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Deep- and shallow-water convergence of the generalized Gibbs measures for the intermediate long wave equation

Andreia Chapouto Guopeng Li Tadahiro Oh

Limites en eaux profondes et peu profondes des équilibres statistiques pour l'équation des ondes longues intermédiaires

Résumé

Cette note est basée sur une présentation donnée par la première autrice lors de la conférence Journées Équations aux dérivées partielles 2025. Nous nous intéressons à l'équation intermédiaire des ondes longues (ILW), qui modélise la propagation des ondes internes à l'interface dans un fluide stratifié de profondeur finie. Cette équation relie le régime des eaux profondes (= le régime BO) et le régime des eaux peu profondes (= le régime KdV). En exploitant l'intégrabilité complète de l'ILW, nous fournissons une description détaillée de ses lois de conservation polynomiales, construisons les mesures de Gibbs généralisées invariantes associées, puis montrons leur convergence vers celles des régimes BO et KdV. Dans le régime des eaux peu profondes, nous établissons un nouveau recoupement 2-à-1 des lois de conservation de l'ILW avec celles de KdV (ainsi qu'un recoupement 2-à-1 des mesures généralisées de Gibbs associées), qui met en évidence la nature singulière de la convergence des eaux peu profondes.

Abstract

This note is based on a talk given by the first author at the conference Journées Équations aux dérivées partielles 2025. We consider the intermediate long wave equation (ILW), modeling the internal wave propagation of the interface in a stratified fluid of finite depth, connecting the deep-water regime (= the BO regime) and the shallow-water regime (= the KdV regime). Exploiting the complete integrability of ILW, we provide a detailed description of its polynomial conservation laws, construct the associated invariant generalized Gibbs measures, and lastly show their convergence to those of BO and KdV. In the shallow-water regime, we establish a novel 2-to-1 collapse of ILW conservation laws to those of KdV (and also a 2-to-1 collapse of the associated generalized Gibbs measures), which exhibits the singular nature of the shallow-water convergence.

1. Intermediate long wave equation

We consider the intermediate long wave equation (ILW) on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\partial_x u - \mathcal{G}_\delta \partial_x^2 u = \partial_x(u^2), \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \tag{ILW}$$

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where the operator $\mathcal{G}_\delta$, characterizing the phase speed, is a Fourier multiplier operator with the multiplier:

$$\widehat{\mathcal{G}_\delta f}(n) = \widehat{\mathcal{G}_\delta}(n)\widehat{f}(n) = -i\big(\coth \delta n - (\delta n)^{-1}\big)\widehat{f}(n), \quad n \in \mathbb{Z} \setminus \{0\},$$

and $\widehat{\mathcal{G}_\delta}(0) = 0$.

This equation was introduced in [21, 27] to describe the propagation of an internal wave at the interface of a stratified fluid of finite depth $\delta > 0$, where u denotes the amplitude of the internal wave. Additionally, (ILW) is an important physical model with applications to atmospheric sciences, oceanography, and quantum physics [4, 14, 26, 42, 46].

From the analytical viewpoint, (ILW) is of great interest due to its rich structure; it is a *dispersive* equation, it is *Hamiltonian*, and it is *completely integrable* with an infinite number of conservation laws [24, 25, 43]. Moreover, it can be seen as an "intermediate" equation of finite depth $0 < \delta < \infty$, bridging the Benjamin–Ono equation (BO):

$$\partial_t u - \mathcal{H}\partial_x^2 u = \partial_x(u^2), \tag{BO}$$

when $\delta = \infty$, and the Korteweg-de Vries equation (KdV) corresponding to $\delta = 0$:

$$\partial_t u + \frac{1}{3}\partial_x^3 u = \partial_x(u^2). \tag{KdV}$$

Here, $\mathcal{H}$ denotes the Hilbert transform, with Fourier multiplier $\widehat{\mathcal{H}}(n) = -i\,\mathrm{sgn}(n)$.

We refer the interested readers to [23, 45] for an overview of the study of (ILW), (BO), and (KdV).

Our main goal in the present note and [11] is to study the *statistical convergence* of (ILW) to (BO) in the deep-water limit and to (KdV) in the shallow-water limit. More precisely, we first construct generalized Gibbs measures associated with higher order conservation laws of (ILW), and then establish their invariance and convergence to the corresponding measures for (BO) and (KdV). This requires a careful analysis of the conservation laws for the three equations, which is of independent interest in the literature.

2. Solutions, convergence, and statistics

We start by clarifying how (BO) and (KdV) arise as limiting equations for (ILW), as we change the depth parameter δ, in the deep-water ($\delta \to \infty$) and shallow-water limits ($\delta \to 0$).

The deep-water limit is motivated by the convergence of the dispersive symbol in (ILW): in [1], the authors observed that

$$0 \le \widehat{\mathcal{H}\partial_x}(n) - \widehat{\mathcal{G}_\delta\partial_x}(n) \le \frac{1}{\delta} \quad \Longrightarrow \quad \lim_{\delta\to\infty} \widehat{\mathcal{G}_\delta\partial_x^2}(n) = \widehat{\mathcal{H}\partial_x^2}(n), \tag{2.1}$$

uniformly in frequency $n \in \mathbb{Z}$, which formally shows the convergence of (ILW) to (BO).

In the shallow-water regime, recalling that the unknown u denotes the amplitude of the internal wave at the interface (which is of order $O(\delta)$) as $\delta \to 0$, we expect u to collapse to 0. Therefore, in order to observe a meaningful limit in this shallow-water regime, we need to magnify the fluid amplitude by a factor of $\frac{1}{\delta}$. This leads to the following scaling [1]:

$$v(t,x) = \frac{1}{\delta}u(\frac{1}{\delta}t, x), \tag{2.2}$$

where the new unknown v solves the following scaled ILW:

$$\partial_t v - \widetilde{\mathcal{G}}_\delta \partial_x^2 v = \partial_x(v^2), \quad \text{where} \quad \widetilde{\mathcal{G}}_\delta = \frac{1}{\delta}\mathcal{G}_\delta. \tag{sILW}$$

From the Mittag-Leffler expansion of $\coth x$, we have

$$\widehat{\widetilde{\mathcal{G}}_\delta\partial_x}(n) = \sum_{k=1}^{\infty} \frac{2n^2}{k^2\pi^2 + \delta^2 n^2} \nearrow \frac{1}{3}n^2 = \widehat{\Big(-\frac{1}{3}\partial_x^2\Big)}(n) \quad \text{as} \quad \delta \to 0, \tag{2.3}$$

for each *fixed* $n \in \mathbb{Z}$. We, however, note that the convergence (2.3) is *not uniform* in frequency, unlike in the deep-water limit. Due to the lack of uniformity above together with the increased strength of dispersion, we see that the shallow-water limit is *singular* (as compared to the deep-water limit). We will see below, that this singular nature appears also in the convergence of the conservation laws of (sILW) to those of (KdV) and of their solutions, through regularity jumps and distinct modes of convergence. See also Remark 4.

Regarding the analytical study of (ILW), there have been extensive developments in its well-posedness and long-time behavior [1, 9, 12, 16, 17, 19, 20, 33, 35, 36, 37] as well as its convergence to (BO) and (KdV) [1, 12, 13, 17, 31, 32, 38]. Let us describe some of the works listed above. In [37], Molinet–Vento showed global well-posedness of (ILW) in $H^{1/2}$, both on the line and the torus, via the improved energy method. This was improved to $L^2(\mathbb{R})$ by Ifrim–Saut [20], and revisited and extended to the torus case in [12] by the authors with D. Pilod. More recently, Gassot–Laurens [17] showed optimal well-posedness in $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$, exploiting complete integrability. The analogous result on the line remains open.

Regarding the convergence problem, Abdelouhab–Bona–Felland–Saut [1] showed convergence of solutions to (ILW) (and to the scaled ILW (sILW), respectively) to those of (BO) (and of (KdV), respectively) in H^s for $s > \frac{3}{2}$ (and $s \geq 2$, respectively). This was extended to $s > \frac{1}{2}$ by the second author [31] in both limits (see also [31, Remark 1.6]), and to $s \geq 0$ only in the deep-water limit in [12] by the authors with D. Pilod. In [17], Gassot–Laurens obtained convergence to (BO) solutions in $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$. Regarding the more singular shallow-water limit, the authors with T. Zhao [13] recently showed convergence in L^2 to (KdV) solutions, in both geometries.

From a statistical viewpoint, in [32], the second and third authors with G. Zheng initiated the study of statistical convergence of (ILW) by constructing its Gibbs measure and proving convergence in the deep- and shallow-water limits, as well as convergence of invariant Gibbs dynamics (without uniqueness). In this note, based on [11], we aim to further this statistical study, by exploiting the complete integrability of (ILW), through the construction of an infinite family of invariant *generalized Gibbs measures* associated with higher order conservation laws.

3. (Generalized) Gibbs measures for Hamiltonian systems

Instead of following the trajectory of an individual solution to (ILW) to that of (BO) and (KdV), we may instead follow a statistical viewpoint, and consider the evolution of an ensemble of solutions. This relates to the study of invariant Gibbs measures for Hamiltonian PDEs, which was initiated by Lebowitz–Rose–Speer [30] and Bourgain [7, 8].

Here, given a Hamiltonian equation with Hamiltonian H, a Gibbs measure refers to the probability measure on periodic functions / distributions, formally given by

$$\rho(\mathrm{d}u) = Z^{-1} e^{-H(u)} \mathrm{d}u, \tag{3.1}$$

which is of central importance in statistical mechanics. We say that the Gibbs measure ρ in (3.1) is *invariant* under the Hamiltonian dynamics if

$$\rho\big(\Phi(-t)A\big) = \rho(A) \quad \text{for any } t \in \mathbb{R} \text{ and any measurable set } A,$$

where $\Phi(t)$ denotes the solution map, sending initial data to solutions at time t. Invariant (Gibbs) measures allow us to infer on the typical behavior of solutions, beyond what can be typically shown deterministically, such as growth estimates and recurrence properties.

The study of invariant Gibbs measures has attracted much attention in the past 30 years; see surveys papers [3, 40] and references therein for an overview of the subject.

As mentioned above, the statistical study for (ILW) was initiated in the work [32], where the authors constructed the Gibbs measure and established its invariance and deep-water and shallow-water convergence.

For completely integrable models such as (ILW), (BO), and (KdV) with infinitely many conservation laws $\{\mathcal{E}_k(u)\}_{k\in\mathbb{N}}$, we may instead consider the following *generalized Gibbs measures* associated with the conservation law $\mathcal{E}_k(u)$:

$$\rho_k(\mathrm{d}u) = Z_k^{-1} e^{-\mathcal{E}_k(u)} \mathrm{d}u.$$

These generalized Gibbs measures allow us to study statistical behavior for *smoother* solutions. When invariant, they provide qualitative information such as recurrence properties on physically phase space of smooth functions, such as described by Poincaré's recurrence theorem.

The construction of generalized Gibbs measures for (KdV) is due to Zhidkov [51, 52] and for (BO) by Tzvetkov, Visciglia, and Deng in a series of works [15, 47, 48, 49].

In [11], we investigated the construction of the generalized Gibbs measures for (ILW) together with their invariance and convergence properties in the deep- and shallow-water limits, from static and dynamical viewpoints. The stepping stone in our argument is a detailed study of the (polynomial) conservation laws for (ILW) and (sILW) via Bäcklund transforms ((6.1) and (6.5)). In particular, in the shallow-water limit, we observe a novel 2-to-1 collapse (Theorem 1), which propagates to the convergence of the corresponding measures and dynamics (Theorems 2 and 3).

4. Infinite families of conservation laws

The first step of our statistical study of (ILW) is the careful derivation of higher order conservation laws along with their convergence properties. The derivation of conservation laws for (ILW) is well known in the completely integrable literature [1, 18, 25, 28, 29, 34, 44], but the existing proofs of convergence as the parameter δ varies are only formal. Thus, our first task is to rigorously derive conservation laws for (ILW) (and (sILW)) in a form suitable for the construction of generalized Gibbs measures (namely, a sign-definite quadratic-in-u part), carefully analyze their structure, and show their convergence in both limits. See Section 6 for details.

In the deep-water regime,[1] we follow the work of Satsuma–Ablowitz–Kodama [44] and derive conservation laws for (ILW) from the Bäcklund transform in (6.1). This yields a recursion formula for microscopic conservation laws χ_n^δ which, after integration and careful combinatorial / algebraic computations, give rise to a sequence $\{E^\delta_{k/2}(u)\}_{k\in\mathbb{Z}_{\geq 0}}$ of macroscopic conservation laws with the form:

$$\begin{aligned} E_0^\delta(u) &= \frac{1}{2}\|u\|^2_{L^2}, \\ E^\delta_{k/2}(u) &= \frac{1}{2}\sum_{\substack{\ell=0\\ \text{even}}}^{k} a_{k,\ell}\|\mathcal{G}_\delta^{\frac{k-\ell}{2}}u\|^2_{\dot H^{k/2}} + R^\delta_{k/2}(u) =: Q^\delta_{k/2}(u) + R^\delta_{k/2}(u), \quad k\in\mathbb{N}. \end{aligned} \tag{4.1}$$

Here, $a_{k,\ell}$ are *positive* constants adding up to 1, $\dot H^s$ denotes the homogeneous L^2-based Sobolev space, and $R^\delta_{k/2}(u)$ denotes the interaction potential with cubic and higher order terms in u. For readers' convenience, we write down the conservation laws $E^\delta_{k/2}(u)$ for $k=1,2$, corresponding to the Hamiltonian for (ILW) and the first higher order conservation law, respectively:

$$\begin{aligned} E^\delta_{1/2}(u) &= \frac{1}{2}\|\mathcal{G}_\delta^{1/2}u\|^2_{\dot H^{\frac12}} + \frac{1}{3}\int_{\mathbb{T}} u^3\,\mathrm{d}x, \\ E^\delta_1(u) &= \frac{1}{8}\|u\|^2_{\dot H^1} + \frac{3}{8}\|\mathcal{G}_\delta u\|^2_{\dot H^1} + \int_{\mathbb{T}}\left[\frac{1}{4}u^4 + \frac{3}{4}u^2\mathcal{G}_\delta\partial_x u + \frac{1}{4\delta}u^3\right]\mathrm{d}x. \end{aligned} \tag{4.2}$$

The positivity of $a_{k,\ell}$ in (4.1) is essential to guarantee that the quadratic part $Q^\delta_{k/2}(u)$ of $E^\delta_{k/2}(u)$ is sign-definite and is at the $H^{k/2}$-level of regularity:

$$Q^\delta_{k/2}(u) \sim_{k,\delta} \|u\|^2_{\dot H^{k/2}}. \tag{4.3}$$

From the convergence of the $\mathcal{G}_\delta$ operator in (2.1), combined with a detailed description of the energies in (4.1), we recover a family of conservation laws for (BO), $\{E^\infty_{k/2}(u)\}_{k\in\mathbb{Z}_{\geq 0}}$:

$$E_0^\infty(u) = \frac{1}{2}\|u\|^2_{L^2}, \quad E^\infty_{k/2}(u) = \frac{1}{2}\|u\|^2_{\dot H^{k/2}} + R^\infty_{k/2}(u), \quad k\in\mathbb{N}, \tag{4.4}$$

where $R^\infty_{k/2}(u)$ consists of cubic and higher order terms in u. For $k=1,2$, we have

$$\begin{aligned} E^\infty_{1/2}(u) &= \frac{1}{2}\|u\|^2_{\dot H^{1/2}} + \frac{1}{3}\int_{\mathbb{T}} u^3\,\mathrm{d}x, \\ E^\infty_1(u) &= \frac{1}{2}\|u\|^2_{\dot H^1} + \int_{\mathbb{T}}\left[\frac{1}{4}u^4 + \frac{3}{4}u^2\mathcal{H}\partial_x u\right]\mathrm{d}x, \end{aligned}$$

[1] In the following, we use the term “deep-water regime” for analysis on the (unscaled) ILW (ILW) and its deep-water limit (BO), while we use the term “shallow-water regime” for analysis on the scaled ILW (sILW) and its shallow-water limit (KdV).

which can be "obtained" from (4.2) by ignoring terms with negative powers of δ (which vanish in the limit) and replacing $\mathcal{G}_\delta$ with $\mathcal{H}$. See Theorem 1 for a detailed statement on the deep convergence ($\delta \to \infty$) of $E^{\delta}_{k/2}(u)$ to $E^{\infty}_{k/2}(u)$.

Next, we discuss the shallow-water regime. We recall that (ILW) and (sILW) only differ by the scaling (2.2). Therefore, given a solution v to (sILW), the scaled quantity $E^{\delta}_{k/2}(\delta v)$ is also conserved for fixed $0 < \delta < \infty$, where $E^{\delta}_{k/2}$, $k \in \mathbb{Z}_{\geq 0}$, is as in (4.1). However, (after suitably dividing by a power of δ to avoid a trivial limit) this definition is not suitable to study shallow-water convergence, as there are multiple divergent terms as $\delta \to 0$. Although it may be possible to manually group divergent terms to obtain cancellation for the first few energies, this strategy becomes intractable for large k. Instead, we start anew and repeat the derivation of conservation laws using the Bäcklund transform but adapted to the shallow-water regime due to Gibbons and Kupershmidt [18, 28]; see (6.5). As in the deep-water regime, this yields a recursive formula for microscopic conservation laws h^{δ}_n. It turns out that these densities have an additional difficulty; half of the corresponding macroscopic conservation laws, $\int h^{\delta}_{2n+1} dx$, *vanish* in the shallow-water limit! Nevertheless, by taking suitable combinations of h^{δ}_n's (see (6.6)), a careful algebraic and combinatorial computation allows us to construct a full family of conservation laws $\{\widetilde{E}^{\delta}_{k/2}(v)\}_{k\in\mathbb{Z}_{\geq 0}}$ for (sILW) with non-trivial shallow-water limits for every k:

$$\begin{aligned} \widetilde{E}^{\delta}_{0}(v) &= \frac{1}{2}\|v\|^2_{L^2}, \\ \widetilde{E}^{\delta}_{k/2}(v) &= \frac{1}{2}\sum_{\substack{\ell=0\\ \text{even}}}^{k} \widetilde{a}_{k,\ell}\,\delta^{\ell}\|\widetilde{\mathcal{G}}^{\ell/2}_{\delta} v\|^2_{\dot{H}^{k/2}} + \widetilde{R}^{\delta}_{k/2}(v) =: \widetilde{Q}^{\delta}_{k/2}(v) + \widetilde{R}^{\delta}_{k/2}(v), \end{aligned} \tag{4.5}$$

where $\widetilde{\mathcal{G}}_\delta$ is the scaled operator as in (sILW), $\widetilde{a}_{k,\ell}$ are non-negative constants adding up to 1, and $\widetilde{R}^{\delta}_{k/2}(v)$ denotes the interaction potential with cubic and higher order terms in v. In taking a limit $\delta \to 0$, we expect the quantities in (4.5) to converge to those of (KdV), namely $\{\widetilde{E}^{0}_{\kappa}(v)\}_{\kappa\in\mathbb{Z}_{\geq 0}}$ given by

$$\widetilde{E}^{0}_{0}(v) = \frac{1}{2}\|v\|^2_{L^2}, \quad \widetilde{E}^{0}_{\kappa}(v) = \frac{1}{2}\|v\|^2_{\dot{H}^{\kappa}} + \widetilde{R}^{0}_{\kappa}(v),$$

where $\widetilde{R}^{0}_{\kappa}(v)$ contains cubic and higher order terms in v. Note that (KdV) has only "half" as many conservation laws as (sILW), only capturing H^{κ}-levels of regularity as opposed to $H^{\kappa/2}$, for $\kappa \in \mathbb{Z}_{\geq 0}$. In fact, in the shallow-water limit, we establish a novel 2-to-1 collapse of these conservation laws, with each pair, $\widetilde{E}^{\delta}_{\kappa-1/2}(v)$ and $\widetilde{E}^{\delta}_{\kappa}(v)$, sharing the same limit $\widetilde{E}^{0}_{\kappa}(v)$. This is in contrast with the 1-to-1 correspondence between $E^{\delta}_{k/2}(u)$ and $E^{\infty}_{k/2}(u)$ in the deep-water regime.

The following theorem summarizes the construction of conservation laws for (ILW) and (sILW), as well as their convergence in the deep- and shallow-water limits, respectively. This constitutes the first main contribution in [11]. We also obtain detailed descriptions of the terms appearing in the energies, needed for the construction and analysis of generalized Gibbs measures but omit details; see [11] for a further discussion.

Theorem 1. *Let $0 < \delta < \infty$. Then, the quantities*

$$\{E^{\delta}_{k/2}(u)\}_{k\in\mathbb{N}}, \quad \{E^{\infty}_{k/2}(u)\}_{k\in\mathbb{N}}, \quad \{\widetilde{E}^{\delta}_{\kappa-1/2}(v), \widetilde{E}^{\delta}_{\kappa}(v)\}_{\kappa\in\mathbb{N}}, \quad \text{and} \quad \{\widetilde{E}^{0}_{\kappa}(v)\}_{\kappa\in\mathbb{N}}$$

are conserved under the dynamics of (ILW), (BO), (sILW), *and* (KdV), *respectively. Moreover, the following limits hold for $k, \kappa \in \mathbb{N}$:*

$$\begin{aligned} \lim_{\delta\to\infty} E^{\delta}_{k/2}(u) &= E^{\infty}_{k/2}(u), & u &\in H^{k/2}(\mathbb{T}), \\ \lim_{\delta\to 0} \widetilde{E}^{\delta}_{\kappa-1/2}(v) = \lim_{\delta\to 0} \widetilde{E}^{\delta}_{\kappa}(v) &= \widetilde{E}^{0}_{\kappa}(v), & v &\in H^{\kappa}(\mathbb{T}). \end{aligned}$$

Theorem 1 constructs an infinite family of conservation laws in each of deep-water and shallow-water regimes, including the limiting cases ($\delta = \infty$ and $\delta = 0$). Moreover, these conservations are constructed in a form suitable for constructing the associated generalized Gibbs measures as weighted Gaussian measures. See the next section for a further discussion. We emphasize that Theorem 1 presents the first construction of a complete family of shallow-water conservation laws with non-trivial shallow-water limits, where we in particular establish a novel 2-to-1 collapse to the (KdV) conservation laws. In the latter, since $\widetilde{E}^{\delta}_{\kappa-1/2}(v)$ is at the $H^{\kappa-1/2}$-level of regularity, we note that there is a regularity jump in its convergence to $\widetilde{E}^{0}_{\kappa}(v)$, substantiating the singular nature of the shallow-water limit. See Figure 4.1.

$$
\begin{array}{ccccccc}
E^{\delta}_{\kappa}(u) & \xrightarrow{\delta \to \infty} & E^{\infty}_{\kappa}(u) & \qquad & \widetilde{E}^{\delta}_{\kappa}(v) & \xrightarrow{\delta \to 0} & \widetilde{E}^{0}_{\kappa}(v) \\
 & & & & & \nearrow_{\delta \to 0} & \\
E^{\delta}_{\kappa-1/2}(u) & \xrightarrow{\delta \to \infty} & E^{\infty}_{\kappa-1/2}(u) & \qquad & \widetilde{E}^{\delta}_{\kappa-1/2}(v) & &
\end{array}
$$

Figure 4.1: In the deep-water regime, there is a 1-to-1 correspondence between energies, with no regularity jump. In the shallow-water limit, while there is no regularity jump in the convergence of $\widetilde{E}^{\delta}_{\kappa}(v)$ to $\widetilde{E}^{0}_{\kappa}(v)$, there is a regularity jump by $\frac{1}{2}$ in the convergence of $\widetilde{E}^{\delta}_{\kappa-1/2}(v)$.

The proof of construction in Theorem 1 is detailed in Section 6. It is based on the Bäcklund transforms in (6.1) and (6.5), which we use to generate microscopic conservation laws. In order to obtain the conservation laws of the form (4.1) and (4.5), we perform lengthy algebraic and combinatorial computations. For the convergence, we use a perturbative viewpoint in both regimes, by introducing the perturbation operators $\mathcal{Q}_\delta$ and $\widetilde{\mathcal{Q}}_\delta$:

$$\mathcal{Q}_\delta = \mathcal{G}_\delta \partial_x - \mathcal{H}\partial_x \quad \text{and} \quad \widetilde{\mathcal{Q}}_\delta = \widetilde{\mathcal{G}}_\delta \partial_x + \frac{1}{3}\partial_x^2, \tag{4.6}$$

which decay to 0 at different speeds, when $\delta \to \infty$ and $\delta \to 0$, respectively. The nicer convergence in (2.1) translates into a "nicer" operator $\mathcal{Q}_\delta$, while the analysis of $\widetilde{\mathcal{Q}}_\delta$ in the shallow-water regime is more delicate, as it has a slower convergence rate which is not uniform in frequency.

5. Generalized Gibbs measures for ILW

In this section, we state our main results on the statistical study of (ILW):

- construction and convergence of generalized Gibbs measures associated with the conservation laws $E^{\delta}_{k/2}(u)$ and $\widetilde{E}^{\delta}_{k/2}(v)$ in (4.1) and (4.5), respectively (Theorem 2),

- invariance of these generalized Gibbs measures and deep-water and shallow-water convergence of the invariant dynamics (Theorem 3).

We first consider the deep-water regime. Given $0 < \delta \le \infty$ and $k \in \mathbb{N}$, let $E^{\delta}_{k/2}(u)$ be as in (4.1) (and in (4.4) for $\delta = \infty$). Our aim is to construct the generalized Gibbs measure $\rho^{\delta}_{k/2}$, formally defined by the density:

$$\rho^{\delta}_{k/2}(\mathrm{d}u) \text{ “=” } Z^{-1}_{\delta,k/2} \exp\big(-E^{\delta}_{k/2}(u)\big)\mathrm{d}u,$$

as a probability measure on periodic functions / distributions. There are several issues with the expression above. The first issue is that there is no infinite-dimensional Lebesgue measure "*du*". In view of the decomposition of the energy $E^{\delta}_{k/2}(u)$ in (4.1) into the (sign-definite) quadratic-in-u part and the interaction potential part $R^{\delta}_{k/2}(u)$, we first aim to construct $\rho^{\delta}_{k/2}$ as a *weighted Gaussian measure* of the following form:

$$\rho^{\delta}_{k/2}(\mathrm{d}u) = Z^{-1}_{\delta,k/2} \exp\big(-R^{\delta}_{k/2}(u)\big)\mathrm{d}\mu^{\delta}_{k/2}(u). \tag{5.1}$$

Here, $\mu^{\delta}_{k/2}$ is the base Gaussian measure defined as the induced probability measure under the map:

$$\omega \in \Omega \quad \longmapsto \quad X^{\delta}_{k/2}(\omega) = \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{g_n(\omega)}{(T_{\delta,k/2}(n))^{\frac{1}{2}}} e_n, \tag{5.2}$$

where $e_n(x) = e^{inx}$, $\{g_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is a sequence of independent standard[2] complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ conditioned that $g_{-n} = \overline{g_n}$, $n \in \mathbb{Z}\setminus\{0\}$, and

$$T_{\delta,\frac{k}{2}}(n) = \sum_{\substack{\ell=0\\ \text{even}}}^{k} a_{k,\ell}|n|^k|\widehat{\mathcal{G}_\delta}(n)|^{k-\ell}. \tag{5.3}$$

Note that $T_{\delta,k/2}(n)$ determines the variances of the Fourier coefficients of the random variable $X^\delta_{k/2}$, and it is well defined thanks to the non-negativity condition on the constants $a_{k,\ell}$ in (4.1)! Moreover, in view of (4.3), one can easily show that $\mu^\delta_{k/2}$ is supported on $H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T}) \setminus H^{\frac{k-1}{2}}(\mathbb{T})$. When $\delta = \infty$, the base Gaussian measure $\mu^\infty_{k/2}$ is induced by the map $\omega \mapsto X^\infty_{k/2}(\omega)$ in (5.2) with $T_{\infty,k/2}(n) = \lim_{\delta\to\infty} T_{\delta,\frac{k}{2}}(n) = |n|^k$.

Now that we gave a meaning of the Gaussian measure $\mu^\delta_{k/2}$ in (5.1), let us discuss the second issue, which comes from the fact that the interaction potential $R^\delta_{k/2}(u)$ in (5.1) is not sign-definite, exhibiting super-Gaussian growth. As a result, the measure in (5.1), as it is written, cannot be normalized to be a probability measure. In order to overcome this issue, we follow the works [7, 30, 41] and consider the following weighted Gaussian measure an L^2-cutoff:

$$\rho^\delta_{k/2}(\mathrm{d}u) = Z^{-1}_{\delta,k/2}\eta(\|u\|_{L^2}/K)\exp\big(-R^\delta_{k/2}(u)\big)\mathrm{d}\mu^\delta_{k/2}(u), \tag{5.4}$$

where $\eta : [0,\infty) \to [0,1]$ is a smooth cutoff supported on $[0,2]$ such that $\eta \equiv 1$ on $[0,1]$.

A similar construction applies to the shallow-water regime, where we consider the generalized Gibbs measures (with an L^2-cutoff) of the form:

$$\widetilde{\rho}^\delta_{k/2}(\mathrm{d}v) = \widetilde{Z}^{-1}_{\delta,k}\eta(\|v\|_{L^2}/K)\exp\big(-\widetilde{R}^\delta_{k/2}(v)\big)\mathrm{d}\widetilde{\mu}^\delta_{k/2}(v), \tag{5.5}$$

where $\widetilde{\mu}^\delta_{k/2}$ is the base Gaussian measure associated with the quadratic part $\widetilde{Q}^\delta_{k/2}(v)$ in (4.5).

The following theorem summarizes the construction and convergence properties of the generalized Gibbs measures, where a stark contrast appears between the modes of convergence in the two regimes.

Theorem 2.

(i) (deep-water regime) *Given* $0 < \delta \le \infty$ *and* $K > 0$, *there exists a family of probability measures* $\{\rho^\delta_{k/2}\}_{k\ge2}$ *as in* (5.4), *which satisfy:*

  (i.a) *for* $0 < \delta < \infty$, $\rho^\delta_{k/2}$ *and* $\rho^\infty_{k/2}$ *are* equivalent.

  (i.b) $\rho^\delta_{k/2}$ *converges to* $\rho^\infty_{k/2}$ in total variation *as* $\delta \to \infty$.

(ii) (shallow-water regime) *Given* $0 < \delta < \infty$, $\kappa \ge 1$, *and* $K > 0$, *there exists a family of probability measures* $\{\widetilde{\rho}^\delta_{k/2}\}_{k\ge2}$ *as in* (5.4) *and* $\{\widetilde{\rho}^0_\kappa\}_{\kappa\ge1}$ *when* $\delta = 0$, *which satisfy:*

  (ii.a) *for* $0 < \delta < \infty$, $\widetilde{\rho}^\delta_{k/2}$ *and* $\widetilde{\rho}^0_{\lceil k/2\rceil}$ *are* mutually singular. *Here,* $\lceil x\rceil$ *denotes the smallest integer* n *such that* $n \ge x$.

  (ii.b) $\widetilde{\rho}^\delta_{k/2}$ *converges* weakly *to* $\widetilde{\rho}^0_{\lceil k/2\rceil}$ *as* $\delta \to 0$.

This result extends the work in [32] from the Gibbs measure (corresponding to the $k = 1$ case) to the generalized Gibbs measures. Again, we note the singularity of the shallow-water limit; the generalized Gibbs measures are mutually singular (compared to the equivalence in the deep-water regime), which results in a weaker mode of convergence (weak convergence as opposed to the stronger convergence in total variation). Moreover, we observe a novel 2-to-1 collapse as $\delta \to 0$, since each pair of (sILW) measures converges to a single (KdV) measure, analogous to the 2-to-1 collapse for the conservation laws stated in Theorem 1.

We note that Theorem 2 includes the construction of the generalized Gibbs measures for (BO) and (KdV), previously studied in [47, 51]. We prove Theorem 2 by using the Boué–Dupuis variational formula [2, 5] (see (7.3) below) which allows us to significantly simplify the construction

[2]By convention, we assume that g_n has mean 0 and variance 2, $n \in \mathbb{Z}\setminus\{0\}$.

argument in [47]; in particular, our construction only employs an L^2-cutoff, while the construction in [47] required cutoffs for all the lower order conservation laws. As a consequence, the proof of invariance stated in Theorem 3 also becomes simpler.

Our last main contribution is a dynamical one. In particular, we establish invariance of the generalized Gibbs measures under the corresponding dynamics.

Theorem 3.

(i) (deep-water regime) *Let $k \geq 3$ and $0 < \delta \leq \infty$. Then, the measure $\rho^\delta_{k/2}$ in* (5.4) *is invariant under* (ILW) *(under* (BO) *when $\delta = \infty$, respectively). Moreover, for the unique solution u^δ to* (ILW) *with* $\text{Law}(u^\delta(t)) = \rho^\delta_{k/2}$, $t \in \mathbb{R}$, *we have*

$$\lim_{\delta \to \infty} u^\delta = u^\infty \quad \text{in } C(\mathbb{R}; H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T})),$$

where u^∞ denotes the unique solution to (BO) *with* $\text{Law}(u^\infty(t)) = \rho^\infty_{k/2}$, $t \in \mathbb{R}$.

(ii) (shallow-water regime) *Let $k \geq 3$ and $0 \leq \delta < \infty$. Then, the measure $\widetilde{\rho}^\delta_{k/2}$ in* (5.5) *is invariant under* (sILW) *(under* (KdV) *when $\delta = 0$, respectively). Moreover, for the unique solution v^δ to* (sILW) *with* $\text{Law}(v^\delta(t)) = \widetilde{\rho}^\delta_{k/2}$, $t \in \mathbb{R}$, *we have*

$$\lim_{\delta \to 0} v^\delta = v^0 \quad \text{in } C(\mathbb{R}; H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T})),$$

where v^0 denotes the unique solution to (KdV) *with* $\text{Law}(v^0(t)) = \widetilde{\rho}^0_{\lceil k/2 \rceil}$, $t \in \mathbb{R}$.

Theorem 3 establishes the first statistical convergence of invariant dynamics for (ILW) and (sILW) (with uniqueness). Note that in [32] for Gibbs invariant dynamics, uniqueness is missing due to a compactness argument. The 2-to-1 collapse observed in Theorem 2 for the generalized Gibbs measures is extended to Theorem 3 for the invariant dynamics in the shallow-water limit.

Recalling that $\operatorname{supp} \rho^\delta_{k/2} \subset H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T})$, Theorem 3 establishes invariance of generalized Gibbs measures supported at high regularity. In particular, increasing k captures statistical information on smoother functions. The restriction to $k \geq 3$ comes from the regularity $s \geq 1/2$ for the deterministic global well-posedness of (ILW) in $H^s(\mathbb{T})$ due to Molinet–Vento in [37] (without relying on the gauge transform). Note that this regularity restriction just misses the support of the measure ρ^δ_1 (for $k = 2$). We expect invariance to also hold for the $k = 2$ case by using the recent development in well-posedness theory of (ILW) in [12] and adapting the argument in [15] for the (BO) case.

We emphasize that the main technical challenge in Theorems 1, 2, and 3 comes from constructing higher conservation laws in the "correct" form, which we detail in Section 6. In Sections 7 and 8, we provide some details on the main tools needed for Theorems 2–3.

Remark 4. In this note, we restrict our attention to the periodic case. In a recent work [10] with B. Harrop-Griffiths, we investigated the singular nature of the shallow-water convergence of (sILW) to (KdV) on the real line. Recall from [6, 36] that the solution map for (sILW) is not C^2 for any $0 < \delta < \infty$, while the solution map for (KdV) is analytic. In [10], by decomposing the dynamics into the low frequency part and the residual part, we showed that, when the depth parameter is sufficiently small, the solution map for the low frequency part is analytic in $L^2(\mathbb{R})$, while the solution map for the residual part fails to be C^2. Moreover, we established shallow-water convergence in $L^2(\mathbb{R})$ of the low frequency dynamics to (KdV), which explains the mechanism on how analytic regularity of the solution map propagates from low frequencies to the entire frequencies in the shallow-water limit. See [10] for a further discussion.

6. Skecth of the construction of the conservation laws

In the deep-water regime, the construction of the deep-water conservation laws $\{E^\delta_{k/2}(u)\}_{k \in \mathbb{Z}_{\geq 0}}$ for (ILW) is based on the following Bäcklund transform in [44]:

$$\begin{aligned} 2u &= \mu(e^U - 1) + (\mathcal{G}_\delta - i)\partial_x U + \delta^{-1} U, \\ \partial_t U &= \mu(e^U - 1)\partial_x U + \mathcal{G}_\delta \partial_x^2 U + \partial_x U \cdot \mathcal{G}_\delta \partial_x U + \delta^{-1} U \partial_x U, \end{aligned} \tag{6.1}$$

where $\mu \in \mathbb{R}$. Taking u to be a smooth solution to (ILW) and U of the form

$$U = \sum_{n=1}^{\infty} \mu^{-n} \chi_n^\delta,$$

such that (u, U) satisfy (6.1), then it follows from the second equation in (6.1) that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} U \mathrm{d}x = \sum_{n=1}^{\infty} \mu^{-n} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} \chi_n^\delta \mathrm{d}x,$$

i.e. U and $\{\chi_n^\delta\}_{n\in\mathbb{N}}$ are conservation laws. By replacing the expression for U in the first equation in (6.1) and collecting powers of μ^{-n}, we then obtain a recursive relation for the microscopic conservation laws χ_n^δ for (ILW):

$$\begin{aligned} \chi_1^\delta &= 2u, \\ \chi_n^\delta &= -\sum_{j=2}^{n} \frac{1}{j!} \sum_{\substack{n_1,\dots,n_j \in \mathbb{N} \\ n_1+\cdots+n_j=n}} \chi_{n_1}^\delta \cdots \chi_{n_j}^\delta - \left((\mathcal{G}_\delta - i)\partial_x + \frac{1}{\delta} \right) \chi_{n-1}^\delta, \quad n \geq 2. \end{aligned} \tag{6.2}$$

From the convergence in (2.1), one also obtains the Bäcklund transform for (BO) from (6.1) and the analogous recursive relations by taking $\delta \to \infty$.

To illustrate the main difficulties in going from the microscopic conservation laws χ_n^δ in (6.2) to the conservation laws $E_{k/2}^\delta(u)$ in (4.1), let us consider χ_4^δ:

$$\int \chi_4^\delta dx \sim 2\|\mathcal{G}_\delta u\|_{\dot{H}^1}^2 + 6\|u\|_{\dot{H}^1}^2 + 12\delta^{-1}\|\mathcal{G}_\delta^{1/2} u\|_{\dot{H}^{1/2}} + 12\delta^{-2}\|u\|_{L^2}^2 + \text{l.o.t.}$$

We make two observations on the quadratic-in-u terms (which also apply to χ_n^δ with $n \geq 4$):

(1) There are terms of different regularities (H^s-regularity with $s = 0, \frac{1}{2}, 1$).

(2) There are multiple terms at the highest level of regularity ($\|\mathcal{G}_\delta u\|_{\dot{H}^1}^2$ and $\|u\|_{\dot{H}^1}^2$).

In order to define the generalized Gibbs measure $\mu_{k/2}^\delta$ in (5.4) as a weighted Gaussian measure, we want the quadratic part of the conservation law to consist of terms at a single regularity with positive coefficients (to guarantee sign-definiteness of the quadratic part) as in $Q_{k/2}^\delta(u)$ appearing in (4.1), which is essential in defining the random variable $X_{k/2}^\delta$ in (5.2) and the base Gaussian measure $\mu_{k/2}^\delta$ in (5.1) via the relation (5.3).

We tackle issues (1)–(2) by explicitly calculating the linear and quadratic (in u) parts of the microscopic conservation laws χ_n^δ in (6.2). This then allows us to recursively define $E_{k/2}^\delta(u)$ in (4.1):

$$E_{k/2}^\delta(u) = \frac{(-1)^{k+1}}{4b_k} \left[\operatorname{Re} \int_{\mathbb{T}} \chi_{k+2}^\delta dx - 4(-1)^{k+1} \sum_{j=0}^{k-1} \frac{1}{\delta^{k-j}} \binom{k+2}{j+2} b_j E_{j/2}^\delta(u) \right], \tag{6.3}$$

where $b_k = \sum_{\substack{\ell=0 \\ \text{even}}}^{k} \binom{k+1}{\ell+1}$. See [11, Proposition 3.8] for further details. In particular, (6.3) defines polynomial conservation laws for (ILW) of the form (4.1), suitable for constructing the generalized Gibbs measures in the deep-water regime.

In order to organize the combinatorially complex structure of the monomials appearing in the deep-water conservation laws, we introduce the notion of the *deep-water rank.* Given a monomial $p(u)$ (depending on u and its derivatives, $\mathcal{G}_\delta$, and δ), its deep-water rank is defined by

$$\operatorname{rank}(p) = \#u + \#\partial_x + \#\delta^{-1}, \tag{6.4}$$

where $\#u$ denotes the homogeneity in u, $\#\partial_x$ the number of explicit instances of ∂_x, and $\#\delta^{-1}$ that of δ^{-1}. In fact, it is not difficult to see that each such monomial $p(u)$ in χ_n^δ and $E_{k/2}^\delta(u)$ have a constant rank of n and $k+2$, respectively. Moreover, each instance of $\mathcal{G}_\delta$ always appears with ∂_x due to (6.2).

This information suffices to describe all the possible terms appearing in the interaction potentials in $R_{k/2}^\delta(u)$ in (4.1).

Lastly, in order to show the deep-water convergence of these conservation laws claimed in Theorem 1, we combine the notion of the deep-water rank and a perturbative approach based on the convergence of the dispersive operator $\mathcal{G}_\delta$ in (2.1). Following the recent works [9, 12, 20], we replace $\mathcal{G}_\delta\partial_x$ appearing in the deep-water conservation laws by

$$\mathcal{G}_\delta\partial_x = \mathcal{H}\partial_x + (\mathcal{G}_\delta\partial_x - \mathcal{H}\partial_x) =: \mathcal{H}\partial_x + \mathcal{Q}_\delta,$$

where the perturbative operator $\mathcal{Q}_\delta$ defined in (4.6) has a strong smoothing property and rapid convergence to 0 as $\delta \to \infty$. Note that after this substitution, the terms in $E^\delta_{k/2}(u)$ with no dependence on δ, namely no explicit factors of δ or $\mathcal{Q}_\delta$ operators, correspond to the terms in the limiting conservation law $E^\infty_{k/2}(u)$ for (BO).

We now turn our attention to deriving the conservation laws $\widetilde{E}^\delta_{k/2}(v)$ for (sILW) in the shallow-water regime. Recalling the scaling in (2.2) and (sILW), we could naively define $\widetilde{E}^\delta_{k/2}(v) = \delta^{-\alpha}E^\delta_{k/2}(\delta v)$, also replacing $\mathcal{G}_\delta$ by $\delta\widetilde{\mathcal{G}}_\delta$, for a suitable choice of $\alpha \in \mathbb{N}$. It turns out that such conservation laws are not suitable in the shallow-water regime due to multiple divergent terms involving powers of δ^{-1}. Instead, we need to rely on the Bäcklund transform for (sILW) derived in [18, 28]:

$$\begin{aligned} v &= \frac{1}{2\delta^2}\big\{2i\mu\delta V - (1 - i\mu^{-1}\delta)(e^{2i\mu\delta V} - 1)\big\} + \mu\partial_x V + i\mu\delta\widetilde{\mathcal{G}}_\delta\partial_x V,\\ \partial_t V &= \frac{1}{\delta^2}\big\{2i\mu\delta V - (1 - i\mu^{-1}\delta)(e^{2i\mu\delta V} - 1)\big\}\partial_x V + \widetilde{\mathcal{G}}_\delta\partial_x^2 V + 2i\mu\delta\partial_x V\cdot\widetilde{\mathcal{G}}_\delta\partial_x V, \end{aligned} \tag{6.5}$$

with $\mu \in \mathbb{R}$, and repeat the earlier process for the deep-water regime. In particular, writing $V = \sum_{n=0}^\infty \mu^n h^\delta_n$, we obtain a sequence of microscopic conservation laws $\{h^\delta_n\}_{n\in\mathbb{Z}_{\geq 0}}$ recursively given by

$$\begin{aligned} h^\delta_0 &= -v,\\ h^\delta_n &= -\frac{1}{2\delta^2}\sum_{j=2}^{n}\frac{(2i\delta)^j}{j!}\sum_{\substack{n_1,\dots,n_j\in\mathbb{Z}_{\geq 0}\\ n_1+\cdots+n_j=n-j}} h^\delta_{n_1}\cdots h^\delta_{n_j}\\ &\quad + \frac{i}{2\delta}\sum_{j=2}^{n+1}\frac{(2i\delta)^j}{j!}\sum_{\substack{n_1,\dots,n_j\in\mathbb{Z}_{\geq 0}\\ n_1+\cdots+n_j=n+1-j}} h^\delta_{n_1}\cdots h^\delta_{n_j} + (1 + i\delta\widetilde{\mathcal{G}}_\delta)\partial_x h^\delta_{n-1}, \quad n \in \mathbb{N}. \end{aligned}$$

As in the deep-water regime, the quadratic-in-u terms in h^δ_n have mixed regularities and multiple contributions of the highest regularity (for n sufficiently large). Additionally, there is a novel difficulty; half of the macroscopic conservation laws vanish in the limit:

$$\lim_{\delta\to 0}\int_{\mathbb{T}} h^\delta_{2n+1}\mathrm{d}x = 0.$$

This reflects the fact that (KdV) only has half as many conservation laws as (sILW), Namely, $\int h^0_{2n+1}dx = 0$ for any $n \in \mathbb{N}$, where h^0_n denotes the microscopic conservation laws obtained from the Bäcklund transform for (KdV), corresponding to the case $\delta = 0$. In order to define shallow-water conservation laws $\widetilde{E}^\delta_{k/2}(v)$ with non-trivial shallow-water limits for all k, we introduce new microscopic conservation laws $\widetilde{h}^\delta_n$:

$$\widetilde{h}^\delta_n = \frac{(-i\delta)^{\mathfrak{p}(n)}}{\delta^2}\sum_{j=1}^{n}\frac{1}{(i\delta)^{n-j}}h^\delta_j, \tag{6.6}$$

where $\mathfrak{p}(n) = 0$ for n even, and 1 otherwise. We then carefully analyze the linear and quadratic terms (in v) of $\widetilde{h}^\delta_n$ to define $\widetilde{E}^\delta_{k/2}(v)$ as a suitable linear combination as follows:

$$\widetilde{E}^\delta_{\kappa-\frac{1}{2}}(v) = (-1)^{\kappa+1}\frac{3}{4\kappa}\operatorname{Re}\int \widetilde{h}^\delta_{2\kappa}\mathrm{d}x \qquad \text{and} \qquad \widetilde{E}^\delta_{\kappa}(v) = (-1)^{\kappa+1}\frac{1}{2}\operatorname{Re}\int \widetilde{h}^\delta_{2\kappa+1}\mathrm{d}x$$

for $\kappa \in \mathbb{N}$, which we can check to be in the form (4.5). See [11, Proposition 4.10] for details.

In order to obtain a detailed description of $\widetilde{E}^\delta_{k/2}(v)$, we introduce an analogous *shallow-water rank*. We say that a monomial $p(v)$, depending on v, ∂_x, $\widetilde{\mathcal{G}}_\delta$, and δ, has the shallow-water rank

rank(p) given by

$$\text{rank}(p) = \#v + \frac{1}{2}(\#\partial_x + \#\widetilde{\mathcal{G}}_\delta - \#\delta).$$

Compare this with the deep-water counterpart in (6.4), which does not account for the operators $\mathcal{G}_\delta$ and is much simpler. With this definition of the shallow-water rank, one can then show that all monomials $p(v)$ appearing in $\widetilde{E}^\delta_{\kappa-1/2}(v)$ and $\widetilde{E}^\delta_\kappa(v)$ have a constant rank $\kappa+2$. Moreover, we have

$$\begin{aligned} &0 \le \#\widetilde{\mathcal{G}}_\delta \le \min(\#\partial_x, \#\delta+\alpha), \qquad \#v + \#\partial_x \le 2\kappa + 2 - \alpha \\ &0 \le \#\delta \le 2\kappa - \alpha, \end{aligned}$$

where $\alpha = 1$ for $\widetilde{E}^\delta_{\kappa-1/2}(v)$ and $\alpha = 0$ for $\widetilde{E}^\delta_\kappa(v)$. This allows us to clearly describe all terms appearing in these shallow-water conservation laws.

In establishing the shallow-water convergence of these conservation laws claimed in Theorem 1, we proceed perturbatively as in the deep-water regime We first isolate the δ-free contributions, which survive in the limit, by collecting the terms with no explicit powers of δ, and replacing $\widetilde{\mathcal{G}}_\delta$ by $\widetilde{\mathcal{G}}_\delta = -\frac{1}{3}\partial_x + (\widetilde{\mathcal{G}}_\delta + \frac{1}{3}\partial_x) = -\frac{1}{3}\partial_x + \widetilde{\mathcal{Q}}_\delta$, where $\widetilde{\mathcal{Q}}_\delta$ is as in (4.6). For $\widetilde{E}^\delta_\kappa(v)$, the contributions with no explicit powers of δ and $\widetilde{\mathcal{G}}_\delta$ terms already agree with the conservation law $\widetilde{E}^0_\kappa(v)$ for (KdV). For the κ-indexed conservation law $\widetilde{E}^\delta_\kappa(v)$, this agreement with $\widetilde{E}^0_\kappa(v)$ goes back to the Bäcklund tranform in (6.5) and its limit as $\delta \to 0$, while for the $(\kappa-1/2)$-indexed conservation law $\widetilde{E}^\delta_{\kappa-1/2}(v)$, this agreement requires a more careful argument based on the uniqueness of the H^κ-conservation law for (KdV). It remains to show the remaining terms in the conservation laws tend to 0 in the shallow-water limit ($\delta \to 0$). As compared to the deep-water limit, the weaker convergence of $\widetilde{\mathcal{Q}}_\delta$, not being uniform in frequency, requires more careful analysis. In fact, for contributions with "insufficient" powers of δ, we consider two frequency regimes $|n| \le \delta^{-\frac{1}{2}}$ and $|n| > \delta^{-\frac{1}{2}}$, where we can gain extra powers of δ in the former and argue by the dominated convergence theorem in the latter (which disappears in the limit $\delta \to 0$).

7. Sketch of the construction of the generalized Gibbs measures

In this section, we give a sketch of the proof of Theorem 2 on the construction and convergence properties of the generalized Gibbs measures. Since the same argument applies to both the deep- and shallow-water regimes, we only focus on the former.

We construct $\rho^\delta_{k/2}$ in (5.4) as a (unique) limit of its frequency-truncated version $\rho^\delta_{k/2,N}$ with density:

$$\begin{aligned} \rho^\delta_{k/2,N}(\mathrm{d}u) &= Z^{-1}_{\delta,k/2,N}\, \eta\big(\|\mathbf{P}_N u\|_{L^2}/K\big) \exp\Big(-R^\delta_{k/2}(\mathbf{P}_N u)\Big)\mathrm{d}\mu^\delta_{k/2}(u) \\ &=: Z^{-1}_{\delta,k/2,N}\, F^\delta_{k/2}(\mathbf{P}_N u)\mathrm{d}\mu^\delta_{k/2}(u), \end{aligned} \tag{7.1}$$

where $\mathbf{P}_N$ denotes the Dirichlet projector onto spatial frequencies $\{|n| \le N\}$. In particular, the crucial ingredient in constructing $\rho^\delta_{k/2}$ for each fixed $0 < \delta \le \infty$ and establishing its deep-water convergence ($\delta \to \infty$) is the following uniform (in N and δ) bounds on the truncated density $F^\delta_{k/2}(\mathbf{P}_N u)$:

$$\sup_{N\in\mathbb{N}} \sup_{0<\delta\le\infty} \big\|F^\delta_{k/2}(\mathbf{P}_N u)\big\|^p_{L^p(d\mu^\delta_{k/2})} < \infty. \tag{7.2}$$

With the current formulation, the base Gaussian measures $\mu^\delta_{k/2}$, appearing in (7.2), are different for distinct values of δ. This initial difficulty can be overcome by working with the random variables $X^\delta_{k/2}$ in (5.2) and instead establish bounds on $F^\delta_{k/2}(\mathbf{P}_N X^\delta_{k/2})$ in $L^p(\Omega)$.

The main tool for showing (7.2) is the Boué–Dupuis variational formula [5, 50], recently popularized by Barashkov–Gubinelli [2]. In our simplified version of this formulation, the L^p-bounds in (7.2) follow once we show

$$\sup_{\Xi} \mathbb{E}\left[-pR^\delta_{k/2}(\mathbf{P}_N X^\delta_{k/2} + \mathbf{P}_N\Xi) - \frac{1}{2}\|\Xi\|^2_{H^{k/2}} - pA\|\mathbf{P}_N X^\delta_{k/2} + \mathbf{P}_N\Xi\|^\alpha_{L^2}\right] < \infty, \tag{7.3}$$

for $A, \alpha \gg 1$ sufficiently large and any $1 \le p < \infty$, uniformly in N and δ. Heuristically, the Boué–Dupuis formula allows us to consider boundedness of the potential $R^\delta_{k/2}$ instead of its exponential (as in (7.2)), at the cost of a variational problem involving a smoother drift $\Xi \in H^{k/2}(\mathbb{T})$. We note that the variational formula (7.3) comes with the helpful second term in (7.3), while the last term in (7.3) comes from the taming introduced by the L^2-cutoff in (7.1). In order to establish these bounds, it is essential to have the detailed understanding of the interaction potentials $R^\delta_{k/2}(u)$ obtained from the careful analysis of the conservation laws $E^\delta_{k/2}(u)$.

Once we have the uniform bounds in (7.2), a standard argument allows us to construct $\rho^\delta_{k/2}$ as the (unique) limit in total variation of $\rho^\delta_{k/2,N}$ as $N \to \infty$ for each fixed $0 < \delta \le \infty$, while the deep-water convergence of the measures follows from the decomposition

$$d(\rho^\delta_{k/2}, \rho^\infty_{k/2}) = d(\rho^\delta_{k/2}, \rho^\delta_{k/2,N}) + d(\rho^\delta_{k/2,N}, \rho^\infty_{k/2,N}) + d(\rho^\infty_{k/2}, \rho^\infty_{k/2,N}),$$

for a suitable distance d, together with the uniform (in N and δ) $L^p(\Omega)$-bounds (7.2). Moreover, the equivalence of $\rho^\delta_{k/2}$ and $\rho^\infty_{k/2}$ follows from that of the base Gaussian measures $\mu^\delta_{k/2}$ and $\mu^\infty_{k/2}$, which is due to Kakutani's theorem (see [22]).

In the shallow-water regime, we can proceed in an analogous manner to construct the generalized Gibbs measure $\widetilde{\rho}^\delta_{k/2}$ and establish its shallow-water convergence. We must, however, account for the different algebraic structure of the conservation laws $\widetilde{E}^\delta_{k/2}(v)$ in showing the analogue of (7.2). Moreover, the singularity of $\widetilde{\rho}^\delta_{k/2}$ and $\widetilde{\rho}^0_{\lceil k/2 \rceil}$ follows from that of the underlying Gaussian measures and Kakutani's theorem.

8. Ideas for invariance of the generalized Gibbs measures

Lastly, we briefly discuss the ideas behind the proof of invariance of $\rho^\delta_{k/2}$ under (ILW) in the deep-water regime claimed in Theorem 3. We omit modifications required for the shallow-water generalized Gibbs measure $\widetilde{\rho}^\delta_{k/2}$ since the strategy is analogous.

We first consider the following truncated dynamics:

$$\partial_t u_N - \mathcal{G}_\delta \partial_x^2 u_N = \mathbf{P}_N \partial_x \big((\mathbf{P}_N u_N)^2 \big). \tag{8.1}$$

The truncated dynamics (8.1) is Hamiltonian with the Hamiltonian $E^\delta_{1/2}(u_N)$. As a result, $E^\delta_{1/2}(u_N)$ is conserved under the truncated dynamics. For $k \ge 2$, however, the truncated conservation law $E^\delta_{k/2}(u_N)$ is no longer conserved for the truncated dynamics (8.1), which leads to non-invariance of the corresponding truncated generalized Gibbs measure. This is a common difficulty in establishing invariance of a generalized Gibbs measure associated with a higher order conservation law; see, for example, [39].

We overcome this difficulty by following the strategy introduced by Tzvetkov–Visciglia in [48, 49]. Namely, we prove *probabilistic asymptotic conservation* of the truncated energy $E^\delta_{k/2}(\mathbf{P}_N u)$ associated with the truncated generalized Gibbs measure $\rho^\delta_{k/2,N}$ in (7.1). Following the argument in [48, 49], we can reduce the issue to what happens at time $t = 0$, where the initial data is distributed according to the base Gaussian measure. In particular, we focus on showing the following probabilistic asymptotic conservation of the truncated energy $E^\delta_{k/2}(\mathbf{P}_N u)$:

$$\lim_{N\to\infty} \bigg\| \frac{\mathrm{d}}{\mathrm{d}t} E^\delta_{k/2}\big(\mathbf{P}_N u_N(t)\big)\Big|_{t=0} \bigg\|_{L^p(d\mu^\delta_{k/2})} = 0 \tag{8.2}$$

for any $0 < \delta \le \infty$ and $1 \le p < \infty$. Together with a change-of-variable formula, the probabilistic asymptotic conservation (8.2) allows us to establish

$$\lim_{N\to\infty} \sup_{t\in[0,T],A} \bigg| \frac{\mathrm{d}}{\mathrm{d}t} \rho^\delta_{k/2,N}\big(\Phi_N(-t)A\big) \bigg| = \lim_{N\to\infty} \sup_{t\in[0,T],A} \bigg| \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Phi_N(t)A} F^\delta_{k/2}(\mathbf{P}_N u) \mathrm{d}\mu^\delta_{k/2}(u) \bigg| = 0$$

for any $T > 0$ and any measurable set $A \subset H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T})$, where $\Phi_N(t)$ denotes the data-to-solution map of the truncated dynamics (8.1). Then, the desired invariance of $\rho^\delta_{k/2}$ follows from a simple argument.

As in the previous parts, the detailed description of the conservation laws $E^\delta_{k/2}(u)$ is essential in establishing the limit (8.2), where we use a perturbative approach, replacing $\mathcal{G}_\delta \partial_x = \mathcal{H}\partial_x + \mathcal{Q}_\delta$ (note that δ is fixed for this argument). In addition to the description in Section 6, we need more explicit form for cubic and quartic (in u) terms, where the precise location of the Hilbert transform $\mathcal{H}$ plays a crucial role, inducing certain cancellations due to its anti self-adjointness. We also emphasize that since $\rho^{\delta}_{k/2,N}$ only depends on an L^2-cutoff and not on other higher order conservation laws (which are not conserved for the truncated dynamics), the study of the transported truncated measures is simplified as compared to that in [48, 49] for the (BO) case where the time derivatives of other conservation laws also needed to be studied.

Lastly, we note that the dynamical convergence in Theorem 3 follows from the following ingredients:

(1) weak convergence of the measures $\rho^{\delta}_{k/2}$ to $\rho^{\infty}_{k/2}$;

(2) Skorokhod representation theorem along a continuous parameter, which guarantees the existence of random functions $\{u^{\delta}_0\}_{2\le\delta\le\infty} \subset H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T})$ such that $\mathrm{Law}(u^{\delta}_0) = \rho^{\delta}_{k/2}$ for each $2 \le \delta \le \infty$ and u^{δ}_0 converges a.s. to u^{∞}_0 in $H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T})$ as $\delta \to \infty$;

(3) deterministic global well-posedness and convergence in [31, 37] of the (ILW) solution u^{δ} to the unique (BO) solution u^{∞} in $C(\mathbb{R}; H^{\frac{k-1}{2}-\varepsilon}(\mathbb{T}))$.

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