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Analyse de l'équation cinétique associée à la chaîne FPUT en dimension un

Résumé

Nous présentons brièvement les principaux résultats de Germain–La–Menegaki (2024) et Escobedo–Germain–La–Menegaki (2025), qui (i) établissent la stabilité non linéaire des équilibres de Rayleigh–Jeans non singuliers pour l'équation cinétique associée à la chaîne β-FPUT, et (ii) classent les maximiseurs d'entropie pour une classe plus générale d'équations cinétiques issues de tels systèmes discrets. Nous esquissons également les idées et stratégies principales des démonstrations.

Abstract

We briefly review the main results from Germain–La–Menegaki (2024) and Escobedo–Germain–La–Menegaki (2025) which (i) proves nonlinear stability of the non-singular Rayleigh–Jeans equilibria for the kinetic wave equation associated with the β-FPUT chain and (ii) classifies the entropy maximizers for a more general class of kinetic equations arising from such discrete systems. We briefly explain the main ideas and strategies for the proofs.

1. Introduction

The goal of weak wave turbulence theory is to understand how energy is distributed and transferred in nonlinear wave systems. By imposing random initial data and passing to a weak nonlinearity limit, one formally derives a wave kinetic equation describing the average evolution of the energy distribution in frequency space. This framework is closely analogous to the classical kinetic theory of gases, where the Boltzmann equation is obtained under the Boltzmann–Grad limit [9, 17, 20].

Although wave kinetic theory provides a statistical description of energy transfer at a mesoscopic level and finds applications in many physical systems [3, 4, 18, 19, 24, 26, 28], rigorous derivations remain limited to a small number of models. In particular, it is now proved for all times for semilinear Schrödinger equation in dimensions $d \geq 3$ in a series of papers, see [7, 8] and references therein. Also important progress has been achieved for the 2D gravity water wave systems, [10, 11], while for one-dimensional settings, see [25] for the MMT model and [27] for the FPUT-β chain. Even less is currently known about the well-posedness and long-time behavior of the resulting kinetic equations. This is the subject of this review. We focus on the kinetic equation arising from one-dimensional Fermi–Pasta–Ulam–Tsingou (FPUT) oscillator chains, which is also referred to as weakly anharmonic crystals.

The FPUT model originated in the celebrated numerical experiments performed at Los Alamos in the 1950s [5, 16]. Contrary to expectations, Fermi and collaborators observed that energy initially

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injected into a small number of modes did not equilibrate, but instead exhibited a quasi-periodic behaviour over long times. This lack of ergodicity, now known as the FPU paradox, highlighted the subtle nature of thermalisation in low-dimensional nonlinear systems.

Understanding energy transport in low-dimensional systems has since become a central topic in both physics and mathematics. One classical approach to understand the microscopic origin of Fourier's law of heat conduction (stating that heat flux is proportional to the temperature gradient, i.e. systems exhibiting regular energy transport) is to study finite one-dimensional chains coupled to thermal reservoirs at their boundaries. In such truncated systems, particles interact with nearest neighbours through Hamiltonian dynamics, while stochastic forcing and dissipation are introduced at the endpoints via Langevin baths at different temperatures T_1 and T_N. Under suitable assumptions on the interaction potential, the system is known to relax rapidly to a non-equilibrium steady state (NESS) [12, 13, 14], and quantitative estimates in terms of the system size have been obtained in certain cases [2, 23]. This steady state is non-equilibrium due to the presence of nonzero heat fluxes on average in the system. The associated microscopic conductivity $\kappa(N)$ is defined through

$$\langle J(N)\rangle_{\mathrm{NESS}} = \frac{T_1 - T_N}{N}\,\kappa(N),$$

where $\langle\,\cdot\,\rangle_{\mathrm{NESS}}$ denotes expectation with respect to the NESS, and Fourier's law corresponds to the existence of a finite, non-zero limit of $\kappa(N)$ as $N \to \infty$.

Numerical simulations indicate that in one-dimensional systems such as the FPUT chain, the conductivity diverges as $\kappa(N) \sim N^\alpha$ with $\alpha \in (0,1)$, signalling anomalous transport [1, 21]. Understanding this phenomenon from a wave turbulence perspective provides a key motivation for the present study.

The kinetic wave equation we consider takes the form

$$\partial_t f(t,p) = \mathcal{C}[f](t,p)$$

where the collision operator is given by

$$\mathcal{C}[f](t,p_0) = \int_{\mathbb{T}^3} \delta(\Sigma)\delta(\Omega) \prod_{\ell=0}^{3} \omega_\ell \prod_{\ell=0}^{3} f_\ell \left(\frac{1}{f} + \frac{1}{f_1} - \frac{1}{f_2} - \frac{1}{f_3} \right) \mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3 \tag{1.1}$$

with the usual notations $p = p_0$, $f = f_0 = f(p_0)$, $f_i = f(p_i)$ if $i = 1,2,3$, similarly for $\omega_i = \omega(p_i)$, and furthermore

$$\omega(p) = \left|\sin\left(\frac{p}{2}\right)\right|, \quad \Sigma = \Sigma(p,p_1,p_2,p_3) = p_0 + p_1 - p_2 - p_3$$
$$\Omega = \Omega(p,p_1,p_2,p_3) = \omega_0 + \omega_1 - \omega_2 - \omega_3$$

where we denote as usual $\mathbb{T}$ for the periodised torus

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

This equation conserves the mass $\mathcal{M}(f) = \int_{\mathbb{T}} f(p)\,\mathrm{d}p$ and the energy $\mathcal{E}(f) = \int_{\mathbb{T}} \omega(p) f(p)\,\mathrm{d}p$, and satisfies an H-theorem in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} -\log f(p)\,\mathrm{d}p \geq 0.$$

In the one-dimensional kinetic model associated with the FPU-β chain, the unique entropy maximizers are the Rayleigh–Jeans distributions (RJ)

$$\mathfrak{f}_{\beta,\gamma}(p) = \frac{1}{\beta\omega(p) + \gamma}, \qquad \gamma \geq 0,\ \beta \geq -\gamma,$$

where the parameters β, γ are fixed by the conservation of energy and mass, respectively. So, these constitute the thermodynamic equilibria under fixed mass and energy.

Organisation of the report. In the next section we classify all entropy maximizers (equilibria) for a more general class of microscopic systems, including the FPUT case. We then restrict to the kinetic equation arising from the FPUT model. We first summarize local well-posedness and unboundedness phenomena for the nonlinear collision operator, together with the main ideas of the proofs. Finally, we discuss the linearized dynamics around non-singular Rayleigh–Jeans equilibria and state the linear and nonlinear stability results, along with sketches of their proofs.

2. Entropy maximizers

Since mass and energy are conserved and the equation satisfies an H-theorem, it is natural to expect that the long-time behaviour is governed by maximizers of the entropy under fixed mass and energy constraints. In this section we work with the entropy functional $\mathcal{H}_{cl}(f) = \int_{\mathbb{T}^d} \ln f(p) \mathrm{d}p$ so that the H-theorem corresponds to monotonicity of $\mathcal{H}_{cl}$ and we classify its maximizers under these constraints. We work in a setting more general than the FPUT case: in all dimensions $d \geq 1$ and many dispersion relations, so that ω is continuous and $\min \omega \geq 0$.

Since $\omega \geq 0$ and by the homogeneity of the problem, we will furthermore assume that

$$\min_{\mathbb{T}^d} \omega = a \geq 0, \qquad \max_{\mathbb{T}^d} \omega = 1. \tag{2.1}$$

Assumption (2.1) implies that for any nonnegative bounded measure λ on $\mathbb{T}^d$

$$a\mathcal{M}(\lambda) \leq \mathcal{E}(\lambda) \leq \mathcal{M}(\lambda). \tag{2.2}$$

In particular, (2.2) is a necessary condition for a pair $(\mathcal{M}, \mathcal{E})$ to be realisable by a nonnegative bounded measure.

Writing the Euler–Lagrange equation for the constrained maximization of the entropy with fixed mass and energy leads to the Rayleigh–Jeans $\mathfrak{f}_{\beta,\gamma}$ equilibria in the classical case

$$\mathfrak{f}_{\beta,\gamma}(p) = \frac{1}{\beta\omega(p) + \gamma}. \tag{2.3}$$

We have $\mathfrak{f}_{\beta,\gamma} \geq 0$ if and only if

$$\text{either } \begin{cases} \gamma \geq 0 \\ \beta \geq -\gamma \end{cases} \qquad \text{or } \begin{cases} \gamma \leq 0 \\ \beta \geq -\frac{\gamma}{a}. \end{cases} \tag{2.4}$$

However, for certain dispersions ω, there exist admissible pairs $(\mathcal{M}_0, \mathcal{E}_0)$ satisfying (2.2) for which no *regular* Rayleigh–Jeans distribution (2.3) attains the entropy maximum; in those regimes the maximizer requires an additional singular part.

By the Radon–Nikodym theorem, λ can be uniquely decomposed into an absolutely continuous part with respect to the Lebesgue measure and a singular part:

$$\mathrm{d}\lambda = f\mathrm{d}p + \mathrm{d}\lambda_{\mathrm{sing}} \tag{2.5}$$

where $f \in L^1(\mathbb{T}^d)$, $f \geq 0$ and λ_{sing} is singular with respect to the Lebesgue measure.

Following [6, 15], the definitions of the mass, energy and entropy can be extended to general measures as follows

$$\mathcal{M}(\lambda) = \int_{\mathbb{T}^d} \mathrm{d}\lambda, \qquad \mathcal{E}(\lambda) = \int_{\mathbb{T}^d} \omega\, \mathrm{d}\lambda, \qquad \mathcal{H}_{cl}(\lambda) = \int_{\mathbb{T}^d} \ln f(p)\, \mathrm{d}p.$$

Defining

$$c_1 := a + (2\pi)^d \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}p}{\omega - a} \right)^{-1}, \qquad c_2 := 1 - (2\pi)^d \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}p}{1 - \omega} \right)^{-1}, \tag{2.6}$$

we see that

$$a \leq c_1 < c_2 \leq 1.$$

The first and second inequalities are equalities if and only if $\int \frac{\mathrm{d}p}{\omega - a} = \infty$ and $\int \frac{\mathrm{d}p}{1-\omega} = \infty$ respectively.

Whether we are able to find such Rayleigh–Jeans that maximize the entropy, given mass and energy constraints, is determined by the ratio of the given energy over the mass. So we have the following classification of the entropy maximizers, as it is also illustrated in Fig. 2.

Theorem 1 (Maximizers of the classical entropy)**.** *Let $\mathcal{M}_0, \mathcal{E}_0 \in (0, \infty)$ be a given mass and energy. The maximizers of the classical entropy $\mathcal{H}_{cl}$ over positive measures subject to the constraints $\mathcal{M}(f) = \mathcal{M}_0$ and $\mathcal{E}(f) = \mathcal{E}_0$ are characterized as follows:*

(i) *If $c_1 < \frac{\mathcal{E}_0}{\mathcal{M}_0} < c_2$, the unique maximizer is the unique RJ equilibrium with this mass and energy.*

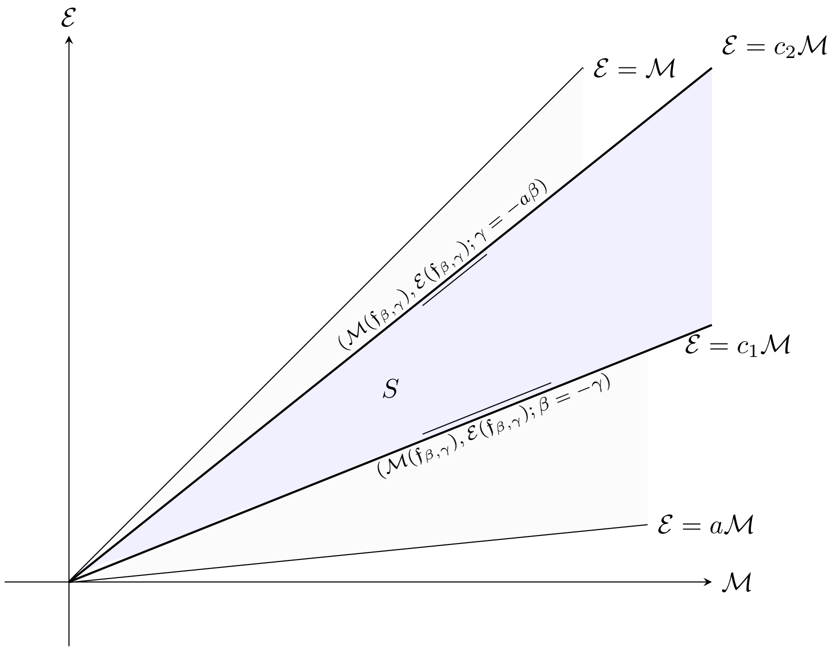

Figure 2.1: For a given mass and energy, $(\mathcal{M}_0, \mathcal{E}_0) \in S$, the entropy maximizer is a regular RJ $\mathfrak{f}_{\beta,\gamma}$. In the remaining light gray regions, an additional singular measure is needed to maximize the entropy.

(ii) *If* $c_2 \leq \frac{\mathcal{E}_0}{\mathcal{M}_0} < 1$*, maximizers are of the type* $\mathfrak{f}_{\beta,\gamma} + \lambda_{\text{sing}}$*, where* (β, γ) *are characterized by*

$$\begin{cases} \mathcal{M}(\mathfrak{f}_{\beta,\gamma}) = \frac{\mathcal{M}_0 - \mathcal{E}_0}{1-c_2} \\ \mathcal{E}(\mathfrak{f}_{\beta,\gamma}) = c_2 \frac{\mathcal{M}_0 - \mathcal{E}_0}{1-c_2}, \end{cases}$$

and λ_{sing} *has mass* $\frac{1}{1-\beta}(\mathcal{E}_0 - \beta\mathcal{M}_0)$ *and is supported on the set where* ω *is maximal.*

(iii) *If* $a < \frac{\mathcal{E}_0}{\mathcal{M}_0} \leq c_1$*, maximizers are of the type* $\mathfrak{f}_{\beta,\gamma} + \lambda_{\text{sing}}$ *with*

$$\begin{cases} \mathcal{M}(\mathfrak{f}_{\beta,\gamma}) = \frac{\mathcal{M}_0 - \mathcal{E}_0}{1-c_1} \\ \mathcal{E}(\mathfrak{f}_{\beta,\gamma}) = c_1 \frac{\mathcal{M}_0 - \mathcal{E}_0}{1-c_1}. \end{cases}$$

Also λ_{sing} *has mass and energy equal to* $\frac{1}{1-c_1}(\mathcal{E}_0 - c_1\mathcal{M}_0)$ *and is supported on the set where* ω *is maximal.*

Sketch of the proof.

Step I (feasibility of Rayleigh–Jeans constraints). One first determines for which ratios $\mathcal{E}_0/\mathcal{M}_0$ there exist parameters (β, γ) such that $\mathfrak{f}_{\beta,\gamma}$ matches the constraints.

Using the identity $\beta\mathcal{E}(\mathfrak{f}_{\beta,\gamma}) + \gamma\mathcal{M}(\mathfrak{f}_{\beta,\gamma}) = (2\pi)^d$ and the parametrization $\beta = \rho\cos\varphi$, $\gamma = \rho\sin\varphi$, our problem is then equivalent to

$$\frac{\mathcal{E}_0}{\mathcal{M}_0} = \frac{(2\pi)^d}{\cos\varphi\mathcal{M}(f_{\infty,\varphi})} - \tan\varphi =: F(\tan\varphi),$$

(after eliminating ρ), or in other words to determine the range of F above, while the domain is determined by the allowed range for β, γ, (2.4). After explicit calculations, we find that the range of F is $(a, 1)$.

This yields the critical slopes $c_1 < c_2$ and identifies the cone $S = \{c_1 < \mathcal{E}_0/\mathcal{M}_0 < c_2\}$ where a regular RJ solves the constraints.

Step II (unique entropy maximizers). Applying $\log x \le x - 1$ with $x = f/\mathfrak{f}_{\beta,\gamma}$ gives the bound

$$H(f) - H(\mathfrak{f}_{\beta,\gamma}) \le \beta\big(\mathcal{E}_0 - \mathcal{E}(\mathfrak{f}_{\beta,\gamma})\big) + \gamma\big(\mathcal{M}_0 - \mathcal{M}(\mathfrak{f}_{\beta,\gamma})\big).$$

- In the regime $c_1 < \mathcal{E}_0/\mathcal{M}_0 < c_2$ one can choose (β, γ) so that the right-hand side vanishes (due to Step I), yielding optimality and uniqueness of the corresponding RJ.
- (appearance of a singular part): Outside the cone S, one chooses a RJ distribution on the boundary line $\mathcal{E} = c_i\mathcal{M}$ ($i = 1$ or 2) so that $\mathcal{E}_0 - \mathcal{E}(\mathfrak{f}_{\beta,\gamma})) = \mathcal{M}_0 - \mathcal{M}(\mathfrak{f}_{\beta,\gamma})$. Then the idea is to compensate the remaining constraints by adding a singular measure λ_{sing} to this RJ. Explicit calculations yield that λ_{sing} should satisfy $\mathcal{E}(\lambda_{\text{sing}}) = \mathcal{M}(\lambda_{\text{sing}})$ or in other words to be necessarily supported on the set where ω attains its maximum (and carrying the missing mass/energy). This produces the maximizers described in (ii)–(iii). □

3. First results on the nonlinear kinetic FPUT equation

To start studying boundedness of this operator we first need to parametrize the resonant manifold. Apart form the non-trivial zeros, one can find [22] that it can be parameterized by

$$p_1 = h(p_0, p_2) \mod 2\pi, \quad \text{where} \quad h(x, z) = \frac{z - x}{2} + 2\arcsin\left(\tan\frac{|z - x|}{4}\cos\frac{z + x}{4}\right).$$

Using this expression, a calculation ([22]) shows that the collision operator can be written

$$\mathcal{C}[f](p_0) = \int_0^{2\pi} \frac{\omega_0\omega_1\omega_2\omega_3}{\sqrt{F_+(p_0, p_2)}} \prod_{\ell=0}^{3} f_\ell\left(\frac{1}{f} + \frac{1}{f_1} - \frac{1}{f_2} - \frac{1}{f_3}\right) \mathrm{d}p_2, \tag{3.1}$$

where it is understood that

$$p_3 = p_0 + p_1 - p_2, \quad p_1 = h(p_0, p_2).$$

and

$$F_+(p_0, p_2) = \sqrt{\left[\cos\left(\frac{p_0}{2}\right) + \cos\left(\frac{p_2}{2}\right)\right]^2 + 4\sin\left(\frac{p_0}{2}\right)\sin\left(\frac{p_2}{2}\right)}.$$

Using lower bounds on F_+ and a fixed point argument, we have the following local-in-time result in weighted L^∞ spaces, for the nonlinear problem:

Theorem 2. *This Cauchy problem is locally well-posed in $\omega^\alpha L^\infty(\mathbb{T})$ if $\alpha \ge -1$. More precisely, if $f_0 \in \omega^\alpha L^\infty$, there exists a unique solution $f \in \mathcal{C}([0, T], \omega^\alpha L^\infty)$, where*

$$T \gtrsim \|f_0\|^{-2}_{\omega^\alpha L^\infty}.$$

However as the following theorem shows, the operator is unbounded in L^p spaces for $p < 3$.

Theorem 3. *The collision operator $\mathcal{C}$ is not bounded on $L^p(\mathbb{T})$ if $p < 3$.*

Idea of proof of Theorem 3. One constructs f^ε as the sum of three packets localized near distinct frequencies p_0^0, p_1^0, p_2^0, scaled so that $\|f^\varepsilon\|_{L^p} = O(1)$. For fixed p_0^0, the point p_2^0 is chosen so that it is a (modulo 2π) maximizer of $z \mapsto h(p_0^0, z)$. As a consequence, on the resonant manifold and for p_0 near p_0^0, the parametrization $p_1 = h(p_0, p_2)$, $p_3 = p_0 + p_1 - p_2$ forces p_3 to lie near p_2^0 whenever p_2 does. So the resonant parametrisation selects essentially a single interaction (resonant packet configuration). This yields a contribution that makes $\|\mathcal{C}[f^\varepsilon]\|_{L^p}$ diverge as $\varepsilon \to 0$ when $p < 3$. □

4. Linearizing around Rayleigh–Jeans equilibria

We now turn to the linearized dynamics around a Rayleigh–Jeans equilibrium. Denote by L the linearized collision operator around $\mathfrak{f}_{\beta,\gamma}$, $\gamma > 0$.

Theorem 4 (Weighted/degenerate Poincaré-type inequality)**.** *There exists a nonnegative weight $a(p)$ satisfying*

$$|a(p)| \sim \left|\sin\left(\frac{p}{2}\right)\right|^{\frac{5}{3}}$$

such that for any $\beta, \gamma > 0$,

$$\textit{for all } g \in L^2 \cap \mathrm{Ker}(L)^\perp, \qquad \langle -Lg, g\rangle \gtrsim \int a(p)|g(p)|^2 \,\mathrm{d}p.$$

In particular, for such initial datum g_0,

$$\int_0^\infty \int_0^{2\pi} a(p)\,|e^{tL} g_0(p)|^2 \,\mathrm{d}p\,\mathrm{d}t \;\lesssim\; \|g_0\|_{L^2}^2.$$

The weight $a(p)$ vanishes at the boundary of the frequency domain, implying that the dissipation degenerates near $p = 0$ and $p = 2\pi$. As a consequence, this inequality does not yield a spectral gap, and one cannot expect exponential decay of the linear semigroup.

Nevertheless, it is still possible to prove polynomial-in-time decay estimates for the linearized dynamics. This is the content of the following result.

Theorem 5. *Assume that $g_0 \in L^\infty(\mathbb{T})$ and that g_0 has zero L^2-projection onto* Ker L. *Then for any $\mu, \nu \in [\frac{1}{6}, \frac{1}{2}]$ and any $\delta > 0$,*

$$\|\omega^\mu e^{tL} g_0\|_{L^\infty} \;\lesssim_\delta\; \langle t\rangle^{-\frac{3}{5}(\mu+\nu)+\delta} \|\omega^{-\nu} g_0\|_{L^\infty}.$$

The proof relies on understanding how the regions near the edges of the frequency domain, where dissipation degenerates, interact with the bulk, where dissipation is lower bounded. At a technical level, the decay is obtained through an iterative scheme that progressively transfers control from the bulk to the edges.

To make this more precise, we introduce the following functionals:

$$m(t) = \int_{\text{Bulk}} |g(t,p)|^2 \,\mathrm{d}p, \qquad n(t) = \int_{\text{Edges}} |g(t,p)|^2 \,\mathrm{d}p, \qquad q(t) = \sup_{p \in \text{Edges}} |g(t,p)|.$$

Here the bulk is defined as

$$\text{Bulk} := \big\{p \in [0, 2\pi] : \langle t\rangle^{-\alpha} \le p \le 2\pi - \langle t\rangle^{-\alpha}\big\},$$

and the edges are its complement. The time-dependent decomposition reflects the fact that dissipation weakens near the edges $p = 0$ and $p = 2\pi$.

For $\alpha < \frac{3}{5}$, the decay follows by iterating the following bounds:

- If $m(0) + n(0) \le 1$ and $q(t) \le \langle t\rangle^e$ for some $e \in \mathbb{R}$, then

  $$m(t) \lesssim_\alpha \langle t\rangle^{2e-\alpha}.$$

  This step relies essentially on the degenerate Poincaré inequality.

- Assume $\|g_0\|_{L^\infty} \le 1$ and $\|g(t)\|_{L^2} \le \langle t\rangle^b$.

  - If $b > -1$, then

    $$|g(t,p)| \lesssim |g_0(p)| + \omega^{\frac{3}{2}} \langle t\rangle^{b+1+}.$$

  - In the bulk region, i.e. for $\omega(p) > \langle t\rangle^{-\alpha}$,

    $$|g(t,p)| \lesssim \omega^{-\frac{1}{6}} \langle t\rangle^{b+}.$$

  These bounds are obtained by exploiting the equation and carefully handling the singularity of the collision kernel, which behaves like $x^{-1/2}$.

5. Global well-posedness for the nonlinear problem

Having established decay estimates for the linearized equation, we now pass to the nonlinear setting via a perturbative argument. We prove global existence and decay for solutions close to a non-singular Rayleigh–Jeans equilibrium. This is accomplished through a careful definition of an appropriate norm that allows us to control the nonlinearity, relying crucially on the structure of the equation.

This is the content of the final Theorem:

Theorem 6. *There exists* $\epsilon_0 > 0$ *such that the following holds. If*

$$f(0,p) = \mathfrak{f}_{\beta,\gamma}(p)\,[1 + g_0(p)],$$

with

$$\int \mathfrak{f}_{\beta,\gamma} g_0 \,\mathrm{d}p = \int \omega\, \mathfrak{f}_{\beta,\gamma} g_0 \,\mathrm{d}p = 0 \quad \textit{and} \quad \|\omega^{-\frac{1}{2}} g_0\|_{L^\infty} = \epsilon < \epsilon_0,$$

then there exists a unique global solution of the form

$$f(t,p) = \mathfrak{f}_{\beta,\gamma}(p)\,[1 + g(t,p)],$$

satisfying

$$\|\omega^{\frac{1}{2}} g(t)\|_{L^\infty_p} \lesssim \epsilon\,\langle t\rangle^{-\frac{3}{5}+\frac{1}{1000}} \qquad \textit{for all } t \geq 0.$$

Main steps of the proof. For the proof, we write the full nonlinear evolution equation for the perturbation g:

$$\partial_t g - Lg = \mathcal{Q}(g) + \mathcal{C}(g),$$

where $\mathcal{Q}$ and $\mathcal{C}$ denote the quadratic and cubic nonlinearities. Using Duhamel's formula,

$$g(t) = S_t g_0 + \int_0^t S_{t-s}\big(\mathcal{Q}[g](s) + \mathcal{C}[g](s)\big)\,\mathrm{d}s.$$

The key norm controlling the evolution is defined, for $T > 0$, by

$$\|g\|_{\mathcal{B}_T} := \left\|\langle t\rangle^{\frac{2}{5}-}\,\omega^{\frac{1}{6}} g\right\|_{L^\infty_{t,p}([0,T]\times\mathbb{T})} + \left\|\langle t\rangle^{\frac{3}{5}-}\,\omega^{\frac{1}{2}} g\right\|_{L^\infty_{t,p}([0,T]\times\mathbb{T})},$$

where "$-$" in the exponents denotes an arbitrarily small loss.

The proof then relies on the following ingredients:

(i) *A priori bound.* For any $T > 0$, if $g \in C([0,T], \omega^{-\frac{1}{6}} L^\infty)$, then

$$\|g\|_{\mathcal{B}_T} \leq C_0\big[\epsilon + \|g\|^2_{\mathcal{B}_T} + \|g\|^3_{\mathcal{B}_T}\big], \qquad \epsilon := \|\omega^{-\frac{1}{2}} g_0\|_{L^\infty}.$$

(ii) *Bootstrap inequality.* If $x(t)$ is continuous on $(0,T)$ and satisfies

$$x(t) \leq C_0\big[\epsilon + x(t)^2 + x(t)^3\big], \qquad \lim_{t\to 0} x(t) \leq C_0\epsilon,$$

then $x(t) \leq 2C_0\epsilon$ for all $t \in (0,T)$.

(iii) *Local well-posedness in* $\omega^{-\frac{1}{6}} L^\infty$ *(blow-up criterion).* There exists $g \in C([0,T_0), \omega^{-\frac{1}{6}} L^\infty)$, where $T_0 > 0$, such that

$$\lim_{t\to T_0} \|\omega^{\frac{1}{6}} g(t)\|_{L^\infty_p} = \infty \quad \text{if } T_0 < \infty.$$

To conclude the statement of the Theorem, assume by contradiction that $T_0 < \infty$. By (i) and (ii) above, we obtain

$$\|g\|_{\mathcal{B}_t} \leq 2C_0\epsilon \qquad \text{for all } t < T_0,$$

which contradicts the blow-up criterion in (iii). Therefore, $T_0 = \infty$, and the solution exists globally. □

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