Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Aussois, 19–23 juin 2024

Alberto Enciso

Schiffer-type problems and nonradial stationary Euler flows with compact support

J. É. D. P. (2024), Exposé n° IV, 10 p.

<https://doi.org/10.5802/jedp.685>

Article mis en ligne dans le cadre du
Centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/

RÉSEAU THÉMATIQUE AEDP DU CNRS

Journées *Équations aux dérivées partielles*
Aussois, 19–23 juin 2024
RT AEDP (CNRS)

Schiffer-type problems and nonradial stationary Euler flows with compact support

Alberto Enciso

Problèmes de type Schiffer et écoulements d'Euler stationnaires non radiaux à support compact

Résumé

Nous passons en revue quelques résultats récents sur l'existence d'écoulements d'Euler stationnaires non radiaux à support compact dans le plan. L'approche que nous adoptons repose sur un problème elliptique surdéterminé, inspiré par la conjecture de Schiffer en géométrie spectrale.

Abstract

We review some recent results on the existence of nonradial stationary planar Euler flows with compact support. The approach we take relies on an elliptic overdetermined problem motivated by Schiffer's conjecture in spectral geometry.

1. Introduction

Consider an ideal fluid in equilibrium. Can the velocity of the fluid be identically zero outside a bounded set?

This naive question turns out to be remarkably difficult. In three dimensions, the answer was only discovered a few years ago, when Gavrilov [15] constructed a family of smooth, compactly supported solutions to the stationary incompressible Euler equations

$$v \cdot \nabla v + \nabla p = 0\,, \qquad \operatorname{div} v = 0 \tag{1.1}$$

on $\mathbb{R}^3$. Gavrilov's extremely clever construction was put in a broader context in the paper [6], which also presents illuminating discussions.

In contrast, in two dimensions the existence of compactly supported solutions to (1.1) is elementary. Indeed, writing the velocity field as the perpendicular gradient of a stream function (i.e., $v := \nabla^\perp \psi := (-\partial_2 \psi, \partial_1 \psi)$), the 2d Euler equations take the equivalent form

$$\nabla \Delta \psi \cdot \nabla^\perp \psi = 0\,. \tag{1.2}$$

Therefore, any radially symmetric $\psi \in C_c^\infty(\mathbb{R}^2)$ defines a compactly supported stationary Euler flow. More generally, one can let ψ be supported on a union of pairwise disjoint disks, which we will refer to as a *locally radial* function.

Thus the natural question to ask in the planar case is whether nontrivial stationary Euler flows exist. There are several recent rigidity results [17, 18, 25] that impose strong constraints on

This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme through the grant agreement 862342. It is also partially supported by the grants CEX2023-001347-S, RED2022-134301-T and PID2022-136795NB-I00 funded by MCIN/AEI.
2020 *Mathematics Subject Classification:* 35N25, 35Q31, 35B32.
Keywords: Eigenvalue problems, radial symmetry, Euler equations.

the structure of planar stationary Euler flows in various contexts. In fact, continuous, compactly supported stationary Euler flows that are not locally radial were constructed only recently [16]. These flows are of vortex patch type, so the velocity field is piecewise smooth but not differentiable. The proof is challenging, relying on several insightful observations and a Nash–Moser iteration scheme.

Our approach to stationary Euler flows in two dimensions is based on overdetermined elliptic problems. The reason for which one often resorts to elliptic equations in the study of planar stationary Euler flows is that any solution of a semilinear equation of the form

$$\Delta\psi = g(\psi) \tag{1.3}$$

satisfies (1.2). As a matter of fact, it has been recently shown [9] that all analytic stationary Euler flows on the plane that do not satisfy (1.3) have a translation symmetry. Furthermore, overdetermined boundary conditions arise because, if a continuous vector field $v = \nabla^\perp\psi$ is zero outside a bounded domain $\Omega \subset \mathbb{R}^2$, then the whole gradient $\nabla\psi$ (instead of just the normal derivative $\partial_\nu\psi$) must vanish on $\partial\Omega$.

In this note we will be mostly interested in the case where the function g is linear, that is, $g(\psi) = \lambda\psi$. This situation is of great interest in itself, as the corresponding overdetermined problem is very closely related to one of the most intriguing problems in spectral geometry: the Schiffer conjecture. In his 1982 list of open problems, S.T. Yau stated it as follows [32, Problem 80]:

Conjecture (Schiffer, 1950s)**.** If a nonconstant Neumann eigenfunction u of the Laplacian on a smooth bounded domain $\Omega \subset \mathbb{R}^2$ is constant on the boundary $\partial\Omega$, then u is radially symmetric and Ω is a ball.

This overdetermined problem is closely related to the *Pompeiu problem* [22], an open question in integral geometry with many applications in remote sensing, image recovery and tomography [2, 29, 31]. The Pompeiu problem can be stated as the following inverse problem: Given a bounded domain $\Omega \subset \mathbb{R}^2$, is it possible to recover any continuous function f on $\mathbb{R}^2$ from knowledge of its integral over all the domains that are the image of Ω under a rigid motion? If this is the case, so that the only $f \in C(\mathbb{R}^2)$ satisfying

$$\int_{\mathcal{R}(\Omega)} f(x)\, dx = 0\,, \tag{1.4}$$

for any rigid motion $\mathcal{R}$ is $f \equiv 0$, the domain Ω is said to have the *Pompeiu property*. Squares, polygons, convex domains with a corner, and ellipses have the Pompeiu property, and Chakalov was apparently the first to point out that balls fail to have the Pompeiu property [4, 5, 33]. In 1976, Williams proved [30] that a smooth bounded domain with boundary homeomorphic to a sphere fails to have the Pompeiu property if and only if it supports a nontrivial Neumann eigenfunction which is constant on $\partial\Omega$. Therefore, the Schiffer conjecture and the Pompeiu problem are equivalent for simply connected domains.

Although the Schiffer conjecture is famously open, some partial results are available. It is known that Ω must indeed be a ball under one of the following additional hypotheses:

1. There exists an infinite sequence of orthogonal Neumann eigenfunctions that are constant on $\partial\Omega$, which is connected [2, 3].

2. The third order interior normal derivative of u is constant on $\partial\Omega$, which is connected [21].

3. When Ω is simply connected and u has no saddle points in the interior of Ω [31].

4. If Ω is simply connected and the eigenvalue μ is among the seven lowest Neumann eigenvalues of the domain [1, 8].

5. If the fourth or fifth order interior normal derivative of u is constant on $\partial\Omega$ [19].

It is also known that the boundary of any reasonably smooth domain Ω with the property stated in the Schiffer conjecture must be analytic as a consequence of a result of Kinderlehrer and Nirenberg [20] on the regularity of free boundaries.

In this paper we consider an analog of the Schiffer conjecture in which replace the disk by an annulus. This corresponds to relaxing the hypotheses by allowing the Neumann eigenfunction to be *locally constant* on the boundary, that is, constant on each connected component of $\partial\Omega$. The question is whether, in this case, Ω must necessarily be a ball or an annulus.

Remarkably, most of the rigidity properties that have been proven for the Schiffer conjecture carry over to this weaker problem [11]. First, essentially the same reasoning demonstrates that $\partial\Omega$ must be analytic. Additionally, following the approach of [2, 3], one can show that if there exists an infinite sequence of orthogonal eigenfunctions that are locally constant on the boundary of $\Omega \subset \mathbb{R}^2$, then Ω must be either a disk or an annulus. When the Neumann eigenvalue is sufficiently low, Ω must also be a disk or an annulus, a result in the vein of [1, 8]. And doubly connected domains with this property turn out to be connected with an integral identity somewhat reminiscent of the Pompeiu property, as detailed in [11]. Yet, however, this overdetermined problem does have nontrivial solutions [11]:

Theorem 1. *There exist parametric families of doubly connected bounded domains $\Omega \subset \mathbb{R}^2$ such that the overdetermined eigenvalue problem*

$$\Delta u + \mu u = 0 \quad \text{in } \Omega\,, \qquad \nabla u = 0 \quad \text{on } \partial\Omega\,,$$

admits, for some $\mu \in \mathbb{R}$, a non-radial solution. More precisely, for any large enough integer l and for all s in a small neighborhood of 0, the family of domains $\Omega \equiv \Omega_{l,s}$ is given in polar coordinates by

$$\Omega := \left\{(r,\theta) \in \mathbb{R}^+ \times \mathbb{T} : a_l + s\, b_{l,s}(\theta) < r < 1 + s\, B_{l,s}(\theta)\right\}, \tag{1.5}$$

where $b_{l,s}, B_{l,s}$ are analytic functions on the circle $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ of the form

$$b_{l,s}(\theta) = \alpha_l \cos l\theta + o(1)\,, \qquad B_{l,s}(\theta) = \beta_l \cos l\theta + o(1)\,,$$

where $a_l \in (0,1)$, α_l and β_l are nonzero constants, and where the $o(1)$ terms tend to 0 as $s \to 0$.

Having this result in hand, we can go back to the study of nonradial stationary Euler flows with compact support. This is because a straightforward application of Theorem 1 yields families of nonradial stationary planar Euler flows with compact support that are continuous and piecewise smooth:

Theorem 2. *Let u and Ω be as in Theorem 1. Then the field defined in terms of the stream function*

$$\psi(x) := \begin{cases} u(x)\,, & x \in \Omega, \\ 0 & x \notin \Omega, \end{cases} \tag{1.6}$$

as $v := \nabla^\perp \psi$ is a compactly supported stationary Euler flow of class $C(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \backslash \partial\Omega)$ that is not locally radial.

Proof. Since $\nabla u = 0$ on $\partial\Omega$, $\psi \in C^1(\mathbb{R}^2)$, so the field $v := \nabla^\perp \psi$ is continuous and piecewise smooth. It is obviously divergence free in the sense of distributions because it is a perpendicular gradient.

Let Ω' be the bounded connected component of $\mathbb{R}^2 \backslash \overline{\Omega}$, and let Γ_1, Γ_2 the components of $\partial\Omega$. We can assume $\partial\Omega' = \Gamma_2$. With the constants $c_j := u|_{\Gamma_j}$, we define the pressure

$$p := \begin{cases} 0 & \text{in } \mathbb{R}^2 \backslash \overline{(\Omega \cup \Omega')}, \\ -\frac{1}{2}(|\nabla u|^2 + \mu u^2 - \mu c_1^2) & \text{in } \overline{\Omega}\,, \\ -\frac{1}{2}\mu(c_2^2 - c_1^2) & \text{in } \Omega'\,. \end{cases}$$

To show u satisfies the stationary Euler equation, we simply integrate by parts to show that

$$\begin{aligned} \int_{\mathbb{R}^2} (v_i\, v_j\, \partial_i w_j + p \operatorname{div} w)\, dx &= \int_\Omega (v_i\, v_j\, \partial_i w_j + p \operatorname{div} w)\, dx \\ &= -\int_\Omega (v \cdot \nabla v + \nabla p) \cdot w\, dx = 0\,, \end{aligned}$$

all $w \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. □

It is natural to wonder whether one can refine the argument to construct smooth nonradial stationary Euler flows with compact support that are not radial. This can indeed be done [10], but the proof involves several new ideas and is much more technical. Although we do not aim to cover the proof of this fact, it is worth pointing out that the ideas behind the proof of Theorem 1 serve as a good starting point for this more sophisticated result:

Theorem 3. *For any positive integer k, there exist compactly supported stationary Euler flows of class $C^k(\mathbb{R}^2)$ that are not locally radial.*

The paper is organized as follows. First, in Section 3 we will discuss a direct attempt towards the proof of Theorem 1. By analyzing why it fails, we will obtain valuable intuition. The actual proof of Theorem 1 will be sketched in Section 4. To conclude, in Section 5 we will discuss the main difficulties that appear when one tries to extend these ideas to establish the much harder Theorem 3.

2. The Dirichlet and Neumann spectrum of an annulus

Since eigenfunctions of annuli play a key role in what follows, let us start by introducing some notation. By the scaling properties of the Laplacian, it is natural to fix the outer radius of the annulus to 1 and let the inner radius range from 0 to 1. In polar coordinates $(r,\theta) \in \mathbb{R}^+ \times \mathbb{T}$, let us thus consider the family of annuli

$$\Omega_a := \{a < r < 1\}\,,$$

labeled by the parameter $a \in (0,1)$.

On Ω_a, it is well known that an orthogonal basis of $L^2(\Omega_a, r\,dr\,d\theta)$ consisting of Neumann eigenfunctions is

$$\{\psi^a_{0,n}(r), \psi^a_{l,n}(r)\cos l\theta, \psi^a_{l,n}(r)\sin l\theta : l \geq 1,\ n \geq 0\}\,.$$

The Neumann spectrum of the annulus Ω_a, counting multiplicities, is then $\{\mu_{l,n}(a)\}_{l,n=0}^\infty$. From now on, we will omit the dependence on a whenever no confusion can arise. Moreover, for each ≤ 0, $\{\psi_{l,n}\}_{n=0}^\infty$ is an orthonormal basis of $L^2((a,1), r\,dr)$ consisting of eigenfunctions of the associated radial operator. Thus these functions (whose dependence on a we omit notationally) satisfy the ODE

$$\psi''_{l,n} + \frac{\psi'_{l,n}}{r} - \frac{l^2\psi_{l,n}}{r^2} + \mu_{l,n}\,\psi_{l,n} = 0 \ \text{ in } (a,1)\,, \quad \psi'_{l,n}(a) = \psi'_{l,n}(1) = 0\,, \tag{N^l_a}$$

for some nonnegative constants

$$\mu_{l,0} < \mu_{l,1} < \mu_{l,2} < \cdots < \mu_{l,n} < \cdots$$

tending to infinity as $n \to \infty$.

Likewise, an orthogonal basis of $L^2(\Omega_a, r\,dr\,d\theta)$ consisting of Dirichlet eigenfunctions in the annulus Ω_a is

$$\{\varphi^a_{0,n}(r), \varphi^a_{l,n}(r)\cos l\theta, \varphi^a_{l,n}(r)\sin l\theta\}_{l=1,n=0}^\infty\,.$$

Also omitting the dependence on a, let us record here that the radial eigenfunctions satisfy the ODE

$$\varphi''_{l,n} + \frac{\varphi'_{l,n}}{r} - \frac{l^2\varphi_{l,n}}{r^2} + \lambda_{l,n}\,\varphi_{l,n} = 0 \ \text{ in } (a,1)\,, \quad \varphi_{l,n} = \varphi_{l,n}(1) = 0\,, \tag{D^l_a}$$

for some positive constants

$$\lambda_{l,0} < \lambda_{l,1} < \lambda_{l,2} < \cdots < \lambda_{l,n} < \cdots$$

tending to infinity as $n \to \infty$. The Dirichlet spectrum of Ω_a is therefore $\{\lambda_{l,n}(a)\}_{l,n=0}^\infty$.

3. A first attempt

Theorem 1 relies on a bifurcation argument. Specifically, we will employ the celebrated Crandall–Rabinowitz theorem [7], which is a prime example of a bifurcation result for partial differential equations and provides a convenient tool for demonstrating the emergence of a pitchfork bifurcation in an infinite-dimensional context. It can be stated as follows:

Theorem 4 (Crandall–Rabinowitz)**.** *Consider a C^2 function $F : X \times (0,1) \to Y$, where X and Y are Banach spaces. Assume that:*

1. $F(0,a) = 0$ *for all* $a \in (0,1)$.

2. $\dim\ker(D_vF)_{(0,a_*)} = \operatorname{codim}\operatorname{Ran}(D_vF)_{(0,a_*)} = 1$ *for some* $a_* \in (0,1)$.

3. $(D_vD_aF)_{(0,a_*)}[\ker(D_vF)_{(0,a_*)}] \not\subset \operatorname{Ran}(D_vF)_{(0,a_*)}$.

Then there is a nontrivial C^1 *curve of solutions,* $\{(v_s,a_s) : |s| < s_0\} \subset X\times(0,1)$ *with* $F(v_s,a_s) = 0$*, such that* $a_0 = a_*$ *and* $v_s \neq 0$ *for all* $s \neq 0$.

Remark 5. The model behavior corresponds to the function $F : \mathbb{R}\times(0,1) \to \mathbb{R}$ given by $F(v,a) := (\frac{1}{2} - a - v^2)v$, with $a_* := \frac{1}{2}$.

Our objective now is to see how one can formalize the proof of Theorem 1 using this result. We can use as a starting point is the family of annuli Ω_a and a suitable family of radial Neumann eigenfunctions, which will eventually be $u_a := \psi^a_{0,2}$ in the notation of the last section. We use the notation $\mu_a := \mu_{0,2}(a)$ for corresponding eigenvalue.

To deform the annulus Ω_a, we can consider a "small" function $\mathbf{B} \in C^{2,\alpha}(\mathbb{T},\mathbb{R}^2)$, whose components we will denote by $\mathbf{B} = (b,B)$. Here α is some fixed number in the interval $(0,1)$. The corresponding deformed annulus is

$$\Omega^{\mathbf{B}}_a := \{a + b(\theta) < r < 1 + B(\theta)\}\,. \tag{3.1}$$

For small enough $\mathbf{B}$, the implicit function theorem guarantees the existence of a Neumann eigenfunction $u^{\mathbf{B}}_a(r,\theta)$ on $\Omega^{\mathbf{B}}_a$, with eigenvalue $\mu^{\mathbf{B}}_a$, which depend continuously on $\mathbf{B}$ and tend to (u_a,μ_a) as $\mathbf{B} \to 0$ (in a suitable sense).

To ensure that $u^{\mathbf{B}}_a$ is locally constant on the boundary of $\partial\Omega^{\mathbf{B}}_a$, we can look for zeros of the function

$$F(\mathbf{B},a) := \big(u^{\mathbf{B}}_a(a + b(\cdot),\cdot)),\, u^{\mathbf{B}}_a(1 + B(\cdot),\cdot))\big)\,.$$

By elliptic regularity, this maps an open subset of $C^{2,\alpha}(\mathbb{T},\mathbb{R}^2)$ to $C^{2,\alpha}(\mathbb{T},\mathbb{R}^2)$ (but no better).

We certainly have $F(0,a) = 0$ for all $a \in (0,1)$. To verify the remaining hypotheses of the Crandall–Rabinowitz theorem, we need to compute the derivative of F. This is fairly straightforward: one can readily see that there are explicit nonzero constants $c_j(a)$ such that

$$(D_{\mathbf{B}}F)_{(0,a)}\mathbf{W} = \big(c_1(a)\psi^a_{\mathbf{W}}(a,\cdot),\, c_2(a)\psi^a_{\mathbf{W}}(1,\cdot)\big)\,,$$

where $\psi^a_{\mathbf{W}}(r,\theta)$ is the only solution to the Neumann problem

$$\Delta\psi^a_{\mathbf{W}} + \mu_a\psi^a_{\mathbf{W}} = 0 \quad \text{in } \Omega_a\,, \qquad \partial_r\psi^a_{\mathbf{W}}(r,\theta) = \begin{cases} W_1(\theta) & \text{on } r = a\,, \\ W_2(\theta) & \text{on } r = 1 \end{cases}$$

and where we have written $\mathbf{W} =: (W_1,W_2)$.

This implies that $(D_{\mathbf{B}}F)_{(0,a)}$ behaves essentially like the Neumann-to-Dirichlet map of the domain, so in particular it maps $C^{2,\alpha}(\mathbb{T},\mathbb{R}^2) \to C^{3,\alpha}(\mathbb{T},\mathbb{R}^2)$. This is a serious issue, which prevents us from applying the Crandall–Rabinowitz theorem (or any other bifurcation result) because $\operatorname{Ran}(D_{\mathbf{B}}F)_{(0,a_*)} \subset C^{2,\alpha}(\mathbb{T},\mathbb{R}^2)$ has infinite codimension. This sort of difficulties, which are usually called *loss of derivatives* in the context of bifurcation theory, can sometimes be circumvented by means of a Nash–Moser iteration scheme. But this is indeed a bad case of loss of derivatives, which we do not know how to overcome by brute force.

In a way, what one should learn from this failure is that there are fundamental (as opposed to merely technical) reasons to never take bifurcation for granted. In fact, many important classes of overdetermined problems are known to be *rigid* in the sense that there are no nontrivial solutions, see for instance to [23, 26, 28]. The deeply geometrical interplay between rigidity and flexibility is perhaps the most distinctive feature of the study of overdetermined problems, and underlies Schiffer's conjecture about Neumann eigenfunctions. As a prime example of this dichotomy, note that while Serrin's symmetry result [24, 26] ensures that the only positive solutions to many overdetermined problems on a bounded domain of $\mathbb{R}^n$ are radially symmetric, nontrivial solutions do bifurcate from radially symmetric ones in the case of periodic unbounded domains [12, 13].

From a conceptual point of view, our key contribution is to identify a novel geometric setting in which our overdetermined problem exhibits some flexibility: annular domains in the plane. This is the first flexibility result for an eigenvalue problem under overdetermined boundary conditions in bounded domains of the Euclidean space. Looking for nontrivial solutions in this setting involves a leap of faith: this kind of domains were completely uncharted territory in the context of Schiffer-type problems and, contrarily to the kind of domains considered in [14], there were no indications that flexibility was to be expected. What is known, in fact, is that annular domains satisfy some fundamental partial rigidity properties [23] that do not hold in the case of domains in the cylinder or large domains in the sphere.

Although we eventually show that the new geometric setting of planar annuli has good flexibility properties, the strong rigidity properties it nonetheless exhibits turn out to be very important too. The "partial rigidity" of the annular domains we take in our paper is reflected in the strong rigidity properties of Neumann eigenfunctions on bounded Euclidean domains that are locally constant on the boundary: morally, this is why the problem we solve has the same known rigidity properties as the Schiffer conjecture.

In any case, to overcome the problem of loss of derivatives and show that one can indeed bifurcate from annuli, we need to develop a different approach to the problem. This will be done in the next section.

4. Sketch of the proof of Theorem 1

Let us go back to Equation (3.1). We will now proceed by mapping the deformed domain $\Omega_a^{\mathbf{B}}$ to the fixed annulus $\Omega_{1/2} := \{\frac{1}{2} < r < 1\}$ by means of the diffeomorphism

$$\Phi_a^{\mathbf{B}} : \Omega_{1/2} \ni (R, \theta) \mapsto (r, \theta) \in \Omega_a^{\mathbf{B}}$$

defined by

$$r := a + (1 - a + B(\theta))(2R - 1) + 2(1 - R)b(\theta)\,.$$

We will denote the nontrivial component of the diffeomorphism by

$$\Phi_a^{\mathbf{B}}(R, \theta) =: (\Phi_a^{\mathbf{B},0}(R, \theta), \theta)\,.$$

As we had anticipated, we will bifurcate from the family of radial Neumann eigenfunctions $\psi_{0,2}^a(r)$. Contrary to what happens in the case of periodic domains discussed above, $\psi_{0,1}^a$ cannot work because of a symmetry result due to Reichel [23]. When we pull these eigenfunctions back to the fixed annulus, we thus obtain the family of radial functions

$$\overline{\psi}_a(R) := \psi_{0,2}^a[\Phi_a^{0,0}(R, 0)]\,.$$

Our basic unknown is not the "boundary deformation" $\mathbf{B}(\theta) \in C^{2,\alpha}(\mathbb{T}, \mathbb{R}^2)$, but a "Dirichlet" function $v(R, \theta) \in C^{2,\alpha}_{\mathrm{D}}(\Omega_{1/2})$, where

$$C^{2,\alpha}_{\mathrm{D}}(\Omega_{1/2}) := \left\{ v \in C^{2,\alpha}(\Omega_{1/2}) : v|_{\partial\Omega_{1/2}} = 0 \right\}.$$

This function obviously contains much more information than a function $\mathbf{B} \in C^{2,\alpha}(\mathbb{T}, \mathbb{R}^2)$, so we use v to parametrize both the boundary, by means of a map

$$C^{2,\alpha}_{\mathrm{D}}(\Omega_{1/2}) \ni v \mapsto \mathbf{B}_v \in C^{2,\alpha}(\mathbb{T}, \mathbb{R}^2)\,,$$

and to correct the eigenfunction (pulled back to the fixed annulus) as $\overline{\psi}_a + w_v$, where w_v is a "Dirichlet–Neumann" function. More precisely, we define a map

$$C^{2,\alpha}_{\mathrm{D}}(\Omega_{1/2}) \ni v \mapsto w_v \in C^{2,\alpha}_{\mathrm{DN}}(\Omega_{1/2}) := \left\{ w \in C^{2,\alpha}_{\mathrm{D}}(\Omega_{1/2}) : \partial_r w|_{\partial\Omega_{1/2}} = 0 \right\}.$$

The specific expressions for these maps are not particularly illuminating: writing $\mathbf{B}_v = (b_v, B_v)$, one eventually takes

$$\begin{aligned} b_v(\theta) &:= -2(1-a)(\overline{\psi}_a''(\tfrac{1}{2}))^{-1}\partial_R v\big(\tfrac{1}{2}, \theta\big)\,, \\ B_v(\theta) &:= -2(1-a)(\overline{\psi}_a''(1))^{-1}\partial_R v(1, \theta)\,, \\ w_v(R, \theta) &:= v(R, \theta) - \frac{\overline{\psi}_a'(R)}{2(1-a)}\Big[2(1-R)b_v(\theta) + (2R-1)B_v(\theta)\Big]\,. \end{aligned}$$

The function that we consider in the bifurcation argument is then

$$F(v, a) := \left\{ [\Delta + \mu_{02}(a)] \left[\psi_{02}^a + w_v \circ (\Phi_a^{\mathbf{B}_v})^{-1} \right] \right\} \circ \Phi_a^{\mathbf{B}_v}\,. \tag{4.1}$$

Thus $F(v, a) = 0$ if and only $u := \psi_{02}^a + w_v \circ (\Phi_a^{\mathbf{B}_v})^{-1}$ is an eigenfunction with eigenvalue $\mu_{0,2}(a)$ on the domain $\Omega_a^{\mathbf{B}_v}$. Also, since w_v has zero Dirichlet and Neumann boundary traces, ∇u is identically zero on $\partial\Omega_a^{\mathbf{B}_v}$.

To use the Crandall–Rabinowitz theorem, one must now analyze the derivative of this map. It is not hard to show that, in fact,

$$(D_v F)_{(0,a)} w = \big\{ [\Delta + \mu_{02}(a)] \big[w \circ (\Phi_a^0)^{-1} \big] \big\} \circ \Phi_a^0 \,. \tag{4.2}$$

Using v as the main unknown instead of $\mathbf{B}$, by itself, does *not* solve the problem of loss of derivatives; indeed, a moment's thought reveals that the range of (4.2) has infinite codimension with the obvious choice of Hölder spaces. However, as first shown by Fall, Minlend and Weth in [14], one can try to compensate this loss of derivatives using *anisotropic* Banach spaces. The basic idea is that, since the Dirichlet-to-Neumann map essentially originates from a radial derivative, including an additional radial derivative in the functional setting may provide some additional control. This is the content of the following key lemma [11]:

Lemma 6 (Fredholmness)**.** *For any integer $l \geq 3$ and any $a \in (0,1)$, $(D_v F)_{(0,a)} : \mathcal{X} \to \mathcal{Y}$ is Fredholm of index* 0*.*

Here and in what follows, $\mathcal{X}$ and $\mathcal{Y}$ are the Banach spaces

$$\mathcal{X} := \{ u \in C^{2,\alpha}_{\mathrm{D}}(\Omega_{1/2}) : \partial_R u \in C^{2,\alpha}(\Omega_{1/2}) \} / \mathbb{Z}_l \,, \qquad \mathcal{Y} := \big(C^{1,\alpha}(\Omega_{1/2}) + C^{0,\alpha}_{\mathrm{D}}(\Omega_{1/2}) \big) / \mathbb{Z}_l \,,$$

endowed with their natural norms. Here $l \geq 3$ is certain integer, and the quotient by the discrete group $\mathbb{Z}_l$ means that we are only considering functions invariant under a $\mathbb{Z}_l$-dihedral symmetry (that is, functions which are invariant under the action of the isometry group of an l-sided regular polygon).

We can now get back to the hypotheses of the Crandall–Rabinowitz theorem. Since we have started with a family of radial eigenfunctions, we certainly have $F(0,a) = 0$, and Lemma 6 ensures that $\dim \ker (D_v F)_{(0,a)} = \operatorname{codim} \operatorname{Ran} (D_v F)_{(0,a)}$ for all $a \in (0,1)$. Thus we only need to verify the existence of some $a_{l,*} \in (0,1)$ for which

$$\dim \ker (D_v F)_{(0,a_{l,*})} = 1 \tag{4.3}$$

and the transversality condition

$$(D_v D_a F)_{(0,a_{l,*})} [\ker (D_v F)_{(0,a_{l,*})}] \not\subset \operatorname{Ran} (D_v F)_{(0,a_{l,*})} \tag{4.4}$$

That we can indeed satisfy these conditions (at least, provided that l is large enough) is the content of the following two technical lemmas, which are the workhorse behind Theorem 1. Although we will not discuss the proofs, we must point out that checking these conditions boils down to the analysis of Dirichlet and Neumann eigenfunctions on annuli.

Lemma 7 (Eigenvalue crossing)**.** *For all $a_{l,*} \in (0,1)$, the function $F : \mathcal{X} \times (0,1) \to \mathcal{Y}$ satisfies the kernel condition* (4.3) *if and only if the eigenvalues of the annulus Ω_a satisfy*

$$\mu_{0,2}(a_{l,*}) = \lambda_{l,0}(a_{l,*}) \neq \lambda_{ml,n}(a_{l,*}) \qquad \forall \ (m,n) \neq (1,0) \,.$$

Moreover, for each $l \geq 4$, there exists some $a_{l,*}$ which satisfies this condition.

Lemma 8 (Transversality)**.** *Let $a_{l,*} \in (0,1)$ be as in Lemma 7. The transversality condition* (4.4) *holds if and only if*

$$\mu'_{0,2}(a_{l,*}) \neq \lambda'_{l,0}(a_{l,*}) \,,$$

where the primes denote the derivative of the eigenvalue with respect to the parameter. Moreover, for every large enough l, there exists some

$$a_{l,*} = 1 - \frac{\sqrt{3}\pi}{l} + O(l^{-2}) \,, \tag{4.5}$$

for which this condition is satisfied.

Choosing l large enough, Theorem 1 essentially follows as a consequence of the Crandall–Rabinowitz theorem and Lemmas 6–8. Since the inner radius in (4.5) is very close to 1, the qualitative picture one obtains is that the nonradial domains bifurcate from thin annuli. (This is not completely accurate, though, since one can prove an analogous result for $l = 4$ and the annulus $\Omega_{a_{4,*}}$ is not particularly thin.)

5. Some comments on Theorem 3

Theorem 3 also relies on a bifurcation argument: nonradial stationary flows with compact support branch out from a suitably chosen family of radially symmetric, compactly supported flows. These radial flows are given by the perpendicular gradient a one-parameter family of radial stream functions ψ_a which are supported on certain annuli Ω_a and which vanishes on $\partial\Omega_a$ to a high order $m \geq 1$. To implement a bifurcation argument, one assumes that $u_a := \psi_a|_{\Omega_a}$ satisfies certain differential equation in Ω_a. The gist of the argument is to show that, for some value of the parameter a, one can consider a smooth small nonradial deformation u of u_a which satisfies the same equation on a slightly deformed domain $\widetilde{\Omega}$. It is then easy to see that if u also vanishes on the boundary of the deformed domain to order m, just as in Theorem 2, then the vector field defined by $v := \nabla^\perp \psi$ with the stream function defined by (1.6) is of class $C^{m-1}(\mathbb{R}^2)$. The equation satisfied by u must therefore ensure that v is a stationary solution to the Euler equations (1.3).

For a bifurcation argument, it is known that one cannot directly use the Euler equation (1.2) because its linerization is a completely unmanageable operator with an infinite-dimensional kernel. Vortex patch solutions are not C^1, so it is not clear how one could adapt the strategy of [16]. Also, a variation of Gavrilov's construction can only give locally radial solutions of compact support [27]. One would naively think that the elliptic equation (1.3) should be the way to go, but in fact this is not true: one can show [10] that any compactly supported stationary flow of class C^2 whose stream function satisfies a semilinear equation of the form (1.3) must be locally radial.

Hence, in this problem, even the choice of the equation one should consider is rather nontrivial. For us, the starting point of the paper is the construction of a *non-autonomous* nonlinearity f_a enabling us to effectively use the equation

$$\Delta u_a + f_a(|x|, u_a) = 0\,, \tag{5.1}$$

to construct compactly supported solutions. To our best knowledge, this is the first time that non-autonomous elliptic equations have been used for a similar purpose.

Still, passing from this rough idea to an actual proof is remarkably hard. This is because the above outline does not address the three essential difficulties that the problem entails:

1. The radial stream function u_a that we consider is a positive solution to an equation of the form (5.1) on an annulus Ω_a, which vanishes to m-th order on $\partial\Omega_a$. The deformation u will satisfy the same equation on the deformed domain $\widetilde{\Omega}$ and tend to zero as $\widetilde{\rho}^m$ on $\partial\widetilde{\Omega}$. However, even with our well chosen nonlinearity, if u is a nonradial solution to (5.1) in $\widetilde{\Omega}$, $\nabla^\perp u$ does *not* satisfy the stationary Euler equations: this is only true if u is close to u_a in a certain sense.

2. Suppose that the function u vanishes to order $m \geq 3$ on $\partial\widetilde{\Omega}$, where $\widetilde{\Omega}$ can be thought of as a slightly deformed annulus. For concreteness, we can think that $u = \widetilde{\rho}^m U$, where U is a smooth function that does not vanish on $\partial\widetilde{\Omega}$ and where $\widetilde{\rho}$ is a boundary defining function, that is, a positive function on $\widetilde{\Omega}$ that vanishes on $\partial\widetilde{\Omega}$ exactly to first order. Since Δu goes like $\widetilde{\rho}^{m-2}$ near $\partial\widetilde{\Omega}$, the nonlinearity $f(r,t)$ can only be Hölder continuous in the second variable and must behave like $|t|^{1-\frac{2}{m}}$ near 0. Thus, the linearization of this equation, which one expects to encounter in any bifurcation argument, will be controlled by an operator of the form
$$L = -\Delta + \frac{c}{\widetilde{\rho}^2} \tag{5.2}$$
for some nonzero constant c (modulo terms that are less singular). The potential term is then critically singular (i.e., it scales like the Laplacian), so it cannot be treated as a perturbation of Δ: a new set of estimates is necessary.

3. To control the deformation of the domain, one must compensate a serious loss of derivatives similar to the one encountered in the proof of Theorem 1 that we have sketched.

The complexity of the problem resides on the fact that these difficulties are strongly interrelated. In particular, while we eventually succeed in overcoming it using anisotropic spaces $\mathcal{X}$, $\mathcal{Y}$, here these spaces must be modeled not on standard Hölder or Sobolev spaces but on weighted spaces that effectively capture the sharp regularity properties of operators of the form (5.2). For this, we cannot use off-the-shelf spaces and estimates; in fact, we need to develop from scratch a sharp regularity theory for this kind of operators that is adapted to the situation at hand. This is because, in the analysis of the linearized operator L, and we crucially need to control functions that are critical in that their asymptotic behavior at the boundary is given by an indicial root of the operator.

Although these difficulties can ultimately be circumvented, they make the proof of Theorem 3 considerably less straightforward than that of Theorem 1. Details and further discussion can be found in [10].

References

[1] P. Aviles, "Symmetry theorems related to Pompeiu's problem", *Am. J. Math.* **108** (1986), no. 5, p. 1023-1036.

[2] C. Berenstein, "An inverse spectral theorem and its relation to the Pompeiu problem", *J. Anal. Math.* **37** (1980), p. 128-144.

[3] C. Berenstein & P. Yang, "An inverse Neumann problem", *J. Reine Angew. Math.* **382** (1987), p. 1-21.

[4] L. Brown, B. M. Schreiber & B. A. Taylor, "Spectral synthesis and the Pompeiu problem", *Ann. Inst. Fourier* **23** (1973), p. 125-154.

[5] L. Chakalov, "Sur un problème de D. Pompeiu", *Ann. Univ. Sofia, Fac. Sci., Livre I* **40** (1944), p. 1-14.

[6] P. Constantin, J. La & V. Vicol, "Remarks on a paper by Gavrilov: Grad–Shafranov equations, steady solutions of the three dimensional incompressible Euler equations with compactly supported velocities, and applications", *Geom. Funct. Anal.* **29** (2019), p. 1773-1793.

[7] M. Crandall & P. Rabinowitz, "Bifurcation from simple eigenvalues", *J. Funct. Anal.* **8** (1971), p. 321-340.

[8] J. Deng, "Some results on the Schiffer conjecture in $\mathbb{R}^2$", *J. Differ. Equations* **253** (2012), no. 8, p. 2515-2526.

[9] T. M. Elgindi, Y. Huang, A. R. Said & C. Xie, "A Classification Theorem for Steady Euler Flows", 2024, `https://arxiv.org/abs/2408.14662`.

[10] A. Enciso, A. Fernández & D. Ruiz, "Smooth nonradial stationary Euler flows on the plane with compact support", 2024, `https://arxiv.org/abs/2406.04414`.

[11] A. Enciso, A. Fernández, D. Ruiz & P. Sicbaldi, "A Schiffer-type problem for annuli with applications to stationary planar Euler flows", to appear in *Duke Math. J.*, 2023, `https://arxiv.org/abs/2309.07977`.

[12] M. M. Fall, I. A. Minlend & T. Weth, "Unbounded periodic solutions to Serrin's overdetermined boundary value problem", *Arch. Ration. Mech. Anal.* **233** (2017), no. 2, p. 737-759.

[13] ———, "Serrin's overdetermined problem on the sphere", *Calc. Var. Partial Differ. Equ.* **57** (2018), no. 1, article no. 3 (24 pages).

[14] ———, "The Schiffer problem on the cylinder and on the 2-sphere", to appear in *J. Eur. Math. Soc.*, 2023, `https://arxiv.org/abs/2303.17036`.

[15] A. V. Gavrilov, "A steady Euler flow with compact support", *Geom. Funct. Anal.* **29** (2019), p. 190-197.

[16] J. Gómez-Serrano, J. Park & J. Shi, "Existence of non-trivial non-concentrated compactly supported stationary solutions of the 2D Euler equation with finite energy", to appear in *Mem. Am. Math. Soc.*, 2024.

[17] J. Gómez-Serrano, J. Park, J. Shi & Y. Yao, "Symmetry in stationary and uniformly-rotating solutions of active scalar equations", *Duke Math. J.* **170** (2021), no. 13, p. 2957-3038.

[18] F. Hamel & N. Nadirashvili, "Circular flows for the Euler equations in two-dimensional annular domains, and related free boundary problems", *J. Eur. Math. Soc.* **25** (2023), p. 323-368.

[19] B. Kawohl & M. Lucia, "Some results related to Schiffer's problem", *J. Anal. Math.* **142** (2020), p. 667-696.

[20] D. Kinderlehrer & L. Nirenberg, "Regularity in free boundary problems", *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4)* **4(2)** (1977), p. 373-391.

[21] G. Liu, "Symmetry results for overdetermined boundary value problems of nonlinear elliptic equations", *Nonlinear Anal., Theory Methods Appl.* **72** (2010), p. 3943-3952.

[22] D. Pompeiu, "Sur certains systèmes d'équations linéaires et sur une propriété intégrale des fonctions de plusieurs variables", *C. R. Acad. Sci., Ser. I* **188** (1929), p. 1138-1139.

[23] W. Reichel, "Radial symmetry by moving planes for semilinear elliptic BVPs on annuli and other non-convex domains", in *Progress in Partial Differential Equations: Elliptic and Parabolic Problems*, Pitman Research Notes in Mathematics Series, vol. 325, Longman Scientific & Technical, 1995, p. 164-182.

[24] D. Ruiz, "Nonsymmetric sign-changing solutions to overdetermined elliptic problems in bounded domains", to appear in *J. Eur. Math. Soc.*, 2022, `https://arxiv.org/abs/2211.14014`.

[25] ———, "Symmetry Results for Compactly Supported Steady Solutions of the 2D Euler Equations", *Arch. Ration. Mech. Anal.* **247** (2023), no. 3, article no. 40 (25 pages).

[26] J. Serrin, "A symmetry theorem in potential theory", *Arch. Ration. Mech. Anal.* **43** (1971), p. 304-318.

[27] V. A. Sharafutdinov, "Two-dimensional Gavrilov flows", *Sib. Èlektron. Mat. Izv.* **21** (2024), no. 1, p. 247-258.

[28] B. Sirakov, "Symmetry for exterior elliptic problems and two conjectures in potential theory", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **18** (2001), no. 2, p. 135-156.

[29] K. T. Smith, D. C. Solmon & S. L. Wagner, "Practical and mathematical aspects of the problem of reconstructing objects from radiographs", *Bull. Am. Math. Soc.* **83** (1977), p. 1227-1270.

[30] S. A. Williams, "A partial solution to the Pompeiu problem", *Math. Ann.* **223** (1976), no. 2, p. 183-190.

[31] N. B. Willms & G. M. L. Gladwell, "Saddle points and overdetermined problems for the Helmholtz equation", *Z. Angew. Math. Phys.* **45** (1994), p. 1-26.

[32] S.-T. Yau, "Problem section", in *Seminars on Differential Geometry*, Annals of Mathematics Studies, vol. 102, Princeton University Press, 1982, p. 669-706.

[33] L. Zalcman, "A bibliographic survey of the Pompeiu problem", in *Approximation by Solutions of Partial Differential equations* (D. Kluwer, ed.), Pitman Res. Notes, 1992, p. 185-194.

Alberto Enciso
Instituto de Ciencias Matemáticas
Consejo Superior de Investigaciones Científicas
28049 MADRID
SPAIN
`aenciso@icmat.es`