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On the exponential decay for real-valued solutions to elliptic equations with singular potentials in the plane

Kévin Le Balc'h

Sur la décroissance exponentielle des solutions à valeurs réelles des équations elliptiques avec des potentiels singuliers dans le plan

Résumé

Dans cette note, nous prouvons que les solutions non nulles à valeurs réelles de l'équation de Schrödinger elliptique avec potentiel singulier ne peuvent pas décroître plus rapidement qu'exponentiellement. La stratégie repose de manière cruciale sur la méthode introduite par Logunov, Malinnikova, Nadirashvili et Nazarov, ainsi que sur de nouveaux arguments introduits par l'auteur et Souza pour résoudre la conjecture de Landis dans le plan pour des solutions à valeurs réelles de l'équation de Laplace perturbée par des termes d'ordre inférieur bornés.

Abstract

In this note, we prove that non-trivial real-valued solutions to $-\Delta u + Vu = 0$ in $\mathbb{R}^2$, where $V \in L^p(\mathbb{R}^2;\mathbb{R})$ with $p \in (1,+\infty]$, cannot decay faster than exponentially. The strategy builds crucially on the method introduced by Logunov, Malinnikova, Nadirashvili, and Nazarov, as well as some new arguments introduced by the author and Souza to solve the Landis conjecture in the plane for real-valued solutions to the Laplace equation perturbed by bounded lower-order terms.

1. Introduction

1.1. Qualitative and quantitative unique continuation at infinity

In the late 1960's, see [13, Section 3.5, p. 171], Landis conjectured the following statement. For V in $L^\infty(\mathbb{R}^N)$ and $\delta > 0$,

$$\Big(-\Delta u + Vu = 0 \text{ in } \mathbb{R}^N \text{ and } |u(x)| \leq \exp(-|x|^{1+\delta}) \text{ in } \mathbb{R}^N \Big) \Rightarrow \Big(u \equiv 0 \text{ in } \mathbb{R}^N. \Big) \tag{1.1}$$

One can see (1.1) as a *qualitative unique continuation* property at infinity. The decay rate $\exp(-|x|^{1+\delta})$ seems to be a natural barrier, by considering the function $u(x) = \exp(-C\sqrt{1+|x|^2})$ for a suitable constant $C > 0$. Moreover, (1.1) holds when $N = 1$ by an ordinary differential argument, see for instance [19] or [14, Introduction].

The Landis conjecture was first disproved by Meshkov in 1991 in the case of complex-valued potentials V. In fact, the work [18] exhibits in the plane $\mathbb{R}^2$ a counterexample to (1.1):

$$\exists\, V \in L^\infty(\mathbb{R}^2;\mathbb{C}) \text{ and } u \not\equiv 0, -\Delta u + Vu = 0 \text{ in } \mathbb{R}^2 \text{ and } |u(x)| \leq \exp(-|x|^{4/3}) \text{ in } \mathbb{R}^2.$$

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[18] also shows that this is the right scale, proving the qualitative unique continuation property at infinity. For $V \in L^\infty(\mathbb{R}^N)$ and $\delta > 0$, we have

$$\Big(-\Delta u + Vu = 0 \text{ in } \mathbb{R}^N \text{ and } |u(x)| \leq \exp(-|x|^{4/3+\delta}) \text{ in } \mathbb{R}^N \Big) \Rightarrow \Big(u \equiv 0 \text{ in } \mathbb{R}^N. \Big)$$

In their work on Anderson localization [3], Bourgain and Kenig establish a *quantitative version* of Meshkov's result, which assumes that $-\Delta u + Vu = 0$ in $\mathbb{R}^N$, with $\|V\|_\infty \leq 1$ and $\|u\|_\infty = |u(0)| = 1$. Then, for $C, C' > 0$ sufficiently large,

$$\sup_{B(x_0,1)} |u(x)| \geq \exp(-CR^{4/3}\log(R)), \ \forall\, R \geq C', \ \forall\, |x_0| = R. \tag{1.2}$$

The case of real-valued potentials has been addressed in [3] and is more challenging. We may first ask if the qualitative Landis conjecture (1.1) holds for real-valued bounded potentials V. Then, we may wonder if the quantitative Landis conjecture holds for real-valued potentials, i.e., if (1.2) holds, replacing 4/3 by 1. The difficulty in tackling such a question comes from the fact that Carleman estimates do not seem to distinguish between real-valued and complex-valued solutions to elliptic equations.

A first breakthrough was achieved in [11], regarding the quantitative unique continuation at infinity in the plane. Assuming that $-\Delta u + Vu = 0$ in $\mathbb{R}^2$ with $0 \leq V \leq 1$ and $\|u\|_\infty = |u(0)| = 1$, then for $C, C' > 0$ sufficiently large,

$$\sup_{B(x_0,1)} |u(x)| \geq \exp(-CR\log(R)) \qquad \forall\, R \geq C', \ \forall\, |x_0| = R. \tag{1.3}$$

Then, subsequent papers established analogous results in the settings of variable coefficients and singular lower-order terms, [5, 6, 12], always assuming a sign condition on the zero-order term V.

A second breakthrough was achieved very recently in the 2-d case in the work [16] by removing the sign condition on the potential V, proving in particular (1.1) in the real-valued case. More precisely, the authors prove that for $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$, there exists $C > 0$ sufficiently large such that

$$\Big(-\Delta u + Vu = 0 \text{ in } \mathbb{R}^2 \text{ and } |u(x)| \leq \exp(-C|x|\log^{\frac{1}{2}}(1+|x|)) \text{ in } \mathbb{R}^2 \Big) \Rightarrow \Big(u \equiv 0 \text{ in } \mathbb{R}^2 \Big).$$

Actually, the authors prove the following quantitative unique continuation at infinity. Assuming that $-\Delta u + Vu = 0$ in $\mathbb{R}^2$, with $-1 \leq V \leq 1$ and $\|u\|_\infty = |u(0)| = 1$, then for $C, C' > 0$ sufficiently large,

$$\sup_{B(x_0,1)} |u(x)| \geq \exp(-CR\log^{\frac{3}{2}}(R)) \qquad \forall\, R \geq C', \ \forall\, |x_0| = R. \tag{1.4}$$

Based on the new idea from [16], I presented at Journées EDP at Aussois, in June 2024, the results obtained in collaboration with Souza in [15] that extend qualitative and quantitative Landis conjecture results for second-order elliptic equations of the form $-\Delta u - \nabla\cdot(W_1 u) + W_2 \cdot \nabla u + Vu = 0$ in $\mathbb{R}^2$ with $W_1, W_2 \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$, $V \in L^\infty(\mathbb{R}^2;\mathbb{R})$.

The aim of this note is to establish the same type of results for real-valued solutions to $-\Delta u + Vu = 0$ in $\mathbb{R}^2$ with $V \in L^p(\mathbb{R}^2;\mathbb{R})$, a singular potential where $p \in (1, +\infty]$. The advantage of considering such an equation is that we use arguments from [15] and [16] without too much technical difficulty, which are inherent in the first-order terms W_1 and W_2. Note that the case $p = 1$ has to be excluded because of the counterexample to unique continuation for $-\Delta u + Vu = 0$ in $\mathbb{R}^2$ with $V \in L^1(\mathbb{R}^2;\mathbb{R})$ by Kenig and Nadirashvili in [10]. We refer to [4] and [7] for the study of complex-valued potentials $V \in L^p(\mathbb{R}^2;\mathbb{C})$.

1.2. Main results

The first main result of this paper provides a positive answer to the qualitative Landis conjecture in the plane for real-valued solutions to the equation $-\Delta u + Vu = 0$ in $\mathbb{R}^2$ for $V \in L^p(\mathbb{R}^2;\mathbb{R})$, $p \in (1, +\infty]$.

Theorem 1.1. *Let $p \in (1, +\infty]$. Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ be a real-valued weak solution to*

$$-\Delta u + Vu = 0 \quad \text{in } \mathbb{R}^2, \qquad V \in L^p(\mathbb{R}^2;\mathbb{R}). \tag{1.5}$$

There exists a constant $C = C(V) \geq 1$ such that if

$$|u(x)| \leq \exp\Big(-C|x|\log^{\frac{1}{2-2/p}}(|x|)\Big) \qquad \forall\, |x| \geq 1, \tag{1.6}$$

then $u \equiv 0$.

Note that the assumption that u is real-valued is actually dispensable, as one can assume that V takes values in $\mathbb{R}$. Indeed, if u is complex-valued, the real and imaginary parts of u, which are both real-valued, satisfy (1.5) and (1.6), thus reducing the proof to the case of real-valued solutions. However, we will utilize the fact that u is real-valued in our proof. Up to the logarithmic loss, this result is likely optimal according to [17]. Furthermore, it extends Landis-type conjecture results [20] in the two-dimensional setting.

Our second main result is the following quantitative unique continuation property at infinity.

Theorem 1.2. *Let $p \in (1, +\infty]$. Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be a real-valued weak solution to*

$$-\Delta u + Vu = 0 \quad \text{in } \mathbb{R}^2, \qquad V \in L^p(\mathbb{R}^2; \mathbb{R}),\ \|V\|_p \leq 1.$$

Assume that

$$\|u\|_\infty = |u(0)| = 1.$$

Then there exists a positive constant $C \geq 1$ such that

$$\sup_{B(x_0,1)} |u(x)| \geq \exp\Big(-CR \log^{\frac{3-2/p}{2-2/p}}(R)\Big) \quad \forall\, R \geq 2,\ \forall\, |x_0| = R. \tag{1.7}$$

Theorem 1.1 and Theorem 1.2 are actually based on local quantitative unique continuation properties and a scaling argument that we present in the next part.

1.3. Local quantitative unique continuation property

For the next part, we introduce the notation $B_r = B(0, r)$ for $r > 0$ and $\log_+(s) = \log(2+s)$ for $s \geq 0$.

The following result relates to the vanishing order of real-valued solutions to second-order elliptic equations.

Theorem 1.3. *Let $p \in (1, +\infty]$. Let $u \in H^1_{\mathrm{loc}}(B_2) \cap L^\infty(B_2)$ be a real-valued weak solution to*

$$-\Delta u + Vu = 0 \quad \text{in } B_2, \qquad V \in L^p(B_2; \mathbb{R}). \tag{1.8}$$

Assume that for $K \geq 2$,

$$\|u\|_{L^\infty(B_2)} \leq e^K \|u\|_{L^\infty(B_1)}. \tag{1.9}$$

Then, there exists a positive constant $C \geq 1$ such that

$$\|u\|_{L^\infty(B_r)} \geq r^{C\left(\|V\|_p^{\frac{1}{2-2/p}} \log_+^{\frac{1}{2-2/p}}(\|V\|_p) + K\right)} \|u\|_{L^\infty(B_2)}, \quad \forall\, r \in (0, 1/2). \tag{1.10}$$

The rescaled version of Theorem 1.3 is the following result.

Theorem 1.4. *Let $p \in (1, +\infty]$. Let $R \geq 2$. Let $u \in H^1_{\mathrm{loc}}(B_{2R}) \cap L^\infty(B_{2R})$ be a real-valued weak solution to*

$$-\Delta u + Vu = 0 \quad \text{in } B_{2R}, \qquad V \in L^p(B_{2R}; \mathbb{R}),\ \|V\|_p \leq 1.$$

Assume that for $K \geq 2$,

$$\|u\|_{L^\infty(B_{2R})} \leq e^K \|u\|_{L^\infty(B_R)}.$$

Then there exists a positive constant $C \geq 1$ such that

$$\|u\|_{L^\infty(B_r)} \geq \left(\frac{r}{R}\right)^{CR \log^{\frac{1}{2-2/p}}(R) + CK} \|u\|_{L^\infty(B_{2R})} \quad \forall\, r \in (0, R/2). \tag{1.11}$$

The end of this part consists of proving the following sequence of implications:

$$\text{Theorem 1.3} \Rightarrow \text{Theorem 1.4} \Rightarrow \text{Theorem 1.1 and Theorem 1.2.} \tag{1.12}$$

Proof of Theorem 1.4 from Theorem 1.3. We apply Theorem 1.3 to $u_R(x) = u(Rx)$, which solves (1.8) with $V_R(x) = R^2 V(Rx) \in L^p(B_2; \mathbb{R})$. Remark that

$$\|V_R\|_p \leq R^{2-2/p},$$

so for every $r \in (0, R/2)$, i.e., $(r/R) \in (0, 1/2)$, we have

$$\begin{aligned} \|u\|_{L^\infty(B_r)} &= \|u_R\|_{L^\infty(B_{r/R})} \\ &\geq \left(\frac{r}{R}\right)^{CR \log^{\frac{1}{2-2/p}}(R) + CK} \|u_R\|_{L^\infty(B_2)} = \left(\frac{r}{R}\right)^{CR \log^{\frac{1}{2-2/p}}(R) + CK} \|u\|_{L^\infty(B_{2R})}, \end{aligned}$$

leading to the expected inequality (1.11). □

We now prove Theorem 1.1 and Theorem 1.2 from Theorem 1.4.

Proof of Theorem 1.1 from Theorem 1.4. Replacing u by $u_\lambda(x) = u(\lambda x)$ for $\lambda > 0$ small enough, one can assume that $\|V\|_p \leq 1$. We then argue by contradiction, assuming that u_λ is not identically equal to 0. Since $|u_\lambda|$ tends to 0 as $|x| \to \infty$, it follows that $|u_\lambda|$ attains its global maximum at some point $x_{\max}$ in the plane. For any $R \geq 2|x_{\max}| + 2$ and any x with $|x| = R/2$, we have

$$\sup_{B(x,2R)} |u_\lambda| = \sup_{B(x,R)} |u_\lambda|,$$

and additionally, by applying Theorem 1.4 to $u_\lambda(x + \cdot\,)$, we have for $C \geq 1$,

$$\sup_{B(x,R/4)} |u_\lambda| \geq \exp(-CR \log^{\frac{1}{2-2/p}}(R)),$$

leading to a contradiction with the decay assumption (1.6). □

Proof of Theorem 1.2 from Theorem 1.4. Take $x_0 \in \mathbb{R}^2$ such that $|x_0| = R$. From the assumption $\|u\|_\infty = |u(0)| = 1$, we have

$$\|u(x_0 + \cdot\,)\|_{L^\infty(B_{2R})} = \|u(x_0 + \cdot\,)\|_{L^\infty(B_R)}.$$

Thus, we can apply (1.11) to the function $u(x_0 + \cdot\,)$ with $r = 1 \leq R/2$ to get

$$\|u(x_0 + \cdot\,)\|_{L^\infty(B_1)} \geq (1/R)^{CR \log^{\frac{1}{2-2/p}}(R)} \geq \exp\Big(-CR \log^{\frac{3-2/p}{2-2/p}}(R)\Big),$$

showing that (1.7) holds. □

1.4. Strategy of the proof of the main local result Theorem 1.3

Notation and parameters. In the following and throughout the paper, $C, C' \geq 1$ denote various large positive numerical constants, $c, c' > 0$ denote various small positive numerical constants, and $\varepsilon > 0$ is a sufficiently small parameter chosen depending on $\|V\|_p$, as detailed below.

In this part, we present the strategy of the proof of Theorem 1.3 and the main arguments of each step. This strategy follows the approach of [16]. We will explain at the end of this section the new difficulties compared to [16]. The proof of Theorem 1.3 is divided into three main steps.

Step 1: Construction of a positive multiplier φ in a suitable perforated domain. We first introduce the set of zeros of u, called the *nodal set* of u:

$$Z := \{x \in B_2 \mid u(x) = 0\}.$$

In this step, we shall first prove that Z satisfies the following fundamental property:

$$\forall\, x_0 \in Z,\ \forall\, \rho \in (0, \varepsilon),\ \partial B(x_0, \rho) \cap (Z \cup \partial B(0,2)) \neq \emptyset, \tag{P-ε}$$

for

$$\varepsilon \leq c + c\|V\|_\infty^{-\frac{1}{2-2/p}}. \tag{1.13}$$

The next point consists of *perforating the domain* $B(0,2)$ using sufficiently small disks (of radius ε) in a sufficiently large number, whose union is denoted by F_ε, avoiding Z, $\partial B(0,2)$, 0, and $x_{\max}$, the point at which $|u|$ is maximal in B_1. The resulting perforated domain $\Omega_\varepsilon = B_2 \setminus (Z \cup F_\varepsilon)$ has a *small Poincaré constant* of the form $C'\varepsilon$, allowing us to construct a positive solution $\varphi \in H^1(\Omega_\varepsilon)$ satisfying

$$-\Delta\varphi + V\varphi = 0 \ \text{ in } \Omega_\varepsilon,$$

and

$$\varphi - 1 \in H^1_0(\Omega_\varepsilon),\ \|\varphi - 1\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^{2-2/p}\|V\|_\infty.$$

In the following, we will refer to this solution φ as a *multiplier*. Note that for the construction of the multiplier, ε is still of the form (1.13).

Step 2: Reduction to harmonic equation in a perforated domain. Thanks to the positive multiplier from the previous step, we first reduce the elliptic equation $-\Delta u+Vu=0$ to a *divergence type elliptic equation* satisfied by $v=u/\varphi$:

$$-\nabla\cdot(\varphi^2\nabla v)=0 \text{ in } \Omega'_\varepsilon=B_2\setminus F_\varepsilon.$$

Note that the divergence elliptic equation is satisfied in the weak sense, through the nodal set of u. We then apply the *theory of quasiconformal mappings* to find $L:B_2\to B_2$, a quasiconformal mapping, to recast the divergence elliptic equation satisfied by $h=v\circ L^{-1}$:

$$-\Delta h=0 \text{ in } L(\Omega'_\varepsilon). \tag{1.14}$$

The next point of this step consists of controlling how the quasiconformal change of variable L transforms Ω'_ε to another perforated domain. In particular, the holes, which were disks before, will be transformed into holes that still cannot be too flattened by this quasiconformal transform. Note that at the end of this step, ε is now fixed, satisfying

$$\varepsilon\le c+c\|V\|_p^{-\frac{1}{2-2/p}}\log_+^{\frac{1}{2-2/p}}(\|V\|_p). \tag{1.15}$$

Step 3: A Carleman estimate for Δ in a perforated domain. We now employ a *Carleman estimate* in $B(0,2)$ for a cut-off version of h, called y, which vanishes in a small neighbourhood of $\partial B(0,2)$, in a r'-neighbourhood of $B(0,r'/2)$ where $B(0,r')\subset L(B(0,r))$, and in a ε-neighbourhood denoted $\mathcal{V}_\varepsilon$ containing $L(F_\varepsilon)$,

$$\int_{B_2}|y|^2e^{-2s\log(|z|)+2|z|^2}dz+\varepsilon^{-4}\int_{\mathcal{V}_\varepsilon}|y|^2e^{-2s\log(|z|)+2|z|^2}dz \\ \le C\int_{B_2}|\Delta y|^2e^{-2s\log(|z|)+2|z|^2}dz,\quad \forall\, s\ge 1.$$

By using *Harnack inequalities*, the cut-off terms near $\mathcal{V}_\varepsilon$ are absorbed by choosing the parameter s in the Carleman estimate such that

$$s\ge C\varepsilon^{-1}.$$

Thus, according to (1.15), the following choice of s is convenient:

$$s\ge C\|V\|_p^{\frac{1}{2-2/p}}\log_+^{\frac{1}{2-2/p}}(\|V\|_p)+C.$$

The cut-off terms near $\partial B(0,2)$ are absorbed using (1.9) and by recalling that the perforation process in Step 1 avoids the point $x_{\max}$. Here, s must be taken such that

$$s\ge C\|V\|_p^{\frac{1}{2-2/p}}\log_+^{\frac{1}{2-2/p}}(\|V\|_p)+CK+C.$$

The cut-off term near $B(0,r'/2)$ will be our observation term, i.e. the left-hand side of (1.10), recalling that $r'=cr^2$ if $r\le C\varepsilon$ or $r'=cr$ if $r>C\varepsilon$. This combination of arguments leads to the expected quantitative unique continuation estimate for u, i.e. (1.10).

Steps 1, 2, and 3 are critically inspired by the methodology in [16], which focuses on the case of the elliptic equation $-\Delta u+Vu=0$. Still, our strategy differs from that in [16] in Steps 1 and 3.

Differences in Step 1 compared to [16, Act 1]**.** The main difference is the presence of the unbounded potential $V\in L^p(B_2;\mathbb{R})$.

We first prove a weak quantitative maximum principle for $\Phi\in H_0^1(\Omega)$ satisfying $-\Delta\Phi=f$ with $f\in L^p(\Omega)$, where Ω is a bounded open set with a small Poincaré constant (see Lemma 2.2 below). This is a generalization of the weak quantitative maximum principle [16, Lemma 6.10] for the Laplace equation $-\Delta\Phi=f$ with $f\in L^\infty(\Omega)$. We implement De Giorgi's method in the associated variational formulation of the elliptic equation $-\Delta\Phi=f$, and because of the L^p-source term, we need to use precise Sobolev inequalities, quantified in terms of the Poincaré constant (see Lemma 2.1 below), which come from [15].

Differences in Step 3 compared to [16, Act 3]**.** Here, we do not follow [16, Act 3] and prefer employing an argument similar to [15, Step 3], which consists of using a Carleman estimate with a singular weight in a perforated domain. Our strategy takes inspiration from [16, Section 6.1] and [9, Step 6]. Indeed, [16, Section 6.1] proposes an alternative strategy to [16, Act 3], based on a simple Carleman estimate with a linear weight, but with the drawback of a logarithmic loss. In this part, still working with the singular weight function of [15, Step 3] and using some ideas from [8], we avoid the logarithmic loss.

1.5. Organization of the paper

In Section 2, we present the Step 1 of the proof of the main local result Section 1.3. In Section 3, we present the Step 2 of the proof of the main local result Theorem 1.3. In Section 4, we present the Step 3 of the proof of the main local result Theorem 1.3. We highlight that we do not give all the full proofs of the results, even if some are new, because they are small adaptations of the arguments coming from [15] and [16].

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2. Step 1: Construction of a positive multiplier

The main goal of this step is to construct a positive multiplier for the equation $-\Delta\varphi + V\varphi = 0$. As explained in Section 1.4, this construction can be made possible by perforating the domain B_2 in a suitable way to reduce the Poincaré constant. Indeed, this will allow us to apply weak maximum principles, quantified as a function of the Poincaré constant and the parameters of the elliptic operator, to prove the existence of such a function φ.

2.1. Weak quantitative maximum principles

The goal of this first part is to prove maximum principles for elliptic operators in an open bounded set Ω, with a small Poincaré constant.

2.1.1. With a L^p-source term

The main result of this part is a weak maximum principle with a L^p-source term, stated in Lemma 2.2 below.

We first state the following Sobolev's inequality, see [15].

Lemma 2.1. *For every $\varepsilon > 0$, $C' \geq 1$, $q \in [2, +\infty)$, there exists $C > 0$, independent of ε, such that for every bounded open set $\Omega \subset \mathbb{R}^2$ with $C_P(\Omega)^2 \leq (C')^2\varepsilon^2$, we have*

$$\|u\|_{L^q(\Omega)} \leq C\varepsilon^{2/q} \|\nabla u\|_{L^2(\Omega)} \qquad \forall\, u \in H_0^1(\Omega). \tag{2.1}$$

The following weak maximum principle is new.

Lemma 2.2. *For every $\varepsilon > 0$, $C' \geq 1$, $p \in (1, +\infty]$, there exist $c > 0$ and $C > 0$, independent of ε, such that for every bounded open set $\Omega \subset \mathbb{R}^2$ with $C_P(\Omega)^2 \leq (C')^2\varepsilon^2$, $f \in L^p(\Omega; \mathbb{R})$, satisfying*

$$\varepsilon \leq c, \tag{2.2}$$

then there exists a unique $\Phi \in H_0^1(\Omega)$ solution of

$$-\Delta\Phi = f \text{ in } \Omega, \tag{2.3}$$

satisfying

$$\|\Phi\|_{L^\infty(\Omega)} \leq C\varepsilon^{2-2/p} \|f\|_{L^p(\Omega)}, \tag{2.4}$$

together with

$$\|\Phi\|_{H_0^1(\Omega)} \leq C\varepsilon^{2/p'} \|f\|_{L^p(\Omega)}, \quad p \in (1,2), \text{ or } \|\Phi\|_{H_0^1(\Omega)} \leq C\varepsilon \|f\|_{L^2(\Omega)}, \quad p \in [2, +\infty]. \tag{2.5}$$

This is a generalization of [16, Lemma 6.10] and the new difficulty lies in handling L^p-source terms. The whole proof is given in Section A.

By using Lemma 2.2, we can now obtain the following result that is the main result of this part.

Proposition 2.3. *For every $\varepsilon > 0$, $C' \geq 1$, $p \in (1, +\infty]$, there exist $c > 0$ and $C > 0$, independent of ε, such that for every bounded open set $\Omega \subset \mathbb{R}^2$ with $C_P(\Omega)^2 \leq (C')^2\varepsilon^2$, $V \in L^p(\Omega; \mathbb{R})$, satisfying*

$$\varepsilon^{2-2/p} \|V\|_{L^p(\Omega)} \leq c,$$

there exists a unique $\varphi \in H^1(\Omega)$ such that

$$-\Delta\varphi + V\varphi = 0 \text{ in } \Omega, \tag{2.6}$$

and $\widetilde{\varphi} = \varphi - 1$ satisfies $\widetilde{\varphi} \in H_0^1(\Omega)$ and

$$\|\widetilde{\varphi}\|_{L^\infty(\Omega)} \leq C\varepsilon^{2-2/p} \|V\|_{L^p(\Omega)}. \tag{2.7}$$

2.2. Properties of the nodal set and perforation process

Take $\varepsilon > 0$ a free parameter satisfying

$$\varepsilon^{2-2/p} \|V\|_{L^p(B_2)} \leq c. \tag{2.8}$$

Let us now give an application of Lemma 2.3 to establish the fundamental property on the nodal set of u, that we called before (P-ε).

Lemma 2.4. *Let u be a real-valued solution to $-\Delta u + Vu = 0$ in a ball $B(x, \varepsilon)$ with $\varepsilon > 0$ satisfying (2.8) and $u \in H^1(B(x, \varepsilon)) \cap C^0(\overline{B(x, \varepsilon)})$. Then, if $u > 0$ on $\partial B(x, \varepsilon)$ then $u > 0$ in $B(x, \varepsilon)$.*

Corollary 2.5. *Let u be as in Theorem 1.3. Then, the nodal set of u,*

$$Z := \{x \in B(0, 2)\ ;\ u(x) = 0\},$$

is closed in $B(0, 2)$ and satisfies the following property

$$\forall\, x_0 \in Z,\ \forall\, \rho \in (0, \varepsilon),\ \partial B(x_0, \rho) \cap (Z \cup \partial B(0, 2)) \neq \emptyset. \tag{P-ε}$$

Let us take $x_{\max} \in \overline{B_1}$ such that

$$|u(x_{\max})| = \sup_{\overline{B_1}} |u|.$$

The next step is to construct a suitable perforation of the domain B_2 which avoids the nodal set Z, $\partial B(0, 2)$, $x_{\max}$ and 0.

From Crollary 2.5, we then get the following lemma, that is stated in [16, Section 3.1] (see also [9, Lemma 2.10]).

Lemma 2.6. *For all $C_0 \geq 5$, for every $\varepsilon > 0$, there exist finitely many $C_0\varepsilon$-separated closed disks of radius ε, whose union is denoted by F_ε, satisfying the following properties:*

- *these disks are $C_0\varepsilon$-separated from each other, from Z, from $\partial B(0, 2)$, from $x_{\max}$ and from 0,*
- *the set $Z \cup F_\varepsilon \cup \partial B(0, 2)$ is a $6C_0\varepsilon$-net in $B(0, 2)$, meaning that for all $x \in B(0, 2)$, $B(x, 6C_0\varepsilon) \cap (Z \cup F_\varepsilon \cup \partial B(0, 2)) \neq \emptyset$.*
- *the set*

  $$\Omega_\varepsilon := B(0, 2) \setminus (Z \cup F_\varepsilon) \tag{2.9}$$

  satisfies $C_P(\Omega_\varepsilon)^2 \leq C^2\varepsilon^2$ for some constant $C > 0$ depending on C_0 but independent of ε, u and V.

In the sequel, it will be useful to choose a very large C_0. For simplicity, from now on, we set $C_0 = 18 \cdot 32^2$. This choice will be made clearer later.

2.3. Construction of the positive multiplier

Note that now $\varepsilon > 0$ is still a free parameter satisfying

$$\varepsilon + \varepsilon^{2-2/p} \|V\|_{L^\infty(B_2)} \leq c, \tag{2.10}$$

where $c > 0$ is a small positive constant depending on the constant C that appears in Lemma 2.6.

We have the following result, that is the main result of this Step 1.

Proposition 2.7. *Let Ω_ε be as in Lemma 2.6. There exists $\varphi \in H^1(\Omega_\varepsilon)$ such that*

$$-\Delta\varphi + V\varphi = 0 \text{ in } \Omega_\varepsilon, \tag{2.11}$$

and $\widetilde{\varphi} = \varphi - 1$ satisfies $\widetilde{\varphi} \in H_0^1(\Omega_\varepsilon)$ and

$$\|\widetilde{\varphi}\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^{2-2/p} \|V\|_{L^p(B_2)}. \tag{2.12}$$

3. Step 2: Reduction to a non-homogeneous $\partial_{\bar{z}}$-equation

The goal of this step is to use the multiplier φ, defined in Ω_ε in the previous step, as introduced in Proposition 2.7, to first transform the equation (1.8) into a divergence elliptic equation in a subset of B_2. Then, by using a quasiconformal change of variable, we will recast this divergence elliptic equation into an elliptic equation of the form $-\Delta h = 0$.

3.1. The new equation satisfied by $v = u/\varphi$

The first step is to rewrite the elliptic problem $-\Delta u + Vu = 0$ in B_2 as an equation in divergence form.

Unfortunately, we cannot do this directly in the whole set B_2, but only in the set

$$\Omega'_\varepsilon = B_2 \setminus F_\varepsilon,$$

which is slightly larger than the set $\Omega_\varepsilon = B_2 \setminus (Z \cup F_\varepsilon)$ defined in (2.9).

Using the equation for φ in (2.11), it is clear that by setting $v = u/\varphi$ in Ω_ε, we have $-\nabla \cdot (\varphi^2 \nabla v) = 0$ in Ω_ε. Extend φ by 1 to B_2. In fact, since $\Omega'_\varepsilon = \Omega_\varepsilon \cup Z$, and u vanishes on Z, an adaptation of [16, Lemma 4.1] shows that the equation $-\nabla \cdot (\varphi^2 \nabla v) = 0$ also holds in Ω'_ε. To be more precise, we obtain the following result.

Lemma 3.1. *The function v defined in Ω'_ε by*

$$v := \frac{u}{\varphi} \text{ in } \Omega'_\varepsilon, \tag{3.1}$$

belongs to $H^1(\Omega'_\varepsilon)$ and satisfies in the weak sense

$$-\nabla \cdot (\varphi^2 \nabla v) = 0 \text{ in } \Omega'_\varepsilon,$$

Note that the computations take care of what happens through the nodal set of u, i.e. Z.

3.2. Quasiconformal change of variable

We then utilize the theory of quasiconformal mappings, which, roughly speaking, guarantees that solutions to homogeneous elliptic divergence equations behave like harmonic functions; see, e.g., [2].

Lemma 3.2. *There exists an homeomorphic mapping L of $\overline{B(0,2)}$ into itself such that*

- *$L \in H^1_{\mathrm{loc}}(B_2)$ satisfies the following Beltrami equation*

$$\partial_{\bar{z}} L = \mu \partial_z L \text{ in } B_2, \tag{3.2}$$

with $\mu \in L^\infty(B_2)$, satisfying $\mu = 0$ in $B_2 \setminus \Omega'_\varepsilon$,

$$\mu = \frac{1-\varphi^2}{1+\varphi^2} \cdot \frac{\partial_x v + i\partial_y v}{\partial_x v - i\partial_y v} \quad \text{if } \nabla v \neq 0, \ \mu = 0 \quad \text{if } \nabla v = 0 \quad \text{in} \quad \Omega'_\varepsilon, \tag{3.3}$$

and

$$\|\mu\|_{L^\infty(B_2)} \leq C\varepsilon^{2-2/p} \|V\|_{L^p(B_2)}, \tag{3.4}$$

- *L is a K-quasiconformal mapping of B_2 into itself, with K satisfying*

$$1 \leq K \leq 1 + C\varepsilon^{2-2/p} \|V\|_{L^p(B_2)}, \tag{3.5}$$

- *$L(0) = 0$,*

- *the function*

$$h = v \circ L^{-1} \text{ in } L(\Omega'_\varepsilon), \tag{3.6}$$

belongs to $H^1_{\mathrm{loc}}(L(\Omega'_\varepsilon))$ and satisfies in the weak sense

$$-\Delta h = 0 \text{ in } L(\Omega'_\varepsilon). \tag{3.7}$$

We conclude this part with the analysis of the distortion of distances through the quasiconformal mapping L, which is precisely given by Mori's theorem; see [1, Chapter III, Section C]. For a K-quasiconformal mapping L of $B(0,R)$ into itself, for all z_1, $z_2 \in B(0,R)$, we have

$$\frac{1}{16}\left|\frac{z_1 - z_2}{R}\right|^K \leq \frac{|L(z_1) - L(z_2)|}{R} \leq 16\left|\frac{z_1 - z_2}{R}\right|^{1/K}. \tag{3.8}$$

Here, $R = 2$.

Based on this result, it is not difficult to show that the balls in F_ε are not overly distorted by the map L, as demonstrated in the subsequent lemma.

Lemma 3.3. *There exists a positive constant $c > 0$ (independent of u and V) such that for every $\varepsilon > 0$ satisfying*

$$\varepsilon^{2-2/p} \|V\|_{L^p(B_2)} \log\left(\frac{2}{\varepsilon}\right) \leq c, \tag{3.9}$$

- *the images of the disks $B(x_j, \varepsilon)$ (recall the definition in Lemma 2.6) are contained in disks of the form $B(L(x_j), 32\varepsilon)$, indexed by $j \in J$, that are $(C_0/32 - 64)\varepsilon$-separated (recall that C_0 was fixed after Lemma 2.6) from each other, from $L(Z)$, from $L(x_{\max})$, from $L(0) = 0$ and from $\partial B(0,2)$,*

- *$L(B(0, r/2))$ contains $B(0, 2r')$ with*

$$r' = 2^{-5}r^2 \text{ if } r \leq 2^{11}\varepsilon, \qquad r' = 2^{-6}r \text{ if } r > 2^{11}\varepsilon. \tag{3.10}$$

Before ending this step of the proof, we now set $\varepsilon > 0$ such that

$$\varepsilon^{2-2/p} \|V\|_{L^p(B_2)} \log\left(\frac{2}{\varepsilon}\right) \leq c, \tag{3.11}$$

We then set $\varepsilon' = 32\varepsilon$ and remark that by construction, and recalling the choice $C_0 = 18 \cdot 32^2$, for which we have $C_0/32 - 64 = 16 \cdot 32$, the disks $B(L(x_j), \varepsilon')$ given by Lemma 3.3 are $16\varepsilon'$-separated from each other, from $L(Z)$, from ∂B_2, from $L(0) = 0$ and from $L(x_{\max})$. We will also use the notation $x'_j = L(x_j)$.

4. Step 3: The Carleman estimate to the Laplacian

The aim of this section is to apply a suitable L^2 Carleman estimate to the equation satisfied by h, as seen in (3.7) above, in order to deduce the vanishing order estimate for u, namely (1.10). The cut-off terms near the disks $B(x'_j, \varepsilon')$ will be absorbed by the left-hand side term of the Carleman estimate by taking the s-parameter sufficiently large as a function of ε and using Harnack's inequality. The boundary terms will be absorbed by leveraging the assumption on u, i.e., (1.9), and by taking the s-parameter sufficiently large as a function of K. In order to deduce from the L^2 Carleman estimate a L^∞ bound on u, specifically an estimate of $|u(x_{\max})|$, we will finally use local elliptic regularity estimates for the operator Δ.

4.1. The Carleman estimate in the perforated domain

The goal of this first part is to state an elementary L^2-Carleman estimate in the two-dimensional setting.

For $s \geq 1$, a parameter, let us introduce the notation

$$\psi_s(z) = -s\log(|z|) + |z|^2$$

First, remark that for every $z \neq 0$,

$$\Delta\psi_s(z) \geq 2.$$

We have the following Carleman estimate, [8, Section 2].

Proposition 4.1. *Then for every $y \in C_c^\infty(B_2 \setminus (\{0\} \cup B(x_j', \varepsilon')))$, we have*

$$\int_{B_2} |y|^2 e^{2\psi_s(z)} dz + \varepsilon^{-4} \sum_{j\in J} \int_{4\varepsilon' \leq |z-x_j'| \leq 8\varepsilon'} |y|^4 e^{2\psi_s(z)} dz \leq C \int_{B_2} |\Delta y|^2 e^{2\psi_s(z)} dz. \quad (4.1)$$

Note that in Proposition 4.1, we crucially use the elementary Carleman estimate stated in [8, Proposition 2.1] for the $\partial_{\bar{z}}$-operator. In contrast with most of the Carleman estimates, this inequality can involve a singularity weight function at several points. In our case, the weight function for obtaining (4.1) will be of the form

$$\Psi_s(z) = e^{\psi_s(z)} \Phi_\varepsilon(z),$$

where Φ_ε satisfies

$$c_1 \leq \Phi_\varepsilon(z) \leq c_2,\ \Delta \log \Phi_\varepsilon \geq 0 \text{ in } B_2 \setminus \cup(B(x_j', \varepsilon')),\ \Delta \log \Phi_\varepsilon \geq c_3 \varepsilon^{-2} \text{ in } 4\varepsilon' \leq |z - x_j'| \leq 8\varepsilon'.$$

4.2. Application of the Carleman estimate

Let us introduce η a cut-off function such that $\eta = 0$ in a r'-neighborhood of $B(0, r')$, in a small neighborhood of ∂B_2 and in a ε'-neighborhood of the disks $B(x_j', \varepsilon')$ and set $y = \eta h$. Then one can establish the following result.

Proposition 4.2. *There exists a constant $C \geq 1$ such that for every $s \geq 1$ satisfying*

$$s \geq C + C\varepsilon^{-1} + CK, \quad (4.2)$$

we have

$$\int_{B_2} |\eta h|^2 e^{2\psi_s(z)} dz \leq C r'^{-4} \int_{B(0,2r')} |h|^2 e^{2\psi_s(z)} dz. \quad (4.3)$$

We then transform the quantitative unique continuation L^2 result (4.3) to a quantitative unique continuation L^∞ result by using standard elliptic regularity estimates to Δ. We finally come back to the variable u by using that $L(x_{\max})$ is $16\varepsilon'$-separated from the disks $B(x_j', \varepsilon')$ and $h = v \circ L^{-1} = (u/\varphi) \circ L^{-1}$. This leads to (1.10) and concludes the proof.

Appendix A. Proof of the weak quantitative maximum principle

Lemma A.1. *There exist $c > 0$ small enough and $C > 0$ large enough such that for every bounded open set Ω contained in $\mathbb{R}^2$, with $C_P(\Omega)^2 \leq c^2$, $f \in L^p(\Omega; \mathbb{R})$, $\|f\|_p \leq 1$, there exists a unique $\Phi \in H_0^1(\Omega)$ such that*

$$-\Delta\Phi = f \text{ in } \Omega, \quad (A.1)$$

and Φ satisfies

$$\|\Phi\|_\infty \leq C. \quad (A.2)$$

By a scaling argument, we can then deduce the following result.

Proof of Lemma 2.2 from Lemma A.1. Let us set c_0 and C_0 the constants provided by Lemma A.1. Let us set

$$\Omega_0 = \frac{c_0}{C'\varepsilon}\Omega, \quad \widetilde{\Phi} = \frac{c_0^2}{C'^2 \varepsilon^{2-2/p} \|f\|_{L^\infty}} \Phi\left(\frac{C'\varepsilon}{c_0}\cdot\right),$$

$$\widetilde{f} = \varepsilon^{2/p} \|f\|_{L^p}^{-1} f\left(\frac{C'\varepsilon}{c_0}\cdot\right),$$

then $C_P(\Omega_0)^2 \leq c_0^2$, $\|\widetilde{W}\|_\infty \leq 1$ provided that $c \leq c_0/C'$, $\|\widetilde{f}\|_p \leq 1$ so one can apply Lemma A.1 that gives $\|\widetilde{\Phi}\|_\infty \leq C_0$, which leads to (2.4). For obtaining (2.5), we test the variational formulation of (2.3) with Φ to get

$$\int_\Omega |\nabla\Phi|^2 = \int_\Omega f\Phi.$$

Now let us prove (2.5).

Let us first consider the case $p \in (1,2)$. By using the variational formulation of (2.3) with Φ, we get by Hölder's inequality together with Sobolev's inequality

$$\int_\Omega |\nabla\Phi|^2 \leq \|f\|_{L^p(\Omega)}\|\Phi\|_{L^{p'}(\Omega)} \leq C\varepsilon^{2/p'}\|\nabla\Phi\|_{L^2(\Omega)}\|f\|_{L^p(\Omega)}.$$

So, we exactly the first part of (2.5).

Now let us consider the case $p \in [2,+\infty)$. Note that $L^p(\Omega) \hookrightarrow L^2(\Omega)$ because Ω is an open bounded set. By the variational formulation of (2.3) with Φ, we get by Cauchy–Schwarz's inequality and Poincaré's inequality

$$\|\nabla\Phi\|^2_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|\Phi\|_{L^2(\Omega)}, \leq C\|f\|_{L^2(\Omega)}\varepsilon\|\nabla\Phi\|_{L^2(\Omega)},$$

leading to the conclusion of the second part of (2.5). □

The rest of the part is then devoted to the proof of Lemma A.1.

Proof of Lemma A.1. We divide the proof into several steps and $c > 0$ is a positive numerical constant that will be fixed later.

Step 1: Existence and uniqueness by Lax–Milgram's lemma. Set $k^2 = C_P(\Omega)^2 \leq c^2$. Let us introduce

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \qquad \forall\, u, v \in H_0^1(\Omega).$$

It is straightforward to prove that a is a continuous, bilinear, coercive form on $H_0^1(\Omega)$. Let us now consider

$$l(v) = \int_\Omega fv \qquad \forall\, v \in H_0^1(\Omega).$$

It is straightforward to prove that l is a continuous, linear form on $H_0^1(\Omega)$.

Therefore, by Lax–Milgram's lemma, there exists a unique $\Phi \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla\Phi \cdot \nabla v = \int_\Omega fv \qquad \forall\, v \in H_0^1(\Omega). \tag{A.3}$$

Step 2: Local estimate on Φ. We only treat the case $p \in (1,2)$ because the other case $p \in [2,+\infty]$ is a simpler adaptation of the following arguments.

Now we want to prove some local estimate, i.e. there exists a positive numerical constant $C > 0$ such that for every unit ball $B \subset \mathbb{R}^2$,

$$\left(\int_{B\cap\Omega} |\Phi|^{p'}\right) \leq Ck^4. \tag{A.4}$$

Up to a translation argument, one can assume that $B = B(0,1)$. Let us introduce

$$\varphi(x) = \exp(-|x|).$$

Then, it is easy to check that φ satisfies the following properties

$$\forall\, 1 \leq p \leq \infty,\ \varphi \in W^{1,p}(\mathbb{R}^2),\ |\nabla\varphi| \leq \varphi,\ \int_{\mathbb{R}^2} \varphi = 2\pi.$$

Moreover, as a consequence for every $\Psi \in H_0^1(\Omega)$, we have that $\varphi\Psi \in H_0^1(\Omega)$. So, one can apply the Sobolev inequality (2.1) to $\varphi\Phi$, this leads to

$$\left(\int_\Omega |\varphi\Psi|^{p'}\right)^{2/p'} \leq k^{4/p'}\int_\Omega |\nabla(\varphi\Psi)|^2 \leq 2k^{4/p'}\int_\Omega \varphi^2|\nabla\Psi|^2 + 2k^{4/p'}\int_\Omega \varphi^2|\Psi|^2, \tag{A.5}$$

then we also have by Poincaré inequality (2.1)

$$\int_\Omega |\varphi\Psi|^2 \le k^2 \int_\Omega |\nabla(\varphi\Psi)|^2 \le 2k^2 \int_\Omega \varphi^2 |\nabla\Psi|^2 + 2k^2 \int_\Omega \varphi^2 |\Psi|^2. \tag{A.6}$$

So by summing the two previous estimates and hence providing $c < c_0$, we get

$$\left(\int_\Omega |\varphi\Psi|^{p'}\right)^{2/p'} + \int_\Omega |\varphi\Psi|^2 \le Ck^{4/p'} \int_\Omega |\nabla\Psi|^2 \varphi^2. \tag{A.7}$$

Now set $v = \psi\Phi$ that also belongs to $H_0^1(\Omega)$ so one can apply the variational formulation (A.3) to v to get

$$\int_\Omega |\nabla\Phi|^2 \psi + \int_\Omega (\nabla\psi \cdot \nabla\Phi)\Phi = \int_\Omega f\psi\Phi. \tag{A.8}$$

We bound the right hand side of (A.8) by using the assumption on f, Hölder's inequality,

$$\left|\int_\Omega f\psi\Phi\right| \le \|\varphi\Phi\|_{L^{p'}(\Omega)}. \tag{A.9}$$

For the second term in the left hand side of (A.8), we proceed as follows using (A.7), providing $c < 1/16$,

$$\begin{aligned} \left|\int_\Omega (\nabla\psi \cdot \nabla\Phi)\Phi\right| &\le 2\int_\Omega \psi|\nabla\Phi||\Phi| \le 2\left(\int_\Omega |\Phi|^2\psi\right)^{1/2}\left(\int_\Omega |\nabla\Phi|^2\psi\right)^{1/2} \\ &\le 4k\left(\int_\Omega |\nabla\Phi|^2\psi\right) \le \frac{1}{4}\int_\Omega |\nabla\Phi|^2\psi. \end{aligned} \tag{A.10}$$

By conjugating (A.8), (A.9), (A.10), we get for $c < c_0$,

$$\int_\Omega |\nabla\Phi|^2\psi \le Ck^{2/p'}\left(\int_\Omega |\nabla\Phi|^2\psi\right)^{1/2},$$

so

$$\int_\Omega |\nabla\Phi|^2\psi \le Ck^{4/p'}. \tag{A.11}$$

By using (A.7) and (A.11), we get the expected result (A.4) with $C = 64$.

Third step: Poincaré constant of thin domains. We have the following result, that is exactly [16, Corollary 6.9].

Lemma A.2. *There exists $c_0 > 0$ small enough such that for every $k > 0$, for every bounded open set $\Omega \subset \mathbb{R}^2$ satisfying*

$$|\Omega \cap Q| \le k^2 \le c_0^2 \qquad \text{for any square } Q \text{ with } 1/2 \text{ side-length},$$

then $C_P(\Omega)^2 \le Ck^2$ for some numerical constant $C > 0$, independent of k.

Step 4: De Giorgi scheme. We now fix $c = \min(1/32, c_0) > 0$ where $c_0 > 0$ is the constant in Lemma A.2. Let $t_0 > 0$ that we will be fixed later and $\Omega_0 = \{\Phi > t_0\} \subset \Omega$ with $k_0^2 = C_P(\Omega_0)^2$. From (A.4), we get

$$\int_{B\cap\Omega} |\Phi|^{p'} \le Ck^4, \tag{A.12}$$

then

$$|\{\Phi > t_0\} \cap B| \le \frac{Ck^4}{t_0^{p'}}.$$

So, by using Lemma A.2,

$$k_0^2 \le \frac{Ck^4}{t_0^{p'}}.$$

Then, let us set $t_0 = (Ck)^{1/p'}$ leading to $k_0^2 \le k^3 \le c^3$.

We now recall the well-known facts: $H_0^1(\Omega_0) \subset H_0^1(\Omega)$ and $\Phi_0 := (\Phi - t_0)^+ \in H_0^1(\Omega_0)$ with $\nabla\Phi_0 = \nabla\Phi 1_{\Omega_0}$, see for instance [21, Proposition 1.3.10]. Applying the variational formulation (A.3) we then get

$$\int_{\Omega_0} \nabla\Phi_0 \cdot \nabla v + \int_{\Omega_0} (W \cdot \nabla\Phi_0)v = \int_{\Omega_0} fv \qquad \forall\, v \in H_0^1(\Omega_0).$$

We then iterate the previous arguments, that is we first prove the local estimate on Φ_0, there exists a positive numerical constant $C > 0$ such that for every unit ball $B \subset \mathbb{R}^2$,

$$\int_{B\cap\Omega_0} |\Phi_0|^{p'} \leq Ck_0^4. \tag{A.13}$$

Let $t_1 > 0$ that we will be fixed later and $\Omega_1 = \{\Phi_0 > t_1\} = \{(\Phi - t_0)^+ > t_1\} \subset \Omega_0$, $k_1^2 = C_P(\Omega_1)^2$. We then obtain from (A.13) for every unit ball $B \subset \mathbb{R}^2$,

$$|\{\Phi_0 > t_1\} \cap B| \leq \frac{Ck_0^4}{t_1^{p'}}.$$

So, by using Lemma A.2,

$$k_1^2 \leq \frac{Ck_0^4}{t_1^{p'}}.$$

Then, let us set $t_1 = (Ck_0)^{1/p'}$ leading to $k_1^2 \leq k_0^3$.

By induction, we can construct

$$t_n = (Ck_{n-1})^{1/p'},\ \Omega_n = \{\Phi_{n-1} > t_n\},\ k_n^2 = C_P(\Omega_n)^2,\ \Phi_n = (\Phi_{n-1} - t_n)^+,$$

for all $n \in \mathbb{N}$, with the convention $k_{-1} = k = C_p(\Omega)$, $\Phi_{-1} = \Phi$, leading to

$$k_{n+1} \leq \left(c^{3/2}\right)^{n+2} \qquad \forall\, n \geq 0.$$

With such a construction, we have because $c \leq 1/2$,

$$\sum_{n=0}^{+\infty} t_n \leq C^{1/p'} \sum_{n=-1}^{+\infty} 2^{-\frac{3(n+1)}{2p'}} := T, \tag{A.14}$$

$$|\{\Phi_n > t_{n+1}\} \cap B| \leq k_n^3 \qquad \text{for every unit ball } B \subset \mathbb{R}^2,\ \forall\, n \in \mathbb{N}, \tag{A.15}$$

$$\Phi \leq t_0 + t_1 + \cdots + t_n + \Phi_n \qquad \forall\, n \in \mathbb{N}. \tag{A.16}$$

Therefore, for every unit ball $B \subset \mathbb{R}^2$, we have from (A.15) that

$$|\{\Phi_n > t_{n+1}\} \cap B| \to 0 \text{ as } n \to +\infty,$$

hence conjugating with (A.14) and (A.16),

$$\begin{aligned} |\{\Phi > 2T\} \cap B| &\leq |\{\Phi > 2T\} \cap \{\Phi_n \leq t_{n+1}\} \cap B| + |\{\Phi > 2T\} \cap \{\Phi_n > t_{n+1}\} \cap B| \\ &\leq |\{\Phi_n > t_{n+1}\} \cap B| \to 0 \text{ as } n \to +\infty. \end{aligned}$$

Then $|\{\Phi > 2T\}| = 0$ so $\Phi \leq 2T$ almost everywhere

By linearity, using that $-\Phi$ solves (A.1) replacing f by $-f$, we then obtain with the same strategy that $-\Phi \leq 2T$ then the expected bound (A.2). □

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