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Unique and nonunique minimizing travelling waves for some Ginzburg–Landau type equations

Eliot Pacherie

Unicité et non unicité d'ondes progressives minimisantes pour des équations de type Ginzburg–Landau

Résumé

Ceci est un résumé d'un travail à venir en collaboration avec Mihai Maris. On étudie l'équation de Schrödinger avec une condition non triviale à l'infini pour des nonlinéarités génériques, où l'éxistence d'ondes progressives a été prouvée dans [22] comme solutions d'un problème de minimisation. On démontre que pour certaines d'entre elles, il n'y a pas unicité du minimiseur, et on étudie comment ses ondes progressives évoluent quand on modifie la vitesse et/ou la nonlinéarité. On discutera aussi d'exemples où il y a unicité du minimiseur, ainsi que des questions autour de la stabilité de ces objets.

Abstract

This is a summary of an upcoming work in collaboration with Mihai Maris. We study the Schrödinger equation with a nonzero condition at infinity for some nonlinearities, for which the existence of travelling waves has been proven in [22] as solutions of a minimization problem. We show that for some of them, we do not have uniqueness of this minimizer, and we study how these travelling changes when we change the speed and/or the nonlinearity. We also discuss some examples where we have the uniqueness of travelling waves that are solutions to minimisation problems, as well as questions related to the stability of these objects.

1. Introduction and presentation of the results

In this proceeding, we are interested in the study of the Ginzburg–Landau type equation

$$\begin{cases} i\partial_t u - \Delta u - F(|u|^2)u = 0 \\ |u| \to 1 \text{ when } |x| \to +\infty \end{cases}$$

on the space $\mathbb{R}^N, N \geqslant 1$ with F smooth, $F(1) = 0, F'(1) < 0$, and u is complexed valued.

The condition $F(1) = 0$ is taken such that $u = e^{i\gamma}, \gamma \in \mathbb{R}$ corresponds to the trivial solutions of the equation. The condition $F'(1) < 0$ means that this equation is defocusing, and, up to a rescaling in space and time, we can always assume that then $F'(1) = -1$.

We are interested in finding particular solutions of this equation, and try to understand their stability. We focus more precisely on constant and travelling wave solutions for this equation. We also want to discuss how this nonzero condition at infinity make these questions different from more classical nonlinear Schrödinger equations where solutions converge to 0 at infinity.

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1.1. Particular solutions for the Gross–Pitaevskii equation

The simplest case for F is to take $F(s) = 1 - s$, and this is the Gross–Pitaevskii equation. This equation appears in the description of Bose–Einstein condensate and in superfluidity as well as in superconductivity, see [1, 13, 20, 24, 25].

1.1.1. Orbital stability of the constant background

A simple yet insightful question to understand this equation is to look at the problem of the orbital stability of the constant solution 1 for the Gross–Pitaevskii equation

$$\begin{cases} i\partial_t u - \Delta u - (1-|u|^2)u = 0 \\ |u| \to 1 \text{ when } |x| \to +\infty \end{cases}$$

on $\mathbb{R}^N, N \geqslant 1$. The associated energy is

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|u|^2)^2.$$

Remark also that $E(u) \geqslant 0$ for any function u as it is a sum of two squares.

It is easy to check that the equation $i\partial_t u - \Delta u - (1-|u|^2)u = 0$ is globally well posed in the affine space $1 + H^1(\mathbb{R}^N)$. We can therefore be interested in the question of the solution 1, and as it is a minimizer of the energy (since $E(1) = 0$), we should expect it to be stable.

A classical approach to prove orbital stability would be to consider an initial data $u_0 = 1 + \varphi_0$ with $\|\varphi_0\|_{H^1(\mathbb{R}^N)} \ll 1$ and defining $u_t = 1 + \varphi_t$ the solution of the Gross–Pitaevskii equation at time $t \in \mathbb{R}$ with this initial condition, we would like to show that $\|\varphi_t\|_{H^1(\mathbb{R}^N)} \ll 1$ for all $t \in \mathbb{R}$. To do so, by conservation of the energy we compute that

$$\begin{aligned} 1 &\ll E(1+\varphi_0) \\ &= E(1+\varphi_t) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\varphi_t|^2 + \mathfrak{Re}^2(\varphi_t) + \int_{\mathbb{R}^N} \mathfrak{Re}(\varphi_t)|\varphi_t|^2 + \frac{1}{4}\int_{\mathbb{R}^2} |\varphi_t|^4 \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\varphi_t|^2 + \mathfrak{Re}^2(\varphi_t) + O(\|\varphi_t\|^2_{H^1(\mathbb{R}^N)}). \end{aligned} \tag{1.1}$$

The quadratic terms are not equivalent to $\|\varphi_t\|^2_{H^1(\mathbb{R}^N)}$ as we are missing a term of the form $\int_{\mathbb{R}^N} \mathfrak{Im}^2(\varphi_t)$. In fact, for instance if $N = 2$, the function

$$\Phi(x) = \frac{1}{\langle x\rangle^{1+\varepsilon}} + \frac{i}{\langle x\rangle^{\varepsilon}}$$

with $\langle x\rangle = \sqrt{1+|x|^2}$ and $\varepsilon > 0$ small is such that

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla\Phi|^2 + \mathfrak{Re}^2(\Phi) < +\infty$$

but

$$\int_{\mathbb{R}^2} \mathfrak{Re}(\Phi)|\Phi|^2 = +\infty.$$

That is, we can construct functions for which the quadratic term is finite, but the cubic term is infinite. This nonequivalence of norms here is a direct consequence of the background 1, and prevents us to use this classical argument to show orbital stability. In fact, we conjecture that the orbital stability of 1 does not hold in the affine space $1+H^1(\mathbb{R}^N)$ for $N = 2$. For $N \geqslant 3$, asymptotic stability has been proven in [18] by non energetic arguments.

To prove the stability of 1, we have to work in a metric space. It was shown in [12] that the Gross–Pitaevskii equation is globally well posed in the space of functions u such that $E(u) < +\infty$ and is continuous for the metric

$$d(u,v) := \int_{\mathbb{R}^N} |\nabla(u-v)|^2 + \int_{\mathbb{R}^N} (|u|^2 - |v|^2)^2$$

for $N = 2, 3$. In this metric space, we can check that there exists a universal constant $K > 0$ such that for any function u with $E(u) < +\infty$,

$$\frac{1}{K}E(u) \leqslant d(1,u) \leqslant KE(u).$$

From this and the conservation of the energy, we deduce directly that there exists $K > 0$ such that if u_t is the solution at time $t \in \mathbb{R}$ of the Gross–Pitaevskii equation with initial condition u_0, then

$$d(1,u_t) \leqslant Kd(1,u_0).$$

Remark that in fact, d is not a distance, since

$$d(1,e^{i\gamma}) = 0$$

for all $\gamma \in \mathbb{R}$. We define

$$d_E(u,v) := d(u,v) + \int_{B(0,1)} |u-v|^2$$

which is now truely a distance, and we can check the following result by modulation arguments.

Proposition 1. *There exists $\varepsilon_0, K > 0$ such that, for any initial data u_0 such that $d_E(1,u_0) \leqslant \varepsilon_0$, there exists a function*

$$t \to \gamma(t) \in C^1(\mathbb{R},\mathbb{R})$$

with $|\gamma'(t)| \leqslant Kd_E(1,u_0)$ such that, if u_t is the solution of the Gross–Pitaevskii equation for the initial data u_0 at time $t = 0$, then

$$d_E(e^{i\gamma(t)},u_t) \leqslant Kd_E(1,u_0).$$

This is the orbital stability of the set of constant solutions $\{e^{i\gamma}, \gamma \in \mathbb{R}\}$ which correspond exactly to the minimizers of the energy. The orbital stability of vortices, that are stationary nontrivial solutions of the Gross–Pitaevskii equation in dimension 2 ([19]), has been proven in adapted metric spaces, see [16]. Let us sketch the proof of this result.

Proof. Remark that

$$d_E(e^{i\gamma(t)},u_t) = d_E(1,e^{-i\gamma(t)}u_t)$$

and we choose $\gamma(t)$ by modulation arguments such that

$$\int_{B(0,1)} \mathfrak{Im}(e^{-i\gamma(t)}u_t) = 0.$$

Furthermore, $E(e^{-i\gamma(t)}u_t) = E(u_t)$ and we decompose, with

$$e(u) = \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-|u|^2)^2$$

that is such that $E(u) = \int_{\mathbb{R}^2} e(u)$ and $e(e^{-i\gamma}u) = e(u)$ for any $\gamma \in \mathbb{R}$, the energy as

$$E(e^{-i\gamma(t)}u_t) = E_1(e^{-i\gamma(t)}u_t) + E_2(u_t) := \int_{B(0,1)} e(e^{-i\gamma(t)}u_t) + \int_{\mathbb{R}^2\setminus B(0,1)} e(u_t).$$

Now, for E_1 we do the decomposition of (1.1) and we write $e^{-i\gamma(t)}u_t = 1 + \varphi_t$, leading to

$$E_1(e^{-i\gamma(t)}u_t) = \frac{1}{2}\int_{B(0,1)} |\nabla\varphi_t|^2 + \mathfrak{Re}^2(\varphi_t) + O(\|\varphi_t\|^3_{H^1(B(0,1))}).$$

By our choice of $\gamma(t)$, we have $\int_{B(0,1)} \mathfrak{Im}(\varphi_t) = 0$, and by Poincaré inequality, this implies that

$$\int_{B(0,1)} \mathfrak{Im}^2(\varphi_t) \leqslant K\int_{B(0,1)} |\nabla\varphi_t|^2$$

and thus

$$\begin{aligned} E_1(e^{-i\gamma(t)}u_t) &\geqslant \kappa\|\varphi_t\|^2_{H^1(B(0,1))} + O(\|\varphi_t\|^3_{H^1(B(0,1))}) \\ &\geqslant \frac{\kappa}{2}\|\varphi_t\|^2_{H^1(B(0,1))} \end{aligned}$$

for some universal constant $\kappa > 0$ provided that $\|\varphi_t\|_{H^1(B(0,1))}$ is small enough. Now, we check that if $\|\varphi_t\|_{H^1(B(0,1))}$ is small enough, there exists a universal constant $K > 0$ such that

$$\frac{1}{K}(\|\varphi_t\|^2_{H^1(B(0,1))} + E_2(u_t)) \leqslant d_E(1, e^{-i\gamma(t)}u_t) \leqslant K(\|\varphi_t\|^2_{H^1(B(0,1))} + E_2(u_t))$$

and we can conclude the proof from here, as the condition $\|\varphi_t\|_{H^1(B(0,1))}$ small can be propagated. □

A remark concerning this result is that we did not have to take $B(0,1)$ in the last integral of the definition of d_E, but for instance $B(x,1)$ for any $x \in \mathbb{R}^2$ would yield a similar stability results, and although the distances would be equivalents, this is not uniform in $x \in \mathbb{R}^2$. Furthermore, the choice of $t \to \gamma(t)$ is not necessarily not the same for all $x \in \mathbb{R}^2$ for the same initial condition.

1.1.2. Travelling wave solutions

In the rest of this proceeding, we will focus on the existence and stability of travelling wave solutions. We discuss here some results for the Gross–Pitaevskii equation in dimension 2.

In addition to the energy, we can define the momentum as

$$\vec{Q}(u) := \int_{\mathbb{R}^2} \mathfrak{Re}(i\nabla u\overline{u-1}).$$

Travelling waves are solutions of the equation

$$\begin{cases} i\vec{c}.\nabla u - \Delta u - (1-|u|^2)u = 0 \\ |u| \to 1 \text{ when } |x| \to +\infty. \end{cases}$$

We can assume that $\vec{c} = c\vec{e}_1$ for some $c > 0$ as the equation is invariant by rotation. This next result concerns the existence of travelling waves for $c > 0$ small by minimizing the energy for large values of the momentum $Q_1 := \vec{Q}.\vec{e}_1$, and the fact that this minimizer is unique.

Theorem 2 ([5, 9])**.** *There exists $c_0 > 0$ and a family of functions*

$$c \to v_c \in C^1(]0, c_0], L^\infty(\mathbb{R}^2, \mathbb{C}))$$

solving

$$\begin{cases} ic\partial_{x_1} v_c - \Delta v_c - (1-|v_c|^2)v_c = 0 \\ |v| \to 1 \textit{ when } |x| \to +\infty. \end{cases}$$

Furthermore, $c \to Q_1(v) \in C^1(]0, c_0],]Q_1(v_{c_0}), +\infty[)$ and is a strictly decreasing function. Finally,

$$E(v_c) = \min\{E(u), Q_1(u) = Q_1(v_c)\}$$

and is orbitally stable.

See [23] for a summary of the proof of this result. The orbital stability proven follows from a Cazenave–Lions approach in a more generic case in [8]. This stability is in some sense weaker than the one in the metric space for the constant background. In an upcoming work, we are going to show the stability in an adapted metric space for this branch of travelling waves in the small speed limit. Numerical simulations [11] show that we should expect here uniqueness of minimizers for all values of the momentum and not simply large ones, but this is an open problem. See [2, 3, 4, 7, 21] for other construction of travelling waves in this context.

Concerning the dimension $N = 1$, the travelling waves are explicit for all speeds in $]0, \sqrt{2}[$ and their stability can be shown using monotony arguments. We refer to [6, 17] and reference therein.

1.2. Travelling waves solutions in the generic case for $N \geqslant 3$

From now on, we go back to the case of a generic nonlinearity F with $F(1) = 0, F'(1) = -1$ and in any dimension $N \geqslant 3$.

With $V(s) := \int_s^1 F(\tau)d\tau$, we define the energy and momentum by

$$E(u) = \frac{1}{2}\left(\int_{\mathbb{R}^N} |\nabla u|^2 - V(|u|^2)\right) \quad \text{and} \quad \vec{Q}(u) := \int_{\mathbb{R}^N} \mathfrak{Re}(i\nabla u\overline{u-1}).$$

We also define $Q_1(u) = \vec{Q}(u).\vec{e}_1$.

Let us first introduce a generic way to construct travelling waves by looking at a minimization problem. To do so, we need to introduce some identities.

1.2.1. Pohozaev identities

Suppose that u is a solution of $-\Delta u - F(|u|^2)u = 0$. Taking the scalar product of the equation with $x.\nabla u$ leads to the Pohozaev identity

$$P_0(u) := \langle -\Delta u - F(|u|^2)u, x.\nabla u\rangle = \frac{(N-2)}{N}\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V(|u|^2) = 0.$$

If instead u solves $ic\partial_{x_1}u - \Delta u - F(|u|^2)u = 0$, then the same computation leads to

$$\begin{aligned} P_c(u) &:= \langle ic\partial_{x_1}u - \Delta u - F(|u|^2)u, x.\nabla u\rangle \\ &= \frac{(N-2)}{N}\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V(|u|^2) + \frac{(N-1)}{N}cQ_1(u) \\ &= 0. \end{aligned}$$

1.2.2. The minimisation problem

We define $D^{1,2}$ to be the closure of C_c^∞ for the norm of $\dot{H}^1$ and

$$\chi := \{\varphi \in D^{1,2}, \nabla\varphi \in L^2(\mathbb{R}^N), |1+\varphi| - 1 \in L^2(\mathbb{R}^N)\}.$$

Contrary to the case of $F(s) = 1 - s$, in a more general setting it is unclear if minimizing the energy at fixed momentum yields nontrivial solutions. We will instead look at another minimisation problem.

Theorem 3 ([22])**.** *Suppose that $N \geqslant 3$, F is continuous on $[0, +\infty[$, C^1 in a neighborhood of 1, with $F(1) = 0, F'(1) = -1$. Suppose furthermore that there exists $C > 0$ and $p_0 < \frac{2}{N-2}$ such that $|F(s)| \leqslant C(1 + s^{p_0})$ for any $s \geqslant 0$. Then, for any $0 < c < \sqrt{2}$,*

$$\inf_{u \in 1+\chi}\{E(u) + cQ_1(u), P_c(u) = 0, u \not\equiv 1\} > 0$$

and is reached by at least one function. Furthermore, any minimizer is a solution of the equation

$$ic\partial_{x_1}u - \Delta u - F(|u|^2)u = 0.$$

Let us make a few remarks about this result. First, we minimize here $E + cQ_1$ here under the constraint $P_c = 0$ and $u \not\equiv 1$. This is because if we try to minimize E at fixed value of Q_1 as done previously, it is possible (even likely) that this minimization problem is reached only by constant solutions.

Remarks that speeds are taken only in the range $]0, \sqrt{2}[$. The speed $\sqrt{2}$ is the speed of sound in this problem, and no travelling waves can exist with a larger speed, see [14, 15]. In dimension $N = 2$, it is still an open problem to show that for any $c \in]0, \sqrt{2}[$ there exists at least one travelling waves with this speed.

1.3. Main result

The energy, momentum and P_c are invariant by translation and multiplication by a complex of modulus one. One key question is to know whether or not, up to these invariances, we have uniqueness of the minimizers. Our goal is to provide a counterexample. That is, we will construct a nonlinearity that satisfy the hypotheses of Theorem 3, but for which there exists $c_* > 0$ where the minimization problem is reached by two distinct functions.

Theorem 4. *For $N \geqslant 3$, there exists a nonlinearity F_1 satisfying the hypotheses of Theorem 3, a speed $c_* > 0$ and a small constant $\gamma > 0$ such that the minimisation problem*

$$\inf_{u \in 1+\chi}\{E(u) + cQ_1(u), P_c(u) = 0, u \not\equiv 1\}$$

is reached by exactly two functions for $c = c_*$ (up to the natural invariances of the problem), and by exactly one function for $c \in [0, c_* + \gamma]\backslash\{c_*\}$.

Furthermore, for any function $f \in C^3(\mathbb{R},\mathbb{R})$ with $f(1) = 0, f'(1) = 0$ and $|f(s)| \leqslant C(1+s^{p_0})$ for some $C > 0, p_0 < \frac{2}{N-2}$, if we consider instead the minimisation problem for the nonlinearity $F_1 + \varepsilon f, \varepsilon \in \mathbb{R}$, then there exists $\varepsilon_0 > 0$ such that for any $|\varepsilon| \leqslant \varepsilon_0$, there exists $c_{*,\varepsilon} > 0$ with $c_{*,\varepsilon} \to c_*$ when $|\varepsilon| \to 0$ such that the minimisation problem is reached by exactly two functions for $c = c_{*,\varepsilon}$ and by exactly one for $c \in [0, c_{*,\varepsilon} + \gamma/2] \backslash \{c_{*,\varepsilon}\}$.

Having multiple distinct minimizers is expected in some cases, see [10]. In addition to giving such an example, we show some sort of "stability" when changing the nonlinearity. Meaning that this does not happen simply for a very specific nonlinearity, but it is true on an "open set" of nonlinearities around F_1. We can furthermore construct several examples of very distinct nonlinearities for which this result holds, and not simply in a neighborhood of a single case F_1. We conjecture that it is possible to have examples where there are three of more minimizers, but it is unclear if it is possible to have nonuniqueness for an interval of values of c rather than isolated ones. If it happens, we believe that this is only for very degenerate cases, and in particular it will not hold in an "open set" of nonlinearities.

2. Key ideas of the proof

We summarize here the three key steps of the proof of Theorem 4.

2.1. Double stationnary solutions

First, we show the result in the case $c_* = 0$. That is, we show the existence of a nonlinearity F such that the equation admits two stationary solutions with the same energy, and such that there are no other stationary solution with less energy (except the constant 1). To be more precise, we define

$$E_{\min} := \inf_{u \in 1+\chi} \{E(u); P_0(u) = 0, u \not\equiv 1\}$$

where P_0 is defined in subsection 1.2.1. The result is as follows

Proposition 5. *There exists a nonlinearity $F \in C^3(\mathbb{R},\mathbb{R})$ satisfying the hypotheses of Theorem 3 such that $E_{\min}$ is reached by exactly two distincts nontrivial functions (up to the natural invariances of the problem) that we denote u_+ and u_-. They satisfy $|u_+| \geqslant 1, |u_-| \leqslant 1$, they are real valued and radial (up to the invariances), converges to 1 exponentially fast in space (and their derivatives do as well but to 0), they do not cancel and solves the equation*

$$-\Delta u_\pm - F(|u_\pm|^2)u_\pm = 0.$$

Furthermore, there exists $\eta > 0$ such that any solution of $-\Delta u - F(|u|^2)u = 0$ with $E(u) \leqslant E_{\min} + \eta$ must be equal to u_+ or u_- (up to the natural invariances of the problem).

Finally, $u_\pm$ satisfy a spectral result as follow. Defining the linearized operator around $u_\pm$ as

$$L_\pm(\varphi) = -\Delta\varphi - \varphi F(|u_\pm|^2) - 2|u_\pm|^2 \mathfrak{Re}(\varphi) F'(|u_\pm|^2),$$

and decomposing in real and imaginary parts $\varphi = \varphi_1 + i\varphi_2$,

$$L_\pm(\varphi) = L_{\pm,1}(\varphi_1) + iL_{\pm,2}(\varphi_2),$$

then

$$\operatorname{Ker} L_{\pm,1} = \operatorname{Span}\{\partial_{x_1} u_\pm, \dots, \partial_{x_N} u_\pm\},$$

and $L_{\pm,1}$ admits exactly one strictly negative eigenvalue $\lambda_\pm < 0$.

The stationary solution u_+ does not see the values of the nonlinearity $F(x)$ for $x < 1$, and as such we can modify F on these values to modify the energy of u_- without affecting the one of u_+ to make sure that they are equals. We can also use this fact to construct $u_\pm$ separatly so that they satisfied the properties on their linearized operator. From it, we can show that sequences of minimizing travelling waves with speed going to 0 must converge to either u_+ or u_-.

Proposition 6. *For the nonlinearity of Proposition 5, if we consider a sequence $c_n \to 0$ and v_n solving the minimisation problem*

$$\inf_{u \in 1+\chi} \{E(u) + c_n Q_1(u), P_{c_n}(u) = 0, u \not\equiv 1\},$$

then up to a subsequence and the natural invariant of the problem, the sequence v_n converges either to u_+ or u_- when $n \to +\infty$ in $\dot{H}^1$ and $W^{2,p}$ for any $p \in \left]\frac{2N}{N-2}, +\infty\right[$, and $|v_n|$ converges to $|u_\pm|$ in L^2.

2.2. A two parameter bifurcation

We consider here a function $f \in C^3(\mathbb{R}, \mathbb{R})$ with $f(1) = 0, f'(1) = 0$. We look now, for $\varepsilon \in \mathbb{R}$, at the equation

$$-\Delta u - (F + \varepsilon f)(|u|^2)u = 0,$$

where F is defined in Proposition 5. For $\varepsilon = 0$, we know two solutions of this equation, namely u_+ and u_-. First, our goal is to show that for $|\varepsilon|$ small, we can construct two solutions of this equations that are close respectively to u_+ and u_-.

Proposition 7. *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, there exists two functions $u_{\pm,\varepsilon} \in C^\infty(\mathbb{R}^N, \mathbb{R})$ solution of*

$$-\Delta u_{\pm,\varepsilon} - (F + \varepsilon f)(|u_{\pm,\varepsilon}|^2)u_{\pm,\varepsilon} = 0.$$

Furthermore,

$$\varepsilon \to u_{\pm,\varepsilon} - 1 \in C^\infty([-\varepsilon_0, \varepsilon_0], C^\infty(\mathbb{R}^N, \mathbb{R}))$$

and $u_{\pm,0} = u_\pm$. These functions are real valued, radial, converges to 1 exponentially fast in space.

This is done by a bifurcation argument. This can also be used to construct small speed travelling waves.

Proposition 8. *There exists $c_0, \varepsilon_0 > 0$, such that, for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and $c \in [0, c_0]$, there exists a function $U_{\pm,\varepsilon,c}$ solution of the equation*

$$ic\partial_{x_1} U_{\pm,\varepsilon,c} - \Delta U_{\pm,\varepsilon,c} - (F + \varepsilon f)(|U_{\pm,\varepsilon,c}|^2)U_{\pm,\varepsilon,c} = 0.$$

Furthermore, $(c, \varepsilon) \to U_{\pm,\varepsilon,c} - 1 \in C^1([0, c_0] \times [-\varepsilon_0, \varepsilon_0], L^\infty(\mathbb{R}^N))$ and $U_{\pm,\varepsilon,0} = u_{\pm,\varepsilon}$ from Proposition 7.

However, now these solutions are no longer radial nor real valued, and no longer converging exponentially fast to one, only algebraically.

We can also show that these functions are the only possible candidates to the minimization problem

$$\inf_{u \in 1+\chi} \{E(u) + cQ_1(u), P_c(u) = 0, u \not\equiv 1\}$$

for the nonlinearity $F + \varepsilon f$. For simplicity, let us state here the result in the case $c = 0$.

Proposition 9. *Up to reducing the value of $\varepsilon_0 > 0$, with $E_{\min}$ and $\eta > 0$ from Proposition 5, if a real valued radial function $u \in H^1(\mathbb{R}^N, \mathbb{R})$ satisfy*

$$-\Delta u - (F + \varepsilon f)(|u|^2)u = 0$$

with $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and

$$E(u) \leqslant E(u_\pm) + \eta/2,$$

then, up to the natural invariances of the problem, we have either $u = u_{+,\varepsilon}$ or $u = u_{-,\varepsilon}$, with $u_{\pm,\varepsilon}$ defined in Proposition 7.

In particular, the quantity

$$E_{\min}(\varepsilon) := \inf_{u \in 1+\chi} \{E_\varepsilon(u); P_0(u) = 0, u \not\equiv 1\}$$

where $E_\varepsilon(u)$ is the energy for the nonlinearity $F + \varepsilon f$ is such that

$$E_{\min}(\varepsilon) = \min(E_\varepsilon(u_{\pm,\varepsilon}), E_\varepsilon(u_{\pm,\varepsilon})).$$

2.3. Modification of the minimized quantity

Now, we have to parameters c, ε that we can use. Let us show how the quantities

$$\mathfrak{S}_\pm(\varepsilon, c) = E_\varepsilon(U_{\pm,\varepsilon,c}) + cQ_1(U_{\pm,\varepsilon,c})$$

that we want to study changes with respect to c, ε. The energy depends on ε itself as it has a term depending on the nonlinearity. Remark that

$$\mathfrak{S}_+(0,0) = E(u_+) = E(u_-) = \mathfrak{S}_-(0,0).$$

Lemma 10. *The functions $(\varepsilon, c) \to \mathfrak{S}_\pm(\varepsilon, c)$ are C^∞ in a neighborhood of $(0,0)$. Furthermore, we have the following identities for $|\varepsilon|$ small enough:*

$$\partial_\varepsilon \mathfrak{S}_\pm(0,0) = -\int_{\mathbb{R}^N} f(|u_\pm|^2)\frac{r\partial_r(|u_\pm|^2)}{N},$$

$$\partial_c \mathfrak{S}_\pm(\varepsilon, 0) = 0,$$

$$\partial_c^2 \mathfrak{S}_\pm(0,0) > 0$$

and

$$\partial_\varepsilon\partial_c^2 \mathfrak{S}_\pm(0,0) = \int_{\mathbb{R}^N} f(|u_\pm|^2)(\partial_c^2(|U_{\pm,\varepsilon,c}|^2))_{|c=0}.$$

Furthermore, for the $-$ case, the function $r \to \int_{C(0,r)}(\partial_c^2(|U_{-,\varepsilon,c}|^2))_{|c=0}$ where $C(0,r) = \{x \in \mathbb{R}^N, |x| = r\}$ is not identically 0 and not colinear to $r \to r\partial_r(|u_-|^2)$.

With the functions $\mathfrak{S}_\pm(\varepsilon, c)$, Theorem 4 is equivalent to the following statement: There exists $\varepsilon \in \mathbb{R}$ and $c_*, \gamma > 0$ with c_*, γ and $|\varepsilon|$ small (so $\mathfrak{S}_\pm(\varepsilon, c)$ is well defined) such that $\mathfrak{S}_\pm(\varepsilon, c) > \mathfrak{S}_\mp(\varepsilon, c)$ for $c \in [0, c_*]$, $\mathfrak{S}_+(\varepsilon, c_*) = \mathfrak{S}_-(\varepsilon, c_*)$ and $\mathfrak{S}_\pm(\varepsilon, c) < \mathfrak{S}_\mp(\varepsilon, c)$ for $c \in [c_*, c_* + \gamma]$.

Since $\mathfrak{S}_\pm$ is smooth in both its variables, we have

$$\mathfrak{S}_\pm(\varepsilon, c) = \mathfrak{S}_\pm(0,0) + \varepsilon\partial_\varepsilon \mathfrak{S}_\pm(0,0) + c^2(\partial_c^2 \mathfrak{S}_\pm(0,0) + \varepsilon\partial_\varepsilon\partial_c^2 \mathfrak{S}_\pm(0,0)) + O_{c\to 0}(c^3) + O_{|\varepsilon|\to 0}(\varepsilon^2) + O_{c,|\varepsilon|\to 0}(c^2\varepsilon^2).$$

We recall that $\mathfrak{S}_+(0,0) = \mathfrak{S}_-(0,0), \partial_c^2 \mathfrak{S}_\pm(0,0) > 0$.

Let us look first at the case $\partial_c^2 \mathfrak{S}_+(0,0) \neq \partial_c^2 \mathfrak{S}_-(0,0)$, say

$$\partial_c^2 \mathfrak{S}_+(0,0) > \partial_c^2 \mathfrak{S}_-(0,0)$$

(the other case can be done similarly). Then, we can choose f with $\|f\|_{C^3(\mathbb{R}^+)} \leqslant 1$ such that $\partial_\varepsilon \mathfrak{S}_+(0,0) = 0, \partial_\varepsilon \mathfrak{S}_-(0,0) = a > 0, \partial_\varepsilon\partial_c^2 \mathfrak{S}_+(0,0) = \partial_\varepsilon\partial_c^2 \mathfrak{S}_-(0,0) = 0$ where $a > 0$ is independent of ε and c. Then,

$$\mathfrak{S}_+(\varepsilon, c) = \mathfrak{S}_+(0,0) + c^2\partial_c^2 \mathfrak{S}_+(0,0) + O_{c\to 0}(c^3) + O_{|\varepsilon|\to 0}(\varepsilon^2)$$

and

$$\mathfrak{S}_-(\varepsilon, c) = \mathfrak{S}_+(0,0) + a\varepsilon + c^2\partial_c^2 \mathfrak{S}_-(0,0) + O_{c\to 0}(c^3) + O_{|\varepsilon|\to 0}(\varepsilon^2).$$

We look now, for some $x > 0$ independent of ε, the functions

$$\mathfrak{S}_+\left(\varepsilon, x\sqrt{\varepsilon}\right) = \mathfrak{S}_+(0,0) + x^2\varepsilon\partial_c^2 \mathfrak{S}_+(0,0) + O_{|\varepsilon|\to 0}(\varepsilon^{3/2})$$

and

$$\begin{aligned}\mathfrak{S}_-\left(\varepsilon, x\sqrt{\varepsilon}\right) &= \mathfrak{S}_+(0,0) + a\varepsilon + x^2\varepsilon\partial_c^2 \mathfrak{S}_-(0,0) + O_{|\varepsilon|\to 0}(\varepsilon^{3/2}) \\ &= \mathfrak{S}_+\left(\varepsilon, x\sqrt{\varepsilon}\right) + \varepsilon(a + x^2(\partial_c^2 \mathfrak{S}_-(0,0) - \partial_c^2 \mathfrak{S}_+(0,0))) + O_{|\varepsilon|\to 0}(\varepsilon^{3/2}).\end{aligned}$$

Since $\partial_c^2 \mathfrak{S}_-(0,0) - \partial_c^2 \mathfrak{S}_+(0,0) > 0$, for ε small enough we have the existence of $x_0 > 0$ independent of ε such that

$$\mathfrak{S}_+(\varepsilon, 0) < \mathfrak{S}_-(\varepsilon, 0)$$

while

$$\mathfrak{S}_+\left(\varepsilon, x_0\sqrt{\varepsilon}\right) > \mathfrak{S}_-\left(\varepsilon, x_0\sqrt{\varepsilon}\right)$$

and $\partial_x\left(\mathfrak{S}_+\left(\varepsilon, x\sqrt{\varepsilon}\right) - \mathfrak{S}_-\left(\varepsilon, x\sqrt{\varepsilon}\right)\right) \geqslant 0$ on $[0, x_0 + \gamma]$ for some $\gamma > 0$. Choosing then $F_1 = F + \varepsilon f$ this conclude the proof of Theorem 4 in this case.

The remaining case is if $\partial_c^2\mathfrak{S}_+(0,0)=\partial_c^2\mathfrak{S}_-(0,0)$. In that case, we take f with $\|f\|_{C^3(\mathbb{R}^+)}\leqslant 1$ such that

$$\partial_\varepsilon\mathfrak{S}_+(0,0)=0, \partial_\varepsilon\mathfrak{S}_-(0,0)=-\mu<0, \partial_\varepsilon\partial_c^2\mathfrak{S}_+(0,0)=\partial_\varepsilon\partial_c^2\mathfrak{S}_-(0,0)=a$$

where $a>0$ is independent of ε and c, and $\mu>0$ is small compared to c_0^2. In that case, we have

$$\mathfrak{S}_+\left(\varepsilon, x\sqrt{\varepsilon}\right)=\mathfrak{S}_+(0,0)+x^2\varepsilon\partial_c^2\mathfrak{S}_+(0,0)+O_{|\varepsilon|\to 0}(\varepsilon^{3/2})$$

and

$$\begin{aligned}\mathfrak{S}_-\left(\varepsilon, x\sqrt{\varepsilon}\right)&=\mathfrak{S}_+(0,0)+\varepsilon((x^2a+\partial_c^2\mathfrak{S}_+(0,0))-\mu)+O_{|\varepsilon|\to 0}(\varepsilon^{3/2})\\&=\mathfrak{S}_+\left(\varepsilon, x\sqrt{\varepsilon}\right)+\varepsilon(x^2a-\mu)+O_{|\varepsilon|\to 0}(\varepsilon^{3/2})\end{aligned}$$

We can conclude as preivously with now $x_0=\sqrt{\frac{\mu}{a}}\ll c_0$.

This completes the first part of the proof of Theorem 4, that is we constructed the nonlinearity F_1. Now, taking $f\in C^3(\mathbb{R},\mathbb{R})$, we check that the nonlinearity $F_1+\varepsilon f$ satisfy the lemmas and propositions above that were satisfied by $F+\varepsilon f$ previously for $|\varepsilon|\leqslant\varepsilon_0$ where $\varepsilon_0>0$ is small enough, since F_1 has the same properties than F that we use for the proofs. We can then apply all the results above to $F_1+\varepsilon f$ for $\varepsilon_0>0$ small enough, in particular the existence of two branches of travelling waves, smooth with respect to ε, that can only be the two minimizers of $E+cQ$. We can then conclude by continuity with respect of ε of these branches.

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