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On Courant and Pleijel theorems for sub-Riemannian Laplacians

Rupert L. Frank   Bernard Helffer

Théorèmes de Courant et de Pleijel pour les Laplaciens sous-Riemanniens

Résumé

Dans cet exposé (présenté oralement aux journées EDP 2024 à Aussois par le deuxième auteur), nous nous intéressons au nombre d'ensembles nodaux de fonctions propres de sous-Laplaciens définis sur des variétés riemanniennes. Plus précisément, nous explorons la validité du théorème de Pleijel qui énonce que le nombre d'ensembles nodaux d'une fonction propre associée à une k-ième valeur propre est strictement (et uniformément en un certain sens) inférieur à k pour k assez grand. Nous réduisons d'abord la question générale à celle pour des ouverts de groupes nilpotents. Nous analysons ensuite en détail le cas où le groupe nilpotent est le produit direct d'un groupe de Heisenberg et d'un espace Euclidien. En cours de route, nous sommes conduits à améliorer des bornes connues des constantes optimales pour les inégalités de Faber–Krahn ou isopérimétriques pour ces groupes. C'est une annonce (détaillée sur ArXiv) de résultats dont les preuves feront l'objet d'un futur article. Cette annonce reprend avec modification et inclusion de nouveaux résultats l'annonce plus détaillée présentée dans [10].

Abstract

We are interested in the number of nodal domains of eigenfunctions of sub-Laplacians on sub-Riemannian manifolds. Specifically, we investigate the validity of Pleijel's theorem, which states that the number of nodal domains of an eigenfunction corresponding to the k-th eigenvalue is strictly (and uniformly, in a certain sense) smaller than k for large k. We first reduce this question from the case of general sub-Riemannian manifolds to that of nilpotent groups. Secondly, we analyze in detail the case where the nilpotent group is a Heisenberg group times a Euclidean space. Along the way we improve known bounds on the optimal constants in the Faber–Krahn and isoperimetric inequalities on these groups. This is an announcement and the proofs will be given in a future paper (see also in ArXiv). This announcement is a modification with inclusion of new results of the more detailed announcement published in [10].

1. Introduction

1.1. Sub-Laplacian and regularity

Motivated by some of the results of Eswarathasan and Letrouit in [6] and related open problems initially discussed with C. Letrouit, we consider in a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary the Dirichlet realization of a sub-Laplacian (also called Hörmander's operator)

$$-\Delta_{\mathbf{X}}^{\Omega} := -\sum_{j=1}^{p} X_j^2 \,,$$

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where the X_j ($j = 1, \ldots, p$) are C^∞ real vector fields on $\overline{\Omega}$ satisfying the so-called Hörmander condition of rank r introduced in [15], which reads:

Assumption $(CH(r))$**.** For some $r \geq 1$ the X_j and the brackets up to order r generate at each point $x \in \overline{\Omega}$ the tangent space $T_x\Omega$.

More generally, we can consider a connected C^∞ manifold M of dimension n (with or without boundary) with a given measure μ (with a C^∞-density with respect to the Lebesgue measure in a local system of coordinates) and a system of p C^∞ ($p \leq n$) vector fields satisfying Assumption $(CH(r))$. In this case

$$-\Delta_{\mathbf{X}}^{M,\mu} := \sum_j X_j^\star X_j \,,$$

where $X_j^\star$ is the formal adjoint obtained by using the L^2 scalar product with respect to the given measure μ.

From [15], these operators are known to be hypoelliptic in M. Coming back to simplify to $\Omega \subset \mathbb{R}^n$ the analysis of their regularity at the boundary can be done under the assumption:

Assumption $NC(\mathbf{X}, \partial\Omega)$**.** A system $\mathbf{X}$ is said non-characteristic for an open set Ω, if for each point $x \in \partial\Omega$ there exists a vector field X_i that is transverse to the boundary at x.

Under these assumptions, it has been shown by M. Derridj [5] that we have hypoellipticity up to the boundary.

We emphasize that we will not need $NC(\mathbf{X}, \partial\Omega)$ for our results. Indeed, topological considerations show that this last assumption is rather strong.

L.P. Rothschild and E.M. Stein have proven in 1976 that these sub-Laplacians are maximally hypoelliptic [26], i.e. satisfy

$$\|X_k X_\ell\, u\| \leq C \left(\|\Delta_{\mathbf{X}}^{\Omega,\mu} u\| + \|u\| \right), \quad \forall\, k, \ell\,, \forall\, u \in C_0^\infty(\Omega)\,,$$

where $\|\cdot\|$ denotes the L^2-norm in Ω with respect to the measure μ.

As a side remark we note that there is a characterization of those polynomials of vector fields that are maximally hypoelliptic using a Rockland's like criterion initially introduced by Helffer–Nourrigat in 1979 [13]. The proof in full generality of this criterion was recently obtained by Androulidakis–Juncken–Mohsen (2022) [1].

We do not need this characterization here, but the pseudo-differential calculus introduced by Rothschild–Stein, in the version given by L. Rothschild in the equiregular case [25] (1979), is important in our proof of a Faber–Krahn inequality.

Under Assumption $(CH(r))$ the sub-Laplacian on a bounded set Ω has compact resolvent and concerning its discrete spectrum and the associated eigenfunctions we can consider all the questions that have been solved along the years concerning the Dirichlet realization $-\Delta^\Omega$ of the Euclidean Laplacian. Here, we focus on two questions: Courant's theorem and Pleijel's Theorem. For further properties we refer, for instance, to [6].

1.2. Courant's theorem

Courant's theorem [4] (1923) states that in the case of the Dirichlet Laplacian in $\Omega \subset \mathbb{R}^n$, an eigenfunction associated with the ℓ-th eigenvalue has at most ℓ nodal domains:

$$\nu_\ell \leq \ell\,.$$

Here ν_ℓ denotes the maximum number of nodal domains of an eigenfunction corresponding to the ℓ-th eigenvalue λ_ℓ.

Following the standard proof of Courant's theorem, this appears as a consequence of

- a restriction statement (the restriction of an eigenfunction to its nodal domain is the ground state of the Dirichlet realization of the Laplacian in this domain),
- the minimax characterization of the eigenvalue,
- the Unique Continuation Theorem (UCT).

Hence the difficulty is to determine under which conditions we can extend these results to the sub-Riemannian Laplacians. Concerning the first item, having only rather limited information about the nodal sets (i.e. the boundary of the nodal domains) we adapt a proof [21] of the restriction statement to the sub-Riemannian case. This permits us to avoid to assume $NC(\mathbf{X}, \partial\Omega)$. The variational characterization then holds. Concerning the unique continuation theorem, K. Watanabe [31] proves (UCT) in the C^∞ category in dimension 2, but H. Bahouri gives a discouraging counter-example with two vector fields in $\mathbb{R}^3$. Nevertheless, J. M. Bony [3] proved that (UCT) holds when the vector fields are analytic. Hence Courant's theorem holds in the analytic category as proved by Eswarathasan–Letrouit in [6]. We can also extend the statement (based on a remark of D. Mangoubi [18]) given in [6] and prove that under $(CH(r))$, we have:

$$\nu_\ell \leq \ell + \mathrm{Mult}(\lambda_\ell) - 1\,.$$

1.3. Pleijel's theorem

In the case of the Dirichlet Laplacian in $\Omega \subset \mathbb{R}^n$, Pleijel's theorem [24] (1956) says that, if $n \geq 2$, there exists an Ω-independent constant $\gamma(n)$ in $(0,1)$ such that

$$\limsup_{\ell \to +\infty} \frac{\nu_\ell}{\ell} \leq \gamma(n)\,. \tag{1.1}$$

In the case of the Euclidean Laplacian, the proof of Pleijel's theorem is a nice combination of Weyl's formula, which gives the asymptotic behavior as $\lambda \to +\infty$ of the counting function

$$N(\lambda, -\Delta^\Omega_{eucl}) \sim W_n |\Omega|\, \lambda^{n/2}\,, \tag{1.2}$$

and of the Faber–Krahn inequality, which states that

$$\lambda_1(-\Delta^\Omega_{eucl}) \geq |\Omega|^{-2/n}\, C^{FK}(\mathbb{R}^n)\,. \tag{1.3}$$

Here, with B_1 denoting the ball of unit volume,

$$C^{FK}(\mathbb{R}^n) := \lambda_1(-\Delta^{B_1})\,. \tag{1.4}$$

Given these two ingredients one can establish (1.1) where, for $n \geq 2$,

$$\gamma(n) = W_n^{-1} C^{FK}(\mathbb{R}^n)^{-n/2} < 1\,. \tag{1.5}$$

Adapting this proof to the sub-Riemannian setting, we are led to the investigation of

- Weyl-type asymptotics,
- Faber–Krahn-type inequalities.

We review these in the following two subsections.

1.4. Métivier's Weyl formula

Following the pioneering work of G. Métivier [20], we consider the following assumption (CEq), which is called in the modern language of sub-Riemannian geometry the equiregularity condition:

Assumption (CEq)**.** For each $j \leq r$ the dimension of the space spanned by the commutators of length $\leq j$ at each point is constant.

In this case, we denote by $\mathcal{D}_j$ the span of all vector fields obtained as brackets of length $\leq j$ of the X_k's. By Assumption, $x \mapsto \dim(\mathcal{D}_j(x))$ is constant (and denoted by n_j) and we let (with $n_0 := 0$)

$$Q := \sum_{j=1}^{r} j\,(n_j - n_{j-1})\,, \tag{1.6}$$

the homogeneous dimension.

Under Assumptions $(CH(r))$ and (CEq), G. Métivier shows:

Theorem 1.1. *There exists a continuous, positive function $x \mapsto c^{\mathrm{Weyl}}(x)$ on M such that the counting function of the Dirichlet selfadjoint realization $-\Delta_{\mathbf{X}}^{M,\mu}$ of $-\Delta_{\mathbf{X}}$ in M satisfies, as λ tends to $+\infty$,*

$$N_{-\Delta_{\mathbf{X}}^{M,\mu}}(\lambda) := \#\{j : \ \lambda_j(-\Delta_{\mathbf{X}}^{M,\mu}) \leq \lambda\} \sim \left(\int_M c^{\mathrm{Weyl}}(x)\ d\mu(x)\right)\lambda^{\frac{Q}{2}} . \tag{1.7}$$

An analogous theorem is obtained in the case with boundary for the Dirichlet realization.

Note that in the case $r = 2$, related results are obtained in [19, 20] and Métivier's theorem (together with many other results) has been revisited recently at the light of sub-Riemannian geometry in [28, 29, 30].

Combining our result about Faber–Krahn inequalities with Métivier's Weyl-type formula, we will obtain a sufficient condition for the validity of a Pleijel-type bound; see Theorem 2.1 below. Our upper bound on $\limsup_{k\to\infty} \nu_k/k$ is of the form

$$\left(\int_M (c^{\mathrm{FK}}(x))^{-\frac{Q}{2}}\ d\mu(x)\right)\left(\int_M c^{\mathrm{Weyl}}(x)\ d\mu(x)\right)^{-1}, \tag{1.8}$$

where

- $c^{\mathrm{FK}}(x)$ is a certain local Faber–Krahn constant, defined in terms of the nilpotentization of $-\Delta_{\mathbf{X}}^{M,\mu}$ at $x \in M$,
- $c^{\mathrm{Weyl}}(x)$ is the local Weyl constant from Theorem 1.1; in fact, it is defined in terms of the same nilpotentization.

This strengthening of our original result (2023) is due to Y. Colin de Verdière, who kindly allowed us to include his argument.

The role of the Borel measure

$$D \mapsto \int_D c^{\mathrm{Weyl}}(x)\ d\mu(x)$$

on M is emphasized in the work of Colin de Verdière–Hillairet–Trélat [30], where it is called the *Weyl measure*.

Similarly, we introduce what may be called the *Faber–Krahn measure*

$$D \mapsto \int_D (c^{\mathrm{FK}}(x))^{-\frac{Q}{2}}\ d\mu(x) .$$

It is interesting to compare (1.8) with the Pleijel formula (1.5), to which it reduces in the case of open subsets of $\mathbb{R}^n$. More generally, in the Riemannian case (where $p = n$ and where μ is the Riemannian volume measure) the expression (1.8) reduces to (1.5) and we recover the result of Bérard and Meyer [2].

However, our result is already new in this case when μ is different from the Riemannian volume measure. In the general sub-Riemannian case, the integration with respect to the measure μ takes into account that the model spaces $\mathcal{G}_x$ may vary with the point $x \in M$.

In this respect it is also interesting to note that (1.8) depends on M and the vector fields $X_1, \dots, X_p$, but does *not* depend on the measure μ. Indeed, both integrals in (1.8) do not depend on μ.

According to (1.8), a sufficient condition for the validity of Pleijel's theorem is the following bound on the "local Pleijel constants":

$$\left(c^{\mathrm{FK}}(x)\right)^{-\frac{Q}{2}}\left(c^{\mathrm{Weyl}}(x)\right)^{-1} < 1 \qquad \text{for all } x \in M ;$$

We emphasize that the latter condition involves the corresponding Faber–Krahn constants for Dirichlet realizations of sub-Laplacians in open set of nilpotent groups.

Hence in the second part of the talk we will describe what we have obtained in this particular case.

1.5. Nilpotent approximation

We recall that we are interested in two aspects of the sub-Laplacian on sub-Riemannian manifolds, namely the existence of Weyl-type asymptotics and the existence of a Faber–Krahn type theorem. Compared to our knowledge about Weyl-type asymptotics, which we recalled in the previous subsection, our knowledge is rather poor concerning the constant in the Faber–Krahn inequality in the sub-Riemannian setting. In the case of the Heisenberg group, one can think of a result by P. Pansu [23] concerning the isoperimetric inequality. C. Léna's approach [17] for treating the Neumann problem for the Laplacian could be helpful if the set in Ω where the system of the X_j is not elliptic is "small" in some sense. We will follow another way by revisiting in a first part the nilpotenzation procedure permitting to deduce Faber–Krahn inequalities for sub-Laplacians from Faber–Krahn inequalities for sub-Laplacians on nilpotent groups.

Concerning the nilpotent approximation we refer to Métivier [20], Rothschild–Stein [26] and the presentation of Rothschild [25] (based on assumptions and definitions given earlier by Folland [8]. Since this period in the seventies, a huge literature has developed the so-called sub-Riemannian geometry analyzing in particular this nilpotent approximation.

We assume that Assumptions $(CH(r))$ and (CEq) are satisfied. To simplify, in this abstract we also assume that the X_j are linearly independent.

We can locally construct a family of vector fields Y_j such that $Y_j = X_j$ for $j = 1, \dots, n_1$ and such that, for $2 \leq i \leq r$, $Y_1, \dots, Y_{n_i}$ gives at each x a basis of $\mathcal{D}_i(x)$. Given these Y_j, we can construct at each $x \in M$ the map θ_x given by

$$\theta_x(y) := u = (u_i) \qquad \text{if } y = \exp\Big(\sum u_i Y_i\Big) \cdot x \,, \tag{1.9}$$

where exp denotes the exponential map defined in some small neighborhood of x. In this way we identify a neighborhood of $x \in M$ with a neighborhood of 0 in $\mathbb{R}^n$. It has been shown by G. Métivier that everything depends smoothly on x. We now introduce the notion of nilpotentized measure $d\widehat{\mu}_x$ at $x \in M$. There is a definition in the formalism of sub-Riemannian geometry but we prefer to explain it "by hand". On $\mathbb{R}^n$ we have the Lebesgue measure $du = \prod_i du_i$, and in these local coordinates the measure $d\mu$ is of the form $d\mu = a(x,u)du$, where $(x,u) \mapsto a(x,u)$ is C^∞ in both variables x and u.

In a small neighborhood of 0, the nilpotentized measure at x can be defined by

$$d\widehat{\mu}_x := a(x,0)du \,. \tag{1.10}$$

Then we denote by $Y_{i,x}$, the image of Y_i by θ_x, which is simply Y_i written in the local canonical coordinates around x.

On $\mathbb{R}^n$, with coordinates $u = (u_i)$, we introduce the family of dilations given by

$$\delta_t(u_i) = (t^{w_i} u_i) \,,$$

where, for each i, w_i is the unique $j \in \{1, \dots, n\}$ such that $n_{j-1} + 1 \leq i \leq n_j$.

With this dilation, we have a natural definition of order for a differential operator and G. Métivier [20, Theorem 3.1] proves (in addition to the regularity of θ_x already mentioned above) the following theorem.

Theorem 1.2. *For any x, $X_{j,x}$ is of order ≤ 1 (for $j = 1, \dots, p$). Furthermore,*

- $$X_{j,x} = \widehat{X}_{j,x} + R_{j,x} \,,$$
  where $\widehat{X}_{j,x}$ is homogeneous of order 1 and $R_{j,x}$ is of order ≤ 0.
- *The $\widehat{X}_{j,x}$ generate a nilpotent Lie algebra $\mathcal{G}_x$ of dimension n and rank r.*
- *The mapping $x \mapsto \widehat{X}_{j,x}$ is smooth.*

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2. Main result for sub-Laplacians in the equiregular case

2.1. Main statement

By the nilpotent approximation, we can associate with each point $x \in M$ a nilpotent group G_x (identified with the algebra $\mathcal{G}_x$ in the exponential coordinates) and a corresponding sub-Laplacian

$$\widehat{\Delta}_x = \sum_{j=1}^{p} \widehat{X}_{j,x}^2$$

in $\mathcal{U}_2(\mathcal{G}_x)$ (i.e. the elements in the enveloping algebra that are homogeneous of degree 2).

Using results of [8, 25, 27], we see that for all $x \in M$, for all $\Omega \subset G_x$ open and for all $v \in C_0^\infty(\Omega)$ we have a Faber–Krahn inequality in the form

$$\langle -\widehat{\Delta}_x v, v\rangle_{L^2(G_x,\widehat{\mu}_x)} \geq c^{\mathrm{FK}}(x)\ \widehat{\mu}_x(\Omega)^{-\frac{2}{Q}} \|v\|^2_{L^2(G_x,\widehat{\mu}_x)}\,. \tag{2.1}$$

By definition $c^{\mathrm{FK}}(x)$ is the largest constant such that (2.1) holds. Note that this constant depends on μ through $\widehat{\mu}_x$ and on $\mathbf{X}$ through the $X_{j,x}$.

Using Sobolev-type inequalities, we can show that

$$c^{\mathrm{FK}}(M, \mathbf{X}, \mu) = \inf_{x\in M} c^{\mathrm{FK}}(x) > 0\,.$$

Our main statement is the following theorem:

Theorem 2.1. *Under Assumptions $(CH(r))$ and (CEq), let $-\Delta = \sum_\ell X_\ell^\star X_\ell$ be an equiregular sub-Riemannian Laplacian on a closed connected manifold M. Then*

$$\limsup_{\ell\to+\infty} \frac{\nu_\ell}{\ell} \leq \Big(\int_M (c^{\mathrm{FK}}(x))^{-\frac{Q}{2}}\, d\mu(x)\Big)\cdot\Big(\int_M c^{\mathrm{Weyl}}(x)\ d\mu(x)\Big)^{-1}, \tag{2.2}$$

where ν_ℓ denotes the maximal number of nodal domains of an eigenfunction of $-\Delta$ associated with eigenvalue λ_ℓ .

Note that, in comparison with the first versions that were presented in ArXiv, this improved statement has been proposed by Yves Colin de Verdière in March 2024.

Corollary 2.2. *If*

$$(c^{\mathrm{FK}}(x))^{\frac{Q}{2}} c^{\mathrm{Weyl}}(x)\ > 1 \qquad \textit{for all } x \in M\,, \tag{2.3}$$

then Pleijel's theorem holds.

2.2. Basic example

In an open set $\Omega \subset \mathbb{R}^3$, we consider

$$X_1 = \frac{\partial}{\partial x} + K_1(x,y)\frac{\partial}{\partial z}\,, \qquad X_2 = \frac{\partial}{\partial y} + K_2(x,y)\frac{\partial}{\partial z}\,,$$

with

$$\operatorname{curl} \vec{K} = \frac{\partial}{\partial x}K_2 - \frac{\partial}{\partial y}K_1 > 0 \text{ in } \overline{\Omega}\,.$$

The measure μ is simply the Lebesgue measure $dxdydz$. At each point $(x,y,z) \in \overline{\Omega}$, the nilpotent group $G_{(x,y,z)}$ is the Heisenberg group $\mathbb{H}_1$, and we have

$$X_{1,(x,y,z)} = \frac{\partial}{\partial u_1} - \frac{1}{2}u_2\frac{\partial}{\partial u_3}\,, \qquad X_{2,(x,y,z)} = \frac{\partial}{\partial u_2} + \frac{1}{2}u_1\frac{\partial}{\partial u_3}\,,$$

and

$$d\widehat{\mu}_{(x,y,z)}(u) = \operatorname{curl} \vec{K}(x,y)\, du\,.$$

It can be shown that

$$c^{\mathrm{Weyl}}(x,y,z) = \frac{\widehat{W}(\mathbb{H}_1)}{\operatorname{curl} \vec{K}(x,y)} \text{ with } \widehat{W}(\mathbb{H}_1) := \frac{1}{32}\,.$$

We denote by $c^{\mathrm{FK}}(\mathbb{H}_1)$ the Faber–Krahn constant on the Heisenberg group $\mathbb{H}_1$. The condition (2.3) reads (note that $Q = 4$)

$$\big(c^{\mathrm{FK}}(\mathbb{H}_1)\big)^2\, \widehat{W}(\mathbb{H}_1) > 1\,. \tag{2.4}$$

Under this condition, Pleijel's theorem holds. Currently we have no proof that (2.4) holds. We can show, however, that it holds, provided a well-known conjecture by Pansu concerning the isoperimetric inequality on the Heisenberg group is true; see Theorem 3.5.

3. Pleijel theorem for particular groups: $\mathbb{H}_n \times \mathbb{R}^k$

In the previous section we have shown how the general case of a sub-Riemannian manifold satisfying Hörmander's assumption and the equiregularity assumption can be reduced to the analysis of the same problem for domains in nilpotent stratified Lie groups (see [26] for the main definitions or [7]). In this setting the vector fields are left invariant on the group and, viewed as elements of the associated Lie algebra, they generate the algebra.

3.1. Main result for $\mathbb{H}_n \times \mathbb{R}^k$

We focus on $\mathbb{H}_n \times \mathbb{R}^k$, where $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $\mathbb{H}_n$ is the Heisenberg group. Typically, we denote coordinates in $\mathbb{H}_n$ by (x, y, z) with $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$, and we denote coordinates in $\mathbb{R}^k$ by w. The measure $dx\, dy\, dz\, dw$ is the Lebesgue measure on $\mathbb{R}^{2n+1+k}$. For the vector fields we use in this section the following normalization,

$$X_j = \partial_{x_j} + 2y_j \partial_z\,, \qquad Y_j = \partial_{y_j} - 2x_j \partial_z\,, \qquad W_j = \partial_{w_j}\,.$$

The sub-Laplacian is

$$\Delta^{\mathbb{H}_n \times \mathbb{R}^k} = \sum_{j=1}^{n} (X_j^2 + Y_j^2) + \sum_{i=1}^{k} W_i^2\,.$$

If $\Omega \subset \mathbb{H}_n \times \mathbb{R}^k$ is an open set of finite measure, then the spectrum of the Dirichlet realization of $-\Delta_\Omega^{\mathbb{H}_n \times \mathbb{R}^k}$ is discrete (we denote by $\lambda_\ell(\Omega)$ the non decreasing sequence of its eigenvalues) and we can apply Métivier's theorem. Due to the left invariance, (1.7) takes the form

$$N(\mu, -\Delta_\Omega^{\mathbb{H}_n \times \mathbb{R}^k}) \sim \mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k)\, |\Omega|\, \mu^{\frac{2n+2+k}{2}}\,. \tag{3.1}$$

Later we will give a (relatively) explicit expression for the constant $\mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k)$.

The Faber–Krahn constant $C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k)$ is the largest constant such that for any open $\Omega \subset \mathbb{H}_n \times \mathbb{R}^k$ of finite measure and for any $u \in C_0^\infty(\Omega)$ one has

$$\int_\Omega \left(\sum_{j=1}^{n} ((X_j u)^2 + (Y_j u)^2) + \sum_{i=1}^{k} (W_i u)^2 \right) dx\, dy\, dz\, dw \\ \geq C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k) |\Omega|^{-\frac{2}{2n+2+k}} \int_\Omega u^2\, dx\, dy\, dz\, dw\,. \tag{3.2}$$

Let us set

$$\gamma(\mathbb{H}_n \times \mathbb{R}^k) := \left(C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k) \right)^{-\frac{2n+2+k}{2}} \left(\mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k) \right)^{-1}\,. \tag{3.3}$$

Following the standard proof of Pleijel's theorem, we get

Theorem 3.1. *For any open $\Omega \subset \mathbb{H}_n \times \mathbb{R}^k$ of finite measure,*

$$\limsup_{\ell \to \infty} \frac{\nu_\ell(\Omega)}{\ell} \leq \gamma(\mathbb{H}_n \times \mathbb{R}^k)\,.$$

Here $\nu_\ell(\Omega)$ denotes the maximum number of nodal domains of an eigenfunction corresponding to $\lambda_\ell(\Omega)$.

It remains to give conditions on n and k for which $\gamma(\mathbb{H}_n \times \mathbb{R}^k) < 1$.

We recall that for $n = 0$ this was shown to be the case for $k = 2$ by Pleijel [24] and for general k by Bérard and Meyer [2]. Moreover, Helffer and Persson Sundqvist [14] showed that, for $n = 0$, $k \mapsto \gamma(\mathbb{R}^k)$ is decreasing.

Our main result in this section is the following:

Theorem 3.2.

- *If $k = 0$, then for all $n \geq 4$ one has $\gamma(\mathbb{H}_n) < 1$.*
- *If $k = 1$, then for all $n \geq 2$ one has $\gamma(\mathbb{H}_n \times \mathbb{R}) < 1$.*
- *If $k \geq 2$, then for all $n \geq 1$ one has $\gamma(\mathbb{H}_n \times \mathbb{R}^k) < 1$.*

3.2. The constant in the Weyl asymptotics in the case of $\mathbb{H}_n \times \mathbb{R}^k$

We start with the case $k = 0$. Here we follow explicit computations of Hansson–Laptev [12], providing an alternative proof of Métivier's theorem in the special case of $\mathbb{H}_n$.

A conceptual way of thinking about the Weyl asymptotics is to write them as $\mu \to \infty$ in the form

$$N(\mu, -\Delta_\Omega^{\mathbb{H}_n}) \sim \int_\Omega \mathbf{1}(-\Delta^{\mathbb{H}_n} < \mu)((x,y,z),(x,y,z))\, dx\, dy\, dz\,, \tag{3.4}$$

where $\mathbf{1}(-\Delta^{\mathbb{H}_n} < \mu)((x,y,z),(x,y,z))$ is the on-diagonal spectral density of the sub-Laplacian on all of $\mathbb{H}_n$.

By translation invariance and dilation covariance, we get

$$\mathbf{1}(-\Delta^{\mathbb{H}_n} < \mu)((x,y,z),(x,y,z)) = \mathcal{W}(\mathbb{H}_n)\ \mu^{\frac{Q}{2}}\,, \tag{3.5}$$

and we obtain the above form of the spectral asymptotics. One can then show that

$$\mathcal{W}(\mathbb{H}_n) = \frac{1}{2(n+1)}\frac{1}{(2\pi)^{n+1}} c_n\,, \tag{3.6}$$

where c_n is defined by

$$c_n := \sum_{m\in\mathbb{N}} \binom{m+n-1}{m} \frac{1}{(2m+n)^{n+1}}\,.$$

Note that $\mathcal{W}(\mathbb{H}) = \frac{1}{128}$ and $\mathcal{W}(\mathbb{H}_2) = \frac{1}{48^2\pi}$ and that we can have a more explicit form of $\mathcal{W}(\mathbb{H}_n)$ for $n \leq 13$ using Mathematica.

For general k, we can prove the formula

$$\mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k) = \mathcal{W}(\mathbb{H}_n)\ (4\pi)^{-\frac{k}{2}} \frac{\Gamma(n+2)}{\Gamma(\frac{2n+k+4}{2})}\,. \tag{3.7}$$

3.3. Faber–Krahn and Sobolev inequalities for $\mathbb{H}_n \times \mathbb{R}^k$

We obtain a bound on the Faber–Krahn constant in terms of the (critical) Sobolev inequality on $\mathbb{H}_n \times \mathbb{R}^k$. By definition, $C^{\mathrm{Sob}}(\mathbb{H}_n \times \mathbb{R}^k)$ is the largest constant such that for all $u \in C_0^\infty(\mathbb{H}_n \times \mathbb{R}^k)$

$$\int_{\mathbb{H}_n\times\mathbb{R}^k} \left(\sum_{j=1}^n ((X_j u)^2 + (Y_j u)^2) + \sum_{i=1}^k (W_i u)^2 \right) dx\, dy\, dz\, dw$$
$$\geq C^{\mathrm{Sob}}(\mathbb{H}_n \times \mathbb{R}^k) \left(\int_{\mathbb{H}_n\times\mathbb{R}^k} |u|^{\frac{2(2n+2+k)}{2n+k}}\, dx\, dy\, dz\, dw \right)^{\frac{2n+k}{2n+2+k}}.$$

By an application of Hölder, we obtain

$$C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k) \geq C^{\mathrm{Sob}}(\mathbb{H}_n \times \mathbb{R}^k).$$

An explicit expression for $C^{\mathrm{Sob}}(\mathbb{H}_n)$ was found by Jerison and Lee [16]; for an alternative proof see [11]. We have

$$C^{\mathrm{Sob}}(\mathbb{H}_n) = \frac{4\pi n^2}{(2^{2n} n!)^{\frac{1}{n+1}}}\,. \tag{3.8}$$

3.4. Upper bound for the Pleijel constant $\gamma(\mathbb{H}_n)$

Let us explain our proof of the part of Theorem 3.2 concerning $k = 0$, that is, the assertion that $\gamma(\mathbb{H}_n) < 1$ for $n \geq 4$. From our previous bounds to get

$$\gamma(\mathbb{H}_n) \leq \left(C^{\mathrm{Sob}}(\mathbb{H}_n)\right)^{-n-1} \mathcal{W}(\mathbb{H}_n)^{-1} = \frac{2^n(n+1)!}{n^{2(n+1)}} \frac{1}{c_n} =: \widetilde{\gamma}_n \,. \tag{3.9}$$

Numerics treats the case $n \leq 13$. The second step is to show that $\widetilde{\gamma}_n/\widetilde{\gamma}_{n-1}$ becomes < 1 as $n \to +\infty$. In a third step, the estimate of the remainder in the asymptotics shows that this holds for $n \geq 13$.

3.5. Sobolev for $\mathbb{H}_n \times \mathbb{R}^k$

In order to deal with the case $k \geq 1$, we will derive a lower bound on the constant $C^{\mathrm{Sob}}(\mathbb{H}_n \times \mathbb{R}^k)$. For this, we will need to use a certain Sobolev interpolation inequality on $\mathbb{R}^k$. Assume $2 \leq q < \infty$ if $k \leq 2$ and $2 \leq q \leq \frac{2k}{k-2}$ if $k > 2$, and denote by $C_q^{\mathrm{GN}}(\mathbb{R}^k)$ the largest possible constant in the inequality,

$$\left(\int_{\mathbb{R}^k} |\nabla u|^2\, dw\right)^{\theta} \left(\int_{\mathbb{R}^k} |u|^2\, dw\right)^{1-\theta} \geq C_q^{\mathrm{GN}}(\mathbb{R}^k) \left(\int_{\mathbb{R}^k} |u|^q\, dw\right)^{\frac{2}{q}} , \tag{3.10a}$$

where

$$\theta = k\left(\frac{1}{2} - \frac{1}{q}\right) . \tag{3.10b}$$

For $k = 1$ the explicit value of the constant $C_q^{\mathrm{GN}}(\mathbb{R})$ is known from work of Nagy [22]:

$$C_q^{\mathrm{GN}}(\mathbb{R}) = \left(\frac{Q^Q}{4\,(Q-1)^{Q-1}}\right)^{\frac{1}{Q+1}} \left(\sqrt{\pi}\,\frac{\Gamma(\frac{Q+1}{2})}{\Gamma(\frac{Q+2}{2})}\right)^{\frac{2}{Q+1}} \quad \text{with } q = \frac{2(Q+1)}{Q-1} \,.$$

For $k \geq 2$ its explicit value is not known, but we are still be able to derive some results by modifying the strategy.

Proposition 3.3. *For all $n, k \in \mathbb{N}$, setting $Q = 2n + 2$ and $q = \frac{2(Q+k)}{Q+k-2}$,*

$$C^{\mathrm{Sob}}(\mathbb{H}_n \times \mathbb{R}^k) \geq C_q^{\mathrm{GN}}(\mathbb{R}^k)\ \left(C^{\mathrm{Sob}}(\mathbb{H}_n)\right)^{\frac{Q}{Q+k}}\ \frac{Q+k}{Q^{\frac{Q}{Q+k}}\, k^{\frac{k}{Q+k}}} \,.$$

Using this proposition and Nagy's value for the optimal constant, we can show that the part of Theorem 3.2 concerning $k = 1$, that is, Pleijel's theorem for $\mathbb{H}_n \times \mathbb{R}$ for all $n \geq 3$.

3.6. A second bound on the Faber–Krahn constant

In order to prove the parts of Theorem 3.2 concerning $n = 1$ and $n = 2$, we use a different approach to lower bounds on the Faber–Krahn constant on $\mathbb{H}_n \times \mathbb{R}^k$, which we briefly discuss in this subsection.

We proceed via the isoperimetric constant on $\mathbb{H}_n \times \mathbb{R}^k$. We define[1] the (horizontal) perimeter of a measurable set $E \subset \mathbb{H}_n \times \mathbb{R}^k$ by

$$\mathrm{per}_{\mathbb{H}_n\times\mathbb{R}^k}(E) := \sup\left\{\int_E \left(\sum_{j=1}^n (X_j\phi + Y_j\phi) + \sum_{i=1}^k W_i\phi\right) dx\, dy\, dz\, dw :\ \phi \in C_c^1(\mathbb{H}_n\times\mathbb{R}^k),\ |\phi| \leq 1\right\}.$$

We denote by $I(\mathbb{H}_n\times\mathbb{R}^k)$ the largest constant such that for every set $E \subset \mathbb{H}_n\times\mathbb{R}^k$ of finite measure one has

$$\mathrm{per}_{\mathbb{H}_n\times\mathbb{R}^k}(E) \geq I(\mathbb{H}_n \times \mathbb{R}^k)\, |E|^{\frac{2n+1+k}{2n+2+k}} \,.$$

Using a rearrangement argument, we can prove a lower bound on the Faber–Krahn constant in terms of the isoperimetric constant.

[1]This is the natural extension of the Euclidean definition due to Caccioppoli.

Proposition 3.4. *For $n \geq 0$ and $k \geq 0$, we have*

$$C^{\mathrm{FK}}(\mathbb{H}_n \times \mathbb{R}^k) \geq I(\mathbb{H}_n \times \mathbb{R}^k)^2 \, (2n+2+k)^{-2} j^2_{\frac{2n+k}{2},1} \, .$$

where $j_{\nu,1}$ denotes the first positive zero of the Bessel function J_ν.

Hence it remains to investigate the best known lower bounds to the isoperimetric constant. In the case of $\mathbb{H}_n$ (that is, $k = 0$), this is related to an old conjecture by Pierre Pansu.

Using an analogue of the Sobolev representation formula involving the Green's function on $\mathbb{H}_n$, we find lower bounds on $I(\mathbb{H}_n)$. Independently, we prove a lower bound on $I(\mathbb{H}_n \times \mathbb{R}^k)$ in terms of $I(\mathbb{H}_n)$ and $I(\mathbb{R}^k)$. Combining these bounds and inserting them into Proposition 3.4, we deduce Pleijel's theorem on $\mathbb{H}_n \times \mathbb{R}^k$ where either $n = 1$ and $k \geq 2$ or $n = 2$ and $k \geq 1$.

This approach also allows us to show:

Theorem 3.5. *If Pansu's conjecture holds on $\mathbb{H}_n$, then Pleijel's theorem holds on $\mathbb{H}_n \times \mathbb{R}^k$ for any $k \geq 0$.*

We refer to [9] for more details.

References

[1] I. Androulidakis, R. Juncken & O. Mohsen, "A pseudodifferential calculus for maximally hypoelliptic operators and the Helffer–Nourrigat conjecture", 2022, `https://arxiv.org/abs/2201.12060`.

[2] P. Bérard & D. Meyer, "Inégalités isopérimétriques et applications", *Ann. Sci. Éc. Norm. Supér. (4)* **15** (1982), no. 3, p. 513-541.

[3] J. M. Bony, "Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour des opérateurs elliptiques dégénérés", *Ann. Inst. Fourier* **19** (1969), no. 1, p. 277-304.

[4] R. Courant & D. Hilbert, *Methods of Mathematical Physics: Partial Differential Equations, Vol. 2*, John Wiley & Sons, 2008.

[5] M. Derridj, "Un problème aux limites pour une classe d'opérateurs du second ordre hypoelliptiques", *Ann. Inst. Fourier* **21** (1971), no. 4, p. 99-148.

[6] S. Eswarathasan & C. Letrouit, "Nodal sets of Eigenfunctions of sub-Laplacians", *Int. Math. Res. Not.* **2023** (2023), no. 23, p. 20670-20700.

[7] V. Fischer & M. Ruzhansky, *Quantization on Nilpotent Lie Groups*, Progress in Mathematics, vol. 314, Birkhäuser, 2015.

[8] G. Folland, "Subelliptic estimates and function spaces on nilpotent groups", *Ark. Mat.* **13** (1975), p. 161-207.

[9] R. L. Frank & B. Helffer, "On Courant and Pleijel theorems for sub-Riemannian Laplacians", 2024, `https://arxiv.org/abs/2402.13953`.

[10] ———, "On Courant and Pleijel theorems for sub-Riemannian Laplacians", in *Pseudo-differential operators and related topics. Extended Abstracts PSORT 2024, Trends in Mathematics, Research Perspectives Ghent Analysis and PDE Center* (D. Rottensteiner, M. Ruzhansky & V. Kumar, eds.), Birkhäuser, 2025, p. 9-23.

[11] R. L. Frank & E. H. Lieb, "Sharp constants in several inequalities on the Heisenberg group", *Ann. Math. (2)* **176** (2012), no. 1, p. 349-381.

[12] A. M. Hansson & A. Laptev, "Sharp spectral inequalities for the Heisenberg Laplacian", in *Groups and analysis. The legacy of Hermann Weyl*, London Mathematical Society Lecture Note Series, vol. 354, Cambridge University Press, 2008, p. 100-115.

[13] B. Helffer & J. Nourrigat, *Hypoellipticité Maximale pour des Opérateurs Polynômes de Champs de Vecteurs*, Progress in Mathematics, Birkhäuser, 1985.

[14] B. Helffer & M. P. Sundqvist, “On nodal domains in Euclidean balls”, *Proc. Am. Math. Soc.* **144** (2017), no. 11, p. 4777-4791.

[15] L. Hörmander, “Hypoelliptic second order differential equations”, *Acta Math.* **119** (1967), p. 147-171.

[16] D. Jerison & J. M. Lee, “Extremal for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem”, *J. Am. Math. Soc.* **1** (1988), no. 1, p. 1-13.

[17] C. Léna, “Pleijel’s nodal domain theorem for Neumann and Robin eigenfunctions”, *Ann. Inst. Fourier* **69** (2019), no. 1, p. 283-301.

[18] M. Levitin, D. Mangoubi & I. Polterovich, *Topics in Spectral Geometry*, Graduate Studies in Mathematics, vol. 237, American Mathematical Society, 2023.

[19] A. Menikoff & J. Sjöstrand, “On the eigenvalues of a class of hypoelliptic operators”, *Math. Ann.* **235** (1978), p. 55-85.

[20] G. Métivier, “Fonction spectrale et valeurs propres d’une classe d’opérateurs non elliptiques”, *Commun. Partial Differ. Equations* **1** (1976), p. 467-519.

[21] E. Müller-Pfeifer, “On the number of nodal domains for eigenfunctions of elliptic differential operators”, *J. Lond. Math. Soc. (2)* **31** (1985), p. 91-100.

[22] B. D. S. Nagy, “Über Integralungleichungen zwischen einer Funktion und ihrer Ableitung”, *Acta Sci. Math.* **10** (1941), p. 64-74.

[23] P. Pansu, “An isoperimetric inequality for the Heisenberg group”, *C. R. Math.* **295** (1982), no. 2, p. 127-130.

[24] A. Pleijel, “Remarks on Courant’s nodal theorem”, *Commun. Pure Appl. Math.* **9** (1956), p. 543-550.

[25] L. P. Rothschild, “A criterion for hypoellipticity of operators constructed of vector fields”, *Commun. Partial Differ. Equations* **4** (1979), no. 6, p. 248-315.

[26] L. P. Rothschild & E. M. Stein, “Hypoelliptic differential operators and nilpotent groups”, *Acta Math.* **137** (1979), p. 248-315.

[27] N. T. Varopoulos, “Analysis on nilpotent groups”, *J. Funct. Anal.* **66** (1986), p. 406-431.

[28] Y. Colin de Verdière, L. Hillairet & E. Trélat, “Spectral asymptotics for sub-Riemannian Laplacians, I: Quantum ergodicity and quantum limits in the 3-dimensional contact case”, *Duke Math. J.* **167** (2018), no. 1, p. 109-174.

[29] ———, “Small-time asymptotics of hypoelliptic heat kernels near the diagonal, nilpotentization and related results”, *Ann. Henri Lebesgue* **4** (2021), p. 897-971.

[30] ———, “Spectral asymptotics for sub-Riemannian Laplacians”, 2022, `https://arxiv.org/abs/2212.02920`.

[31] K. Watanabe, “Sur l’unicité du prolongement des solutions des équations elliptiques dégénérées”, *Tôhoku Math. J.* **34** (1982), p. 239-249.

Rupert L. Frank
Mathematisches Institut, Ludwig-Maximilians Universität München
Theresienstr. 39, 80333 München, Germany;
Munich Center for Quantum Science and Technology
Schellingstr. 4, 80799 München, Germany;
Mathematics 253-37, Caltech
Pasadena, CA 91125, USA
r.frank@lmu.de

Bernard Helffer
Laboratoire de Mathématiques Jean Leray
Nantes Université
44 000 Nantes, France
bernard.helffer@univ-nantes.fr