$.

3. Regularity of solutions

Benefitting from the above reductions we address now first local and then global issues.

3.1. Local regularity of a weak solution

Consider a weak solution to the problem (1)-(3) assuming conditions of theorem 1.2 to be satisfied. By a suitable galilean transformation, and using $\Omega(t_0, s_0) \neq 0$, we may suppose that the speeds of the flows on the both sides of the interface are equal in modulus but have different signs in (x_0, y_0), i.e. $v(t_0, s_0) = 0$, and that w is non-vanishing near (t_0, s_0). By Lemma 2.4 there exists $\lambda(t, s)$, such that $\lambda \in C^\nu(t, C_s^{1+\alpha}) \cap C^{1+\nu}(t, C_s^\alpha)$ and

$$\lambda_s = w, \tag{38}$$

$$\lambda_t = (a - v_s)w + [\rho]U, \tag{39}$$

with

$$a = \partial_t r \cdot \tau, \quad U = \frac{1}{2}|v|^2 - \frac{\Omega^2}{8} + g.r, \tag{40}$$

Introducing this new parameter λ with allows us to reformulate the problem: the parametrization of the vortex sheet $(x(t,\lambda), y(t,\lambda))$ satisfies the following system (we move Ω to $\Omega\frac{ds}{d\lambda}$)

$$\begin{cases} x_t = F_1(x, y) \\ y_t = F_2(x, y) \end{cases} \tag{41}$$

where the nonlinear first order operators are defined by

$$F_1(x, y) = v_1 - [\rho]Ux_\lambda, \quad F_2(x, y) = v_2 - [\rho]Uy_\lambda, \tag{42}$$

here

$$(v_1 - iv_2)(t, \lambda) = \frac{1}{2\pi i}\text{v.p}\int \frac{\varphi(x(t,\lambda'), y(t,\lambda'))\Omega(t,\lambda')}{z(t,\lambda) - z(t,\lambda')}d\lambda' + E(t, x(t,\lambda), y(t,\lambda)), \tag{43}$$

$$U(t, \lambda) = \frac{1}{2}(v_1^2 + v_2^2) - \frac{\Omega^2}{8(x_\lambda^2 + y_\lambda^2)} + \gamma_1 x + \gamma_2 y \tag{44}$$

where (γ_1, γ_2) is the gravity, and Ω satisfies

$$\Omega = \frac{1}{\langle\rho\rangle}(1 - [\rho](x_\lambda v_1 + y_\lambda v_2)). \tag{45}$$

We have used notation $z(t,\lambda) = x(t,\lambda) + iy(t,\lambda)$. The error term $E(t, x, y)$ belongs to $C_t^\nu(C^\infty(V))$ as soon as the cutoff function $\varphi \in C_0^\infty(U)$ is equal to 1 in vicinity of $V \subset\subset U$.

Using the lemma 2.2 and the basics of the paradifferential calculus we show, that (41) is a nonlinear elliptic system and obtain the following result, which implies theorem 1.2.

Theorem 3.1. *Suppose* $z = x + iy \in C^\nu(t; C^{1+\alpha}_\lambda), \Omega \in C^\nu(t; C^\alpha_\lambda)$ *near* (t_0, λ_0) *satisfies the conditions*

$$\exists c: \ \forall t, \lambda, \lambda' \ \ |z(t,\lambda) - z(t,\lambda')| \geq c|\lambda - \lambda'|, \tag{46}$$

$$\Omega(t_0, \lambda_0) \neq 0. \tag{47}$$

and (x, y) *is a weak solution of the system* (41) *with* $E \in C^\nu(t; C^\infty_{x,y})$ *near* $(t_0, z(t_0, \lambda_0))$. *Then near* (t_0, λ_0) *we have*

$$x, y \in C^{1+\nu}(t; C^\infty_\lambda), \quad \Omega \in C^\nu(t; C^\infty_\lambda) \tag{48}$$

3.2. Regularity of a global solution

Let us first address C^∞ regularity of a solution to a global problem: the following theorem takes place:

Theorem 3.2. *In assumption of the theorem 1.1 one has* $\Sigma, \Omega \in C^\infty$.

Proof. Let us see, what we gain by applying the theorem 1.2 in conditions of theorem 1.1. Let $\nu > 0, \mu > 0$ s.t $\mu + \nu < \alpha$; by lemma 2.3 we have $\Omega \in C^\nu(t; C^\mu_s)$, so $u \in C^\nu(t; \mathcal{D}')$ and we may apply theorem 1.2 with μ instead of α. We immediately obtain $r \in C^{1+\nu}((-T,T); C^\infty)$ and $\Omega, v, w \in C^\nu((-T,T); C^\infty)$. Using (29), we get $w \in C^{1+\nu}((-T,T); C^\infty)$ Next, recalling that Ω is solution of the an essentially spacial integral equation (31), we deduce, that $\Omega \in C^{1+\nu}((-T,T); C^\infty)$, and therefore we can now apply the local theorem with $\nu + 1$ and so by induction we prove the theorem. ☐

Next, applying standard formal series techniques and the estimates close to the ones in [Leb02] one obtains analyticity property of the global solution.

4. Local time existence of analytic solution

Suppose now, that the interface Σ_t is a graph of a periodic function $y(x)$: in other words $r(s,t) = (x(s), y(x(s),t))$, and changing variables in (29),(30) we obtain:

$$W_t - \partial_x\left(-v_1 W + [\rho]\left(\frac{1}{2}(v_1^2 + v_2^2) - \frac{1}{8}\frac{1}{1+y_x^2}\Omega^2\right)\right) = [\rho]\gamma y_x \tag{49}$$

$$y_t = -y_x v_1 + v_2 \tag{50}$$

We assumed here, that $g = (0, \gamma)$. The representations (18) take form:

$$v_1(x,t) \ = \ -\frac{1}{2\pi}\text{v.p}\int_{\mathbb{R}} \frac{y(x,t) - y(x',t)}{(x-x')^2 + (y(x,t) - y(x',t))^2}\Omega(x',t)\,dx', \tag{51}$$

$$v_2(x,t) \ = \ \frac{1}{2\pi}\text{v.p}\int_{\mathbb{R}} \frac{x - x'}{(x-x')^2 + (y(x,t) - y(x',t))^2}\Omega(x',t)\,dx'. \tag{52}$$

Equations (49)-(52) together with

$$W = \langle\rho\rangle\Omega + [\rho](v_1 + y_x v_2) \tag{53}$$

properly define an evolution equation, as it was demonstrated in the previous section.

We seek π-periodic analytic solutions to the stated problem. Following [SSBF81] we fix arbitrary $\alpha \in (0,1)$ and define the scale of Banach spaces B_s of holomorphic in $b_s = \{x + i\zeta, x \in \mathbb{R}/\pi\mathbb{Z},\ |\zeta| < s\}$ functions, with the Hölder α norm:

$$\|u\|_s := \|u; B_s\| := \sup_{|\zeta|<s} |u(\cdot + i\zeta)|_\alpha \tag{54}$$

here and further

$$|f|_\alpha := \|f; C^\alpha(\mathbb{R}/\pi\mathbb{Z})\| \tag{55}$$

We will show, that the Cauchy problem is well posed in B_s, i.e. that for small time there exists a solution $(y, W) \in B_s^2$, holomorphic in t as well. To do this we will use an abstract Cauchy-Kovalevski theorem in Nishida's [Nis77] formulation.

We state the problem formally for the triple (W, y, y_x), the equations are (49)-(52) with an equation for y_x, which is obtained by differentiating (50):

$$y_{xt} = -y_{xx}v_1 - y_x v_{1,x} + v_{2,x}. \tag{56}$$

In [SSBF81] (Prop. 4.1 and further) was proved

Lemma 4.1. *For* $|\operatorname{Im}\ y_x|_{0,b_s}, |\operatorname{Im}\ \tilde{y}_x|_{0,b_s} < \frac{1}{2}$ *and* $\|y_x\|_s, \|\Omega\|_s, \|\tilde{y}_x\|_s, \|\tilde{\Omega}\|_s$ *bounded,*

$$\|v_i\{\Omega, y\} - v_i\{\tilde{\Omega}, \tilde{y}\}\|_s \le C(\|\Omega - \tilde{\Omega}\|_s + \|y_x - \tilde{y}_x\|_s), \quad i = 1, 2. \tag{57}$$

In the right-hand side of the estimate (57) we find s-norm of y_x, this explains, why we had to add (56) to the problem formulation. This lemma insures existence to the problem in the case $\rho_\pm = 1$ in which $[\rho] = 0$ and $W = \Omega$. In the general case we have

$$v_1 + y_x v_2 = v_{(x)} = v_{(x)}\{W, y\} \tag{58}$$

$$\Omega = \Omega\{W, y\} = \frac{1}{\langle\rho\rangle}W - \frac{[\rho]}{\langle\rho\rangle}v_{(x)}\{W, y\} \tag{59}$$

So, all we have to prove to obtain existence of the holomorphic solution is to estimate v_s via W and y, let us prove a weak

Lemma 4.2. *Suppose* $\|W\|_s, \|\tilde{W}\|_s$ *are bounded and* $y, \tilde{y}$ *are in vicinity of some* $y_0^0 \in B_s$ *and are such that norms of imaginary parts* $\|\operatorname{Im}\ y_x\|_s, \|\operatorname{Im}\ \tilde{y}_x\|_s$ *are small enough (i.e. less then a positive constant depending on the* $C^{1+\alpha}$ *norms of* y *and* $\tilde{y}$ *on the real axis), then we have*

$$\|v_{(x)}\{W, y\} - v_{(x)}\{\tilde{W}, \tilde{y}\}\|_s \le C(\|W - \tilde{W}\|_s + \|y_x - \tilde{y}_x\|_s). \tag{60}$$

Proof. As we have shown in the previous sections for real $x \in \mathbb{R}/\pi\mathbb{Z}$ we have

$$v_{(x)}\{W, y\}(x) = -\left(\mathbb{I} - \frac{[\rho]}{\langle\rho\rangle}K\{y^0\}\right)^{-1} \frac{1}{\langle\rho\rangle}K\{y^0\}W^0(x) \tag{61}$$

here we used notation $f^0 = f|_{\text{Im } z=0}$ for the restriction to real axis. So, the key point to prove (60) is to understand the properties of the function $y^0 \mapsto K\{y^0\}$ as a mapping $C^{1+\alpha} \to L(C^\alpha)$. This mapping is bounded and Lipschitz. These properties can be obtained f.e. as a consequence of Lemma 4.1. Next, as we can restrict ourselves with consideration of vicinity of y_0^0, so we have an estimate

$$\| \left(\mathbb{I} - \frac{[\rho]}{\langle\rho\rangle}K\{y^0\}\right)^{-1} K\{y^0\}; L(C^\alpha)\| \le C(\|y^0_{0,x}\|_s, \|y^0_{0,x} - y^0_x\|_s). \tag{62}$$

And this is already enough to obtain "(60) on the real axis":

$$|v^0_{(x)}\{W^0, y^0\} - v^0_{(x)}\{\tilde{W}^0, \tilde{y}^0\}|_\alpha \le C(|W^0 - \tilde{W}^0|_\alpha + |y^0_x - \tilde{y}^0_x|_\alpha). \tag{63}$$

The lemma statement is a small perturbation consequence of (63). □

So, the final result is

Theorem 4.3. *For all initial data $y_0, W_0 \in B_{s_0}$, such that $\|\text{ Im } y_{0,x}\|_{s_0}$ and $\|\text{ Im } y_{0,x}\|_{s_0}$ are sufficiently small, there exists a constant β s.t. for $|t| < \beta(s_0 - s)$ the system has a unique solution $(W, y) \in B_s^2$, which is a holomorphic function of t.*

Remark. A similar result was obtained in [SS85], but in a slightly different framework: although a more general problem was considered in [SS85], nevertheless the final result was obtained in assumption of relative smallness of initial data, which we have managed to avoid due to lemmas 4.2 and 2.1.

References

[Bon81] J.-M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. Ec. Norm. Sup.,IV série*, 14:209–246, 1981.

[Leb02] G. Lebeau. Régularité du problème de Kelvin-Helmholtz pour l'équation d'Euler 2d. *ESAIM: COCV*, 08:801–825, 2002.

[Wu] Sijue Wu Recent progress in mathematical analysis of vortex sheets. *Proceedings of the International Congress of Mathematicians*, Vol. III (Beijing, 2002), 233–242, Higher Ed. Press, Beijing, 2002.

[LK04] G. Lebeau. and V. Kamotski On 2D Rayleigh-Taylor instabilities. to appear in *Asymptotic Analysis* , 2004

[SS85] C. Sulem and P.L. Sulem. Finite time analyticity for the two- and three-dimensional Rayleigh- Taylor instability. *Trans. Am. Math. Soc.*, 287(1):127–160, 1985.

[SSBF81] C. Sulem, P.L. Sulem, C. Bardos, and U. Frisch. Finite time analyticity for the two and three dimensional Kelvin-Helhmoltz instability. *Comm. in Math. Phys.*, 80:485–516, 1981.

[Nis77] T. Nishida. A note on a theorem of Nirenberg. *J. Differ. Geom.*, 12:629–633, 1977.