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Résumé

Nous nous intéressons ici à démontrer des résultats d'observabilité pour le système de Stokes et pour différentes conditions au bord. En particulier nous traitons les cas des conditions de Dirichlet, de Navier et de Neumann. Après un rappel des approches du même type de problème pour les équations paraboliques et une courte revue de la littérature pour le problème de Stokes, nous donnons les résultats principaux ainsi que quelques idées de la démonstration.

Abstract

Here, we are interested in proving observability results for the Stokes system and for various boundary conditions. In particular, we address the Dirichlet, Navier, and Neumann conditions. After reviewing approaches to the same type of problem for parabolic equations and a brief review of the literature for the Stokes problem, we present the main results and some ideas from the proof.

1. Introduction

Suppose Ω is a smooth and bounded open set in $\mathbb{R}^d$. The goal of this article is to study the uniqueness, control, and observation of the Stokes system

$$\begin{cases} \partial_t U - \Delta U + \nabla q = F, \text{ in } \mathbb{R}\times\Omega \\ \operatorname{div} U = 0 \text{ on } \mathbb{R}\times\Omega, \end{cases} \tag{1.1}$$

where U is the flow velocity of a fluid and q is the pressure.

Suppose ω is an open set such that $\omega \subset \Omega$, as sketched in Figure 1.1. We want to prove an estimate of the following form

$$\|U\|_1 + \|q\|_2 \lesssim \|F\|_3 + \|U_{I\times\omega}\|_4,$$

where $I \subset (0,T)$. The norms $\|\cdot\|_j$ are to be specified and depend on the problems or equations being studied. Such an estimate is called an *observability estimate*. A weaker form of this estimate is the following: assume $F = 0$ on $I\times\Omega$ and $U = 0$ on $I\times\omega$. Does this imply that $U = 0$ and $q = 0$? In such a case, one says that the problem satisfies a *uniqueness result*.

The system has been studied for some particular boundary conditions, such as homogeneous Dirichlet and Navier conditions. Here, we wish to analyse more general boundary conditions. Before stating the main results, we provide a brief review of the existing literature on parabolic equations and on the Stokes system. Describing how certain parabolic problems can be treated allows us to introduce some methods of proof for the Stokes problem.

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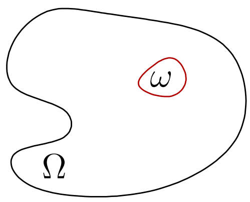

Figure 1.1: Example of geometry for the observability or uniqueness problems

1.1. Parabolic operator and uniqueness property

In this section, we denote by P an elliptic operator of second order with real coefficients in the principal symbol, for instance $P = \Delta + V(x)$ where V is a $\mathscr{C}^\infty$ function and the parabolic associated operator $\partial_t - P$.

For this operator, the uniqueness problem is: assuming $\partial_t u - Pu = 0$ on $I \times \Omega$ and $u = 0$ on $I \times \omega$, does one have $u = 0$ on $I \times \Omega$. A positive answer was first given by Mizohata in his seminal article [20]. See also a previous result by Ito and Yamabe on the uniqueness of a parabolic equation under different assumptions [12]. The Mizohata proof follows the Calderón approach based on Carleman estimates [2]. Other results were proven in this spirit. For instance Saut and Scheurer [22] provend uniqueness hold for operators with $\mathscr{C}^1$ coefficients. For some more recent results on unique continuation, let us mention those is [4], [14], and [23].

1.1.1. The Fursikov and Imanuvilov method

Fursikov and Imanuvilov introduced a innovative Carleman weight [8]; see also [11]. More precisely they proved a Carleman estimate of the following type

$$\|e^{\tau\varphi} u\| \lesssim \|e^{\tau\varphi}(\partial_t u - Pu)\|$$

for u supported in $(0,T) \times (\Omega \setminus \omega)$ and $\varphi = -\frac{\psi}{t(T-t)}$ where $\psi > 0$. In the above works that proved uniqueness, φ was smooth in both x and t. Here, the weightfunction φ has a singularity at $t = 0$ and $t = T$. This kind of estimates has been applied to prove null controllability for linear or nonlinear parabolic equations.

1.1.2. Spectral estimates

Spectral estimates concern elliptic operators. However, from such estimates one can deduce control or observation for the associated parabolic equation.

Assume P is an selfadjoint elliptic operator of order 2. Let u_j and λ_j be such that $-\Delta u_j = \lambda_j u_j$ (+ boundary condition). Let $E_\mu = \mathrm{Span}\{u_j, \lambda_j \le \mu\}$ the space spanned by the eigenfunctions associated with eigenvalues less than μ. One says that a spectral estimate holds if there exists $C > 0$ such that for all $\mu > 0$ and $u \in E_\mu$

$$\|u\|_{L^2(\Omega)} \le C e^{C\sqrt{\mu}} \|u_{|\omega}\|_{L^2(\omega)}.$$

A first proof was given by Jerison and Lebeau [13], including the optimality with respect to the power $\sqrt{\mu}$ for such an estimate. This estimate was later applied for the pupose of proving null controllability by Lebeau and Zuazua [18]; see also [17] where null controllability was proven by a different but related method. Miller [19] proved that one can directly deduce an observability estimate from a spectral estimate.

1.2. The Stokes system

We describe the results obtained for the Stokes system, in the spirit of the methods presented above for heat-type equations.

Uniqueness result. For Stokes system (1.1), an uniqueness result was proven by Fabre and Lebeau [5]. As for heat equation this implies an approximate controllability result.

Fursikov and Imanuvilov method, Dirichlet boundary condition. Observability for the Stokes system under Dirichlet boundary condition was first proven by Fernández-Cara, Guerrero, Imanuvilov, and Puel [6]. Actually, their interest lied in the controllability of the Navier–Stokes system but an observability estimate for the Stokes system is an important step of the proof. We also mention the recent result of Buffe and Takahashi [1] where one may find an improvement on the previous result. It allows one to estimate the cost of a small-time control with a rate similar to what one obtains for the heat equation. We mention [7] where the same authors as in [6] proved a controllability result using less controls than in their the first work

Fursikov and Imanuvilov method, Robin boundary condition. By methods related to [6], Guerrero [9] proved observability under Navier-slip boundary conditions. With Montoya [10] they studied the same boundary conditions but with less observations.

Spectral estimate. As for heat equation, another approach has been followed. For the Stokes system with homogenous Dirichlet boundary conditions a spectral estimate was poven by Chaves-Silva and Lebeau [3]. From this estimate one can deduce the null controllability of the Stokes system. Mention that for other boundary conditions such as Navier or Neumann boundary conditions the validity of a spectral estimates is an open problem.

2. Main results

It is useful to explain the Stokes system in the context of Riemannian geometry. In this framework U is a vector field and q is a function. In the proof we perform changes of variables. Yet, Riemannian geometry allows one to follow what the metric and vector fields are in local coordinates. In the following, if taking the metric $g = \mathrm{Id}$, M an open set of $\mathbb{R}^d$, one finds the classical Stokes system.

Consider a compact manifold with boundary $\mathcal{M}$ with a Riemannian metric g. The metric depends moreover smoothly on a parameter $t \in \mathbb{R}$, to be understood as a time variable in what follows. Positivity of the metric g is assumed on $\mathcal{M}$, uniformly in t.

The Stokes system reads

$$\begin{cases} \partial_t U - \Delta_g U + \nabla_{\mathrm{g}} q = F, \\ \operatorname{div}_{\mathrm{g}} U = 0 \end{cases}$$

with, in addition, some boundary conditions, that have the form

$$(B_j U + c_j q)_{|\partial\mathcal{M}} = f_j, \qquad j = 1, \dots, d,$$

where $B_j = b_j(t, x, D_x)$ is a differential operator of order 0 or 1 with smooth coefficients, and c_j is a smooth function. Some example as presented in [21] are given here.

(1) The Dirichlet condition $U_{|\mathbb{R}\times\partial\mathcal{M}} = 0$.

(2) The Navier condition is $\nu \cdot U = 0$ and $\big(\big((\nabla U) + {}^t(\nabla U)\big)\cdot \nu\big)_{\tan} = 0$ at the boundary in the case of a Euclidean metric. The geometrical form is

$$\begin{aligned} &(U, \nu)_g = 0 \ \text{ on } \ \mathbb{R}\times\partial\mathcal{M}, \\ &(D_\nu U, w)_g + (D_w U, \nu)_g = 0 \ \text{ on } \ \mathbb{R}\times\partial\mathcal{M}, \ \forall\, w \in T\mathcal{M} \text{ such that } (\nu, w)_g = 0. \end{aligned}$$

The so-called Hodge condition only differs from the Navier condition by a zero-order term.

The Robin condition reads

$$\begin{aligned} &(U, \nu)_g = 0 \ \text{ on } \ \mathbb{R}\times\partial\mathcal{M}, \\ &(D_\nu U, w)_g + (D_w U, \nu)_g = \alpha(U, w)_g \ \text{ on } \ \mathbb{R}\times\partial\mathcal{M}, \ \forall\, w \in T\mathcal{M} \text{ such that } (\nu, w)_g = 0. \end{aligned}$$

(3) The Neumann condition is $\big({}^t(\nabla U) + \mathsf{n}(\nabla U)\big)\nu - q\nu = 0$ in the case of a Euclidean metric, where $\mathsf{n} \in (-1, 1]$ is a parameter. The geometrical form is

$$(D_\nu U, w)_g + \mathsf{n}(D_w U, \nu)_g - q(\nu, w)_g = 0 \ \text{ on } \ \mathbb{R} \times \partial\mathcal{M}, \quad \forall\, w \in T\mathcal{M}.$$

These examples are covered by the class of boundary conditions we consider here. The precise description of the adapted Lopatinskiĭ–Šapiroconditions associated with Stokes system is provided below after some microlocal analysis.

Before stating the result, let us introduce some notation. For $\psi(x) > 0$ and $T > 0$ we set

$$\eta(t) = \frac{T^2}{t(T-t)} \quad \text{and} \quad \phi(x,t) = \eta(t)\psi(x),$$

following Fursikov–Imanuvilov [8]. Define the norms

$$\|v\|^2 = \int_0^T \int_\Omega |v|^2 \,\mathrm{d}x\mathrm{d}t \quad \text{and} \quad |w|_s^2 = \int_0^T |w(t)|_s^2 \,\mathrm{d}t,$$

where $|w(t)|_s$ is the H^s norm on $\partial\Omega$.

Theorem 1. *There exist ψ, $C > 0$, and $\tau > 0$ such that for a solution (U, q) of (1.1) and for all $\tau \geq \tau_0(1 + 1/T)$ we have*

$$\|e^{\tau\phi}\tau\eta U\| + \|e^{\tau\phi}\nabla U\| + \|e^{\tau\phi}(\tau\eta)^{1/2}q\| + |(e^{\tau\phi}U)_{|\partial\Omega}|_{3/2} + |(e^{\tau\phi}q)_{|\partial\Omega}|_{1/2} \\ \leq C\Big(\|e^{\tau\phi}F\| + \|(e^{\tau\phi}\tau\eta U)_{|\omega}\| + \|(e^{\tau\phi}(\tau\eta)^{1/2}q)_{|\omega}\|\Big).$$

Remark 2. The Carleman estimate obtained in Theorem 1 is analogous to that in [1] for Dirichlet boundary conditions.

3. Local setting near the boundary

Near a boundary point, normal geodesic coordinates yield the domain to be locally given by $\{x_d > 0\}$, the boundary by $\{x_d = 0\}$, and one has $g_{dj} = 0$ for $j \neq d$, and $g_{dd} = 1$. Consequently near a point $m^0 \in \partial\mathcal{M}$, the Laplace–Beltrami operator Δ_g takes the form

$$-\Delta_g = D_d^2 + R(t, x, D_{x'}) + Q_1(x, D),$$

where $R(t, x, D_{x'})$ is a second-order differential operator in the x' variables and $Q_1(x, D)$ is a first-order differential operator; see for instance [15, Section 17.6]. Recall that we assume here that the Riemannian metric depends on time t here; hence the t variable that appears in $R(t, x, D_{x'})$. The divergence operator has the form

$$\begin{aligned}\operatorname{div}_g U(t,x) &= (\det g)^{-1/2} \sum_{1 \leq i \leq d} \partial_{x_i}\big((\det g)^{1/2} U^i\big)(x) \\ &= \sum_{1 \leq i \leq d} \left(\partial_{x_i} + \frac{\partial_{x_i} \det g}{2 \det g}\right) U^i.\end{aligned}$$

We thus write the Stokes system in the form

$$\begin{cases} PU + \nabla_g q = \widetilde{F}, \\ \operatorname{div} U = \widetilde{h}, \end{cases}$$

with

$$P = \partial_t + D_d^2 + R(t, x, D_{x'}), \quad \text{and} \quad \widetilde{F} = F - Q_1(x, D)U,$$

$$\operatorname{div} U = \sum_{1 \leq i \leq d} \partial_{x_i} U^i, \quad \text{and} \quad \widetilde{h} = -\sum_{1 \leq i \leq d} \frac{\partial_{x_i} \det g}{2 \det g} U^i.$$

Denote by $r(t, x, \xi')$ the principal symbol of $R(t, x, D_{x'})$, that is quadratic in ξ' and write

$$r(t, x, \xi') = {}^t\xi' \mathsf{R}(t, x)\xi',$$

with $\mathsf{R}(t,x)$ the $(d-1)\times(d-1)$ real symmetric matrix given by $\mathsf{R}^{ij}(t,x)=g^{ij}(t,x)$ for $1\le i,j\le d-1$ with $g^{ij}(t,x)$ the inverse of the metric $g_{ij}(t,x)$. In particular $\mathsf{R}(t,x)$ is a uniformly definite positive matrix.

For further use, we give the expression of the classical boundary conditions listed in the introduction in local coordinates.

(1) The Dirichlet condition is given locally on $x_d=0$ by $U=0$.

(2) The Navier condition is given locally on $x_d=0$ by

$$\begin{aligned}&g_{kj}\nu^kU^j=0\\&(\nu^k\partial_kU^j+\Gamma^j_{k\ell}\nu^kU^\ell)g_{ji}w^i+(w^k\partial_kU^j+\Gamma^j_{k\ell}w^kU^\ell)g_{ji}\nu^i=0,\end{aligned}$$

for all $w\in T\mathcal{M}$ such that $g_{ji}\nu^iw^j=0$. In the present normal geodesic coordinates it becomes

$$U^d=0\ \text{ and }\ (\partial_dU^j+\Gamma^j_{d\ell}U^\ell)g_{ji}+\Gamma^d_{i\ell}U^\ell=0\ \text{ for }\ i=1,\dots,d-1. \tag{3.1}$$

(3) The Neumann condition is given locally by

$$(\nu^k\partial_kU^j+\Gamma^j_{k\ell}\nu^kU^\ell)g_{ji}w^i+\mathsf{n}(w^k\partial_kU^j+\Gamma^j_{k\ell}w^kU^\ell)g_{ji}\nu^i-qg_{ij}w^i\nu^j=0,$$

for all $w\in T\mathcal{M}$. In the present normal geodesic coordinates it becomes

$$\begin{aligned}&-(1+\mathsf{n})(\partial_dU^d+\Gamma^d_{d\ell}U^\ell)+q=0\\&-(\partial_dU^j+\Gamma^j_{d\ell}U^\ell)g_{ji}-\mathsf{n}(\partial_iU^d+\Gamma^d_{i\ell}U^\ell)=0,\quad\text{for}\ \ i=1,\dots,d-1,\end{aligned}$$

which reads also

$$\begin{aligned}&-(1+\mathsf{n})(\partial_dU^d+\Gamma^d_{d\ell}U^\ell)+q=0\\&\partial_dU^k+\Gamma^k_{d\ell}U^\ell+\mathsf{n}(\partial_iU^d+\Gamma^d_{i\ell}U^\ell)g^{ik}=0,\quad\text{for}\ \ k=1,\dots,d-1.\end{aligned} \tag{3.2}$$

4. Conjugation

4.1. Weight function

For the proofs of a Carleman estimate, we conjugate the Stokes operator with the weight function $\exp(\tau\eta(t)\phi(x))$ for $\tau>0$ taken sufficiently large in what follows. However, to work on $\mathbb{R}$ for the time variable, instead of the finite interval $(0,T)$, we make the following change of variable

$$s(t)=\tan\left(\frac{\pi t}{T}-\frac{\pi}{2}\right).$$

We note that $\partial_t=\frac{a(s)}{T}\partial_s$, with $a(s)=\pi\langle s\rangle^2$. The tangential operator $R(t,x,D_{x'})$ becomes $R(t(s),x,D_{x'})$. In what follows we keep writting $R(t,x,D_{x'})$ and $r(t,x,\xi')$ simply to keep in mind that the action of the variable s for this operator and this symbols is through $t(s)$ that lies in the compact set $[0,T]$, a fact that becomes important in certain places.

With $P=D_d^2+\frac{a(s)}{T}\partial_s+R(t,x,D_{x'})$, the Stokes system reads

$$\begin{cases}PU+\nabla_{\mathrm{g}}q=\widetilde{F},\\ \operatorname{div}U=\widetilde{h}.\end{cases}$$

By abuse of notation we keep the same notation for all functions in the new variable s. The function $\eta(t)$ changes into

$$\eta(s)=\pi^2\left(\frac{\pi}{2}+\arctan(s)\right)^{-1}\left(\frac{\pi}{2}-\arctan(s)\right)^{-1}.$$

In particular, one has

$$C\langle s\rangle\le\eta(s)\le C'\langle s\rangle,\quad s\in\mathbb{R},\qquad\text{and}\quad|\eta^{(k)}(s)|\le C'\langle s\rangle^{1-k},\quad k\in\mathbb{N}.$$

We consider a smooth function $\widetilde{\psi}$ on $\mathcal{M}$ such that $\widetilde{\psi} \geq c_0 > 0$ with moreover $\partial_\nu \widetilde{\psi}_{|\partial\mathcal{M}} < 0$, with ν the outgoing normal vector field at $\partial\mathcal{M}$. For $\widetilde{K}$ and K well chosen, let $\psi = \widetilde{\psi} + \widetilde{K}$ and we set

$$\phi(x) = e^{\gamma\psi_\varepsilon(x)} - e^{\gamma K}, \quad \varphi(x) = e^{\gamma\psi_\varepsilon(x)}, \quad \text{and} \quad \psi_\varepsilon(x) = \psi(\varepsilon x', x_d)$$

with γ and ε as parameters, satisfying $\gamma \geq 1$, $\varepsilon \in [0,1]$. We also set

$$\widehat{\tau}(x) = \tau\,\gamma\,\varphi(x) \ \text{ and } \ \widetilde{\tau}(s,x) = \widehat{\tau}(x)\,\eta(s).$$

For the functions η, φ, $\widehat{\tau}$, and $\widetilde{\tau}$ we often keep their variables implicit in what follows.

4.2. Operator conjugation

Set

$$P^\phi = e^{\tau\eta\phi} P e^{-\tau\eta\phi},$$
$$\nabla_{\mathrm{g}}^\phi = e^{\tau\eta\phi}\,\nabla_{\mathrm{g}}\, e^{-\tau\eta\phi}, \quad \text{and} \quad \operatorname{div}^\phi = e^{\tau\eta\phi}\operatorname{div} e^{-\tau\eta\phi},$$

and

$$U^\phi = e^{\tau\eta\phi}U, \quad q^\phi = ie^{\tau\eta\phi}q(s,x), \quad F^\phi = e^{\tau\eta\phi}\widetilde{F} \ \text{ and } \ h^\phi = e^{\tau\eta\phi}\widetilde{h}.$$

One has

$$\begin{cases} P^\phi U^\phi - i\,\nabla_{\mathrm{g}}^\phi\, q^\phi = F^\phi, \\ \operatorname{div}^\phi U^\phi = h^\phi. \end{cases}$$

Set $\widetilde{\tau} = \tau\gamma\eta\varphi$. One has

$$\partial_d e^{\tau\eta\phi} = \widetilde{\tau}(\partial_d\psi)_\varepsilon e^{\tau\eta\phi} \ \text{ and } \ \partial_j e^{\tau\eta\phi} = \widetilde{\tau}\varepsilon(\partial_j\psi)_\varepsilon e^{\tau\eta\phi}, \qquad j = 1, \dots, d-1.$$

yielding

$$D_d^\phi := e^{\tau\eta\phi} D_d e^{-\tau\eta\phi} = D_d + i\widetilde{\tau}(\partial_d\psi)_\varepsilon,$$
$$D_{x'}^\phi := e^{\tau\eta\phi} D_{x'} e^{-\tau\eta\phi} = D_{x'} + i\widetilde{\tau}\varepsilon(d_{x'}\psi)_\varepsilon.$$

with respective symbols $\zeta_d = \xi_d + i\widetilde{\tau}(\partial_d\psi)_\varepsilon$ and $\zeta' = \xi' + i\widetilde{\tau}\varepsilon(d_{x'}\psi)_\varepsilon$. Note that

$$D_d^\phi D_{x'}^\phi = D_{x'}^\phi D_d^\phi.$$

For a precise description of the operators that are manipulated, new operator classes and symbols need to be introduced. For the sake of exposition, we simply say *operators of order m* or *symbols of order m*. Because of the parabolic nature of the Stokes problem, note that ∂_s is to be understood as a second order operator, comparable in order to ∂_x^2. This can be done through the Weyl-Hörmander calculus; we refer to [16]. Here, we avoid introducing technical definitions.

Set $R^\phi = e^{\tau\eta\phi} R(t, x, D_{x'}) e^{-\tau\eta\phi}$, an operator of order 2. Its principal symbol of order 2 is

$$r^\phi(s, x, \xi', \tau) = r(t, x, \zeta'). \tag{4.1}$$

One has $P^\phi = (D_d^\phi)^2 + M^\phi$ with $M^\phi = T^{-1}a(s)\big(\partial_s - \tau\eta'\phi\big) + R^\phi$ an operator of order 2 whose principal symbol is

$$m^\phi(s, x, \xi', \tau) = i\widetilde{\sigma} + r^\phi(s, x, \xi', \tau), \tag{4.2}$$

where

$$\widetilde{\sigma} = T^{-1}a(s)(\sigma + i\tau\eta'\phi).$$

The conjugated divergence operator reads

$$\operatorname{div}^\phi V = \sum_{1\leq i\leq d} D_i^\phi V_i.$$

One writes the Stokes system as follows

$$\begin{cases} \big((D_d^\phi)^2 + M^\phi\big)U^\phi - i\,\nabla_{\mathrm{g}}^\phi\, q^\phi = F^\phi, \\ \operatorname{div}^\phi U^\phi = h^\phi. \end{cases} \tag{4.3}$$

4.3. First-order reduction at the boundary

Set $U^{\phi\prime} = (U_1^\phi, \dots, U_{d-1}^\phi)$. Set

$$\begin{aligned} \widetilde{V}^\phi &= {}^t(\widetilde{V}^{\phi\prime}, \widetilde{V}_d^\phi) \text{ with } \widetilde{V}^{\phi\prime} = U^{\phi\prime}, \text{ and } \widetilde{V}_d^\phi = U_d^\phi, \\ \widehat{V}^\phi &= {}^t(\widehat{V}^{\phi\prime}, \widehat{V}_d^\phi) \text{ with } \widehat{V}^{\phi\prime} = D_d^\phi U^{\phi\prime} = D_d^\phi \widetilde{V}^{\phi\prime}, \text{ and } \widehat{V}_d^\phi = q^\phi, \end{aligned} \tag{4.4}$$

and $V^\phi = {}^t(\widetilde{V}^\phi, \widehat{V}^\phi)$.

More generally for $W \in \mathbb{C}^d$, one denotes W' the vector in $\mathbb{C}^{d-1}$ with the $(d-1)$ first components of W. With this notation, for a function f, one sets $\nabla_{\mathrm{g}}^{\phi\prime} f = (\nabla_{\mathrm{g}}^\phi f)'$.

From System (4.3) one writes

$$D_d^\phi \widehat{V}^{\phi\prime} + M^\phi \widetilde{V}^{\phi\prime} - i\,\nabla_{\mathrm{g}}^{\phi\prime}\,\widehat{V}_d^\phi = F^{\phi\prime}. \tag{4.5}$$

Using the divergence equation in (4.3) one also writes

$$D_d^\phi \widetilde{V}_d^\phi = -ih^\phi + i\operatorname{div}^{\phi\prime}\widetilde{V}^{\phi\prime}, \tag{4.6}$$

where $\operatorname{div}^{\phi\prime} = e^{\tau\eta\phi}\operatorname{div}' e^{-\tau\eta\phi}$ with the divergence operator div' only acting in the tangent variables. Applying D_d^ϕ and using that $[D_d^\phi, \operatorname{div}^{\phi\prime}] = 0$, one obtains

$$(D_d^\phi)^2 \widetilde{V}_d^\phi = -iD_d^\phi h^\phi + i\operatorname{div}^{\phi\prime}\widehat{V}^{\phi\prime}.$$

One also has

$$\big((D_d^\phi)^2 + M^\phi\big)\widetilde{V}_d^\phi + D_d^\phi \widehat{V}_d^\phi = F_d^\phi,$$

which we write

$$D_d^\phi \widehat{V}_d + i\operatorname{div}^{\phi\prime}\widehat{V}^{\phi\prime} + M^\phi \widetilde{V}_d^\phi = F_d^\phi + iD_d^\phi h^\phi. \tag{4.7}$$

Collecting (4.5)–(4.7) together we write

$$D_d^\phi V^\phi = A^\phi V^\phi + G^\phi,$$

with A a matrix tangential operator given by

$$A^\phi = \begin{pmatrix} 0 & 0 & \mathrm{Id} & 0 \\ i\operatorname{div}^{\phi\prime} & 0 & 0 & 0 \\ -M^\phi\,\mathrm{Id} & 0 & 0 & i\,\nabla_{\mathrm{g}}^{\phi\prime} \\ 0 & -M^\phi & -i\operatorname{div}^{\phi\prime} & 0 \end{pmatrix}, \tag{4.8}$$

and

$$G^\phi = \begin{pmatrix} 0 \\ -ih^\phi \\ F^{\phi\prime} \\ F_d^\phi + iD_d^\phi h^\phi \end{pmatrix}.$$

Note that all entries of A^ϕ are block matrices of different dimensions.

5. Some ideas to obtain Carleman estimate for a system

5.1. Ideas

Let $D_d^\phi V = AV + G$ be a system. Let a the symbol of A is a matrix $N \times N$ and V valued in $\mathbb{C}^N$. Assume we have ${}^tW$ a left eigenvector of a, i.e. ${}^tWa = \mu{}^tW$. We set $z = \mathrm{Op}({}^tW)V$. Computing

$$\begin{aligned} D_d^\phi z &= D_d^\phi\,\mathrm{Op}({}^tW)V = \mathrm{Op}({}^tW)D_d^\phi V + [D_d^\phi, \mathrm{Op}({}^tW)]V \\ &= \mathrm{Op}({}^tWA)V + \mathrm{Op}({}^tW)G + \text{“error terms”} = \mathrm{Op}(\mu)z + \mathrm{Op}({}^tW)G + \text{“error terms”}. \end{aligned}$$

We have two mains estimates depending of localisation in $\mathbb{C}$ of the root of the symbol of $D_d^\phi - \mathrm{Op}(\mu)$. As the symbol of $D_d^\phi - \mathrm{Op}(\mu)$ is $\xi_d + i\tau\partial_d\phi - \mu$.

(1) If $-\tau\partial_d\phi + \operatorname{Im}\mu < 0$, one has an elliptic estimate of the form (called *perfectly elliptic estimate*)

$$\|z\|_1 + |z_{|x_d=0}|_{1/2} \lesssim \|G\| + \|\text{“error terms”}\|$$

(2) If one has a sub-elliptic assumption that is

$$\tau\partial_d\phi - \operatorname{Im}\mu = 0 \Rightarrow \{\xi_d - \operatorname{Re}\mu, \tau\partial_d\phi - \mu\} > 0$$

one has an estimate with 1/2 lost derivative of the following form

$$\|z\|_{1/2} \lesssim \|G\| + |z_{|x_d=0}|_{1/2} + \|\text{“error terms”}\|.$$

The main observations are the following. In the first case, the trace of z is estimated by the “error terms” and the data. In the second case, the trace of z is needed to estimate z in the domain. Recall that the norm $\|\cdot\|$ is defined with an integral over $x_d > 0$ and $|\cdot|$ with an integral on $x_d = 0$. Roughly speaking, subscript denote a Sobolev index.

5.2. Outline and problems

As the operator matrix A^ϕ in (4.8) has blocs of different orders, it is convenient to change state variables to homogenize the operators. We set

$$\widetilde{V} = \Theta\widetilde{V}^\phi \qquad \widehat{V} = \widehat{V}^\phi, \text{ and } V = {}^t(\widetilde{V}, \widehat{V}), \tag{5.1}$$

where Θ is an elliptic operator of order 1. Denote its symbol by θ. The form $D_d^\phi V^\phi = A^\phi V^\phi + G^\phi$ of the system is preserved, yet with a new operator A that is of order 1. Denote its matrix symbol by a. Out of habit, we compute right eigenvectors, that is, eigenvectors for the transposed matrix symbol

$${}^t a(\varrho') = \begin{pmatrix} 0 & -\zeta' & -\theta^{-1}m^\phi(\varrho') & 0 \\ 0 & 0 & 0 & -\theta^{-1}m^\phi(\varrho') \\ \theta(\varrho') & 0 & 0 & \zeta' \\ 0 & 0 & -{}^t\zeta'\mathsf{R} & 0 \end{pmatrix},$$

where $R = (g^{ij})_{1\le i,j\le d-1}$.

The outline is clear with the following steps.

(1) Compute the eigenvectors and eigenvalues μ of ${}^t a$.

(2) Study the sign of $-\tau\partial_d\phi + \operatorname{Im}\mu$.

(3) Check sub-elliptic holds if $-\tau\partial_d\phi + \operatorname{Im}\mu \ge 0$.

(4) Study particular cases that may arise.

The last point is of importance, as ${}^t a$ depends on many parameters, x, ξ', s, σ, τ. As these parameters vary, we may encounter eigenvalues with variable multiplicities, resulting in nonsmoothness of the eigenvalues and ${}^t a$ may exhibit non trivial Jordan blocs.

6. Symbol spectral analysis

The matrix ${}^t a$ has four eigenvalues μ such that $\mu^2 = -m^\phi$, see (4.2) and ν such that $\nu^2 = -r^\phi$, see (4.1). The particular cases are the following:

(1) r^ϕ small, in this case ν is not necessary smooth;

(2) m^ϕ small implying r^ϕ small and ν and μ are not necessary smooth;

(3) r^ϕ is close to m^ϕ, in this case ${}^t a$ exhibits two 2×2 Jordan bloc.

The full analysis is too extensive to include here. We only describe the generic case that corresponds to $|m^\phi|$, $|r^\phi|$, and $|m^\phi - r^\phi|$ large. The vectors

$$W^\pm(\vartheta) = \begin{pmatrix} \mu^{-1}\theta^{-1}\big(({}^t\zeta'\mathsf{R}\vartheta)\zeta' - m^\phi\vartheta\big) \\ -\theta^{-1}({}^t\zeta'\mathsf{R}\vartheta) \\ \vartheta \\ -\mu^{-1}({}^t\zeta'\mathsf{R}\vartheta) \end{pmatrix}, \qquad \vartheta \in \mathbb{C}^{d-1}.$$

are eigenvectors of ${}^t a$ for μ such that $\mu^2 = -m^\phi$. As μ and $-\mu$ are solutions and ϑ is in a space of dimension $d-1$. These eigenvectors span a space of dimension 2d-2.

The following vector

$$W_\nu^\pm = \begin{pmatrix} 0 \\ -\theta^{-2}m^\phi \\ \theta^{-1}\zeta' \\ \theta^{-1}\nu \end{pmatrix}.$$

is an eigenvector associated with ν such that $\nu^2 = -r^\phi$. These eigenvectors span a space of dimension 2.

7. Boundary condition

The main interest of this approach is to obtain Carleman estimates for different boundary conditions. In what follows we show how one obtains an estimate at the boundary for Dirichlet, Navier, and Neumann boundary conditions.

From the formal analysis of Section 5, if μ and ν are chosen such that $-\tau\partial_d\phi + \operatorname{Im}\mu < 0$ and $-\tau\partial_d\phi + \operatorname{Im}\nu < 0$ one can derive an estimate at the boundary for $\operatorname{Op}_{\mathsf{T}}({}^tW(\vartheta))V$ and $\operatorname{Op}_{\mathsf{T}}({}^tW_\nu)V$. The remaining question is then whether these trace estimation, together with knowledge of the boundary conditions, allow one to estimate the trace of the vector field V.

7.1. Dirichlet boundary condition

With the definitions (5.1) and (4.4), the Dirichlet boundary conditions are given by $\widetilde{V} = 0$ at the boundary. It is convenient to write, for $j = 1, \dots, d$, ${}^tL_jV = 0$ where $L_j = {}^t(e_j, 0)$ where e_j is the canonical basis of $\mathbb{C}^d$. To obtain an estimate of the trace it is enough to prove that $(L_j)_{j=1,\dots,d}, W(\vartheta), \vartheta \in \mathbb{C}^{d-1}$, and W_ν span $\mathbb{C}^{2d}$. From the form of L_j it is sufficient to prove that

$$\begin{pmatrix} \vartheta \\ -\mu^{-1}({}^t\zeta'\mathsf{R}\vartheta) \end{pmatrix}, \quad \begin{pmatrix} \theta^{-1}\zeta' \\ \theta^{-1}\nu \end{pmatrix}.$$

span $\mathbb{C}^d$ for $\vartheta \in \mathbb{C}^{d-1}$. Taking $\vartheta = \theta^{-1}\zeta'$ the last variable of the difference of the two vectors is $\theta^{-1}(\mu^{-1}\nu^2 - \nu) \neq 0$ as $m^\phi \neq r^\phi$. Clearly the first vectors give $\mathbb{C}^{d-1} \times \{0\}$.

7.2. Navier boundary condition

From (3.1), with the definitions (4.4) and (5.1), the Navier boundary condition reads $\widetilde{V}^d = 0$ and $\widehat{V}^j = 0$. As for Dirichlet boundary condition it is sufficient that the first and last components of $W(\vartheta)$ and W_ν span $\mathbb{C}^d$. That is, do the vectors

$$\begin{pmatrix} \mu^{-1}\theta^{-1}\big(({}^t\zeta'\mathsf{R}\vartheta)\zeta' - m^\phi\vartheta\big) \\ -\mu^{-1}({}^t\zeta'\mathsf{R}\vartheta) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \theta^{-1}\nu \end{pmatrix}.$$

span $\mathbb{C}^d$? Clearly the last component is obtained by the second vector. This reduces compute the space spanned by $X(\vartheta) = \mu^{-1}\theta^{-1}\big(({}^t\zeta'\mathsf{R}\vartheta)\zeta' - m^\phi\vartheta\big)$.

If ϑ is such that ${}^t\zeta'\mathsf{R}\vartheta = 0$ this space of dimension $d-1$ spans itself.

If we take $\vartheta = \zeta'$ one obtains a vector colinear to $(r^\phi - m^\phi)\zeta'$. As $r^\phi \neq m^\phi$, this means that we also have ζ' in the image of $X(\vartheta)$. We conclude that $X(\vartheta)$ spans $\mathbb{C}^{d-1}$, as ϑ varies.

7.3. Neumann boundary condition

From the Neumann boundary conditions (3.2) and from (4.4) and (5.1) the conditions read

$$\begin{aligned} & i(1+\mathsf{n}) D_d^\phi U_d^\phi + i\widehat{V}_d = 0 \\ & i\widehat{V}_k + i\mathsf{n} g^{ik} D_i^\phi \Theta^{-1} \widetilde{V}^d = 0, \quad \text{for} \quad k = 1, \dots, d-1, \end{aligned}$$

up to operators of lower orders. Using the divergence condition one obtains

$$- i(1+\mathsf{n}) \sum_{j=1}^{d-1} \Theta^{-1} D_j^\phi \widetilde{V}_j + i\widehat{V}_d = 0,$$

up to operators of lower orders. Set

$$b^{Ne} = \begin{pmatrix} \theta^{-1}(1+\mathsf{n})\zeta' \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad b^{Ne\prime}(\vartheta) = \begin{pmatrix} 0 \\ \mathsf{n}\theta^{-1}({}^t\zeta' \mathsf{R} \vartheta) \\ \vartheta \\ 0 \end{pmatrix}, \ \vartheta \in \mathbb{C}^{d-1}.$$

Neumann boundary conditions are equivalent to

$$\mathrm{Op}_{\mathsf{T}}({}^t b^{Ne}) V = 0 \text{ and } \mathrm{Op}_{\mathsf{T}}({}^t b^{Ne\prime}(\vartheta)) V = 0.$$

The question is whether $W(\vartheta)$, W_ν, b^{Ne}, and $b^{Ne\prime}(\vartheta')$ span $\mathbb{C}^{2d}$?

First we set $\vartheta = \vartheta' = \theta^{-1}\zeta'$. We obtain the vectors

$$\begin{pmatrix} \mu^{-1}\theta^{-2}\big(r^\phi - m^\phi\big)\zeta' \\ -\theta^{-2} r^\phi \\ \theta^{-1}\zeta' \\ -\mu^{-1}\theta^{-1} r^\phi \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -\theta^{-2} m^\phi \\ \theta^{-1}\zeta' \\ \theta^{-1}\nu \end{pmatrix}, \quad \begin{pmatrix} \theta^{-1}(1+\mathsf{n})\zeta' \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \mathsf{n}\theta^{-2} r^\phi \\ \theta^{-1}\zeta' \\ 0 \end{pmatrix}.$$

One can write these vectors in the basis

$$\begin{pmatrix} \theta^{-1}\zeta' \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \theta^{-1}\zeta' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

This gives the matrix

$$M = \begin{pmatrix} 1+\mathsf{n} & 0 & \mu^{-1}\theta^{-1}(r^\phi - m^\phi) & 0 \\ 0 & \mathsf{n}\theta^{-2} r^\phi & -\theta^{-2} r^\phi & -\theta^{-2} m^\phi \\ 0 & 1 & 1 & 1 \\ -1 & 0 & -\mu^{-1}\theta^{-1} r^\phi & \theta^{-1}\nu \end{pmatrix}.$$

A computation gives

$$\det M = \frac{\nu^3(x-1)}{x\theta^3}\big(x^3 + x^2 + (2\mathsf{n}+1)x - \mathsf{n}^2\big), \quad \text{with } x = \nu^{-1}\mu. \tag{7.1}$$

and we can prove that $\det M \neq 0$.

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