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Regularization by noise for some nonlinear dispersive PDEs

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Abstract

In the context of ODEs or transport PDEs, there are examples where adding a rough stochastic perturbation to the equation at hand actually improves the well-posedness theory. In these notes, we review some results showing how a distributional modulation of the dispersion can also produce a regularization by noise effect for a rather large class of nonlinear dispersive PDEs.

1. Introduction

In this note, we present some results regarding the Cauchy problem for some nonlinear dispersive PDEs with modulated dispersion:

$$\partial_t u = \frac{\mathrm{d}W}{\mathrm{d}t}\mathcal{L}u + \mathcal{N}(u),\ t \in \mathbb{R},\ x \in \mathcal{M}, \tag{1.1}$$

where $\mathcal{M}$ is some spatial domain, the operator $\mathcal{L}$ is skew-adjoint, $\mathcal{N}$ is a nonlinearity, and $W_t : \mathbb{R} \to \mathbb{R}$ is a given continuous but not differentiable function. Thus the term $\frac{\mathrm{d}W}{\mathrm{d}t}$ in (1.1) above has to be interpreted in the sense of distributions.

When $\mathcal{L} = i\Delta$ and $\mathcal{N}(u) = \pm i|u|^{p-1}u$ for some $p > 1$, (1.1) corresponds to the non-linear Schrödinger (NLS) equation with modulated dispersion

$$i\partial_t u + \frac{\mathrm{d}W}{\mathrm{d}t}\Delta u \pm |u|^{p-1}u = 0. \tag{1.2}$$

In case $\mathcal{M} = \mathbb{R}$ and W_t is a Brownian motion, (1.2) was proposed by de Bouard and Debussche [1] as an effective model for the propagation of a signal in an optical fibre with dispersion management.

While the well-posedness of the usual NLS equation

$$i\partial_t u + \Delta u \pm |u|^{p-1}u = 0 \tag{1.3}$$

has been extensively studied on various domains $\mathcal{M}$, the irregularity of W_t in (1.2) raises the question of its effect on the well-posedness theory of this equation. In particular, as far as rough *spatially* dependent coefficients are concerned, e.g. $\sum_{j,k} \partial_j(a(x)\partial_k)$ with rough a in place of Δ, it is known [21] that the roughness of the spatial coefficients can alter the Strichartz estimates on the Schrödinger semi-group, hence an alteration of the well-posedness theory for the corresponding nonlinear equation compared to the case of constant coefficients.

However, there is now a large body of literature dealing with singular stochastic nonlinear heat or wave equations [2, 16], or random data nonlinear dispersive PDEs [4, 11], for which stochastic source terms or random initial data of *super-critical* regularity can be dealt with. In these works, one usually treats the stochastic source terms/rough initial data *perturbatively* with respect to the deterministic well-posedness theory of the nonlinear equation, thus also perturbatively with respect to the linear dynamics of the equation. This is done by cooking up some appropriate ansatz for a solution, with a first part consisting of rather explicit objects built on the stochastic

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source terms/random initial data, and a remainder term which solves a suitable perturbation of the equation so as to fall under the scope of the deterministic sub-critical well-posedness theory.

In [1], de Bouard and Debussche showed that despite the irregularity of the Brownian motion W_t, the modulated NLS equation (1.2) is indeed still locally well-posed in the same (sub-critical) range as the usual NLS equation (1.3). On the other hand, the "noise" term in (1.1) above being in front of the dispersion, we expect that it will not act perturbatively, but instead affect also the linear dynamics. Actually, there are several examples of ODEs and PDEs for which adding a non-perturbative rough stochastic term in the equation can actually *improve* the well-posedness of the equation: this is referred to as a *regularization by noise* phenomenon. In recent years, it has become clearer that in many cases this phenomenon is actually entirely deterministic, namely that it comes solely from the irregularity property of W_t but not of its stochastic properties, which is referred to as *noiseless* regularization by noise. The question we address in this note is therefore that of observing such a phenomenon for some nonlinear dispersive PDEs with modulated dispersion such as (1.1), under some appropriate assumptions on the dispersion $\mathcal{L}$ and the nonlinearity $\mathcal{N}$.

Although we can formulate an abstract condition on $\mathcal{L}$ and $\mathcal{N}$ for which this phenomenon indeed occurs, here we will illustrate our results on two toy-models: the periodic fractional Korteweg - de Vries (KdV) equation

$$\partial_t u + \partial_x D^\alpha u + u\partial_x u = 0, \tag{1.4}$$

corresponding to $\mathcal{M} = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\mathcal{L} = \partial_x D^\alpha$ for some $\alpha > 0$, and $\mathcal{N}(u) = u\partial_x u$, and the periodic Wick-ordered fractional cubic NLS

$$i\partial_t u + D^\alpha u \pm \big(|u|^2 - 2\|u\|^2_{L^2_x}\big)u = 0, \tag{1.5}$$

corresponding to $\mathcal{M} = \mathbb{T}$, $\mathcal{L} = iD^\alpha$ for some $\alpha > 2$, and $\mathcal{N}(u) = \big(|u|^2 - \frac{1}{2\pi}\|u\|^2_{L^2_x}\big)u$.

After briefly reviewing the literature on regularization by noise in Section 2, we present in Section 3 the underlying mechanisms for the current well-posedness theory for the noiseless equations (1.3), (1.4), and (1.5). Then in Section 4 we discuss the regularization by noise phenomenon for the modulated equations (1.1) corresponding to (1.3)–(1.4)–(1.5). We present some further interesting directions in Section 5.

2. Regularization by noise phenomena

Regularization by noise in the context of stochastic differential equations has been observed since the 80's. For example, consider a general ODE in integral form:

$$x(t) = x_0 + \int_0^t b(x(s))\,\mathrm{d}s. \tag{2.1}$$

If b is locally Lipschitz continuous, then (2.1) has a unique continuous solution for any given initial data x_0, while if b is only continuous, one has existence of a solution to (2.1) but loses uniqueness in general.

However, Veretennikov [23], and Krylov and Röckner [17], showed for example that by adding a rough stochastic source term:

$$x(t) = x_0 + \int_0^t b(x(s))\,\mathrm{d}s + W_t, \tag{2.2}$$

with W_t a Brownian motion, then one can recover existence and uniqueness for b being merely bounded and measurable.

A similar regularization by noise for linear transport PDEs with rough drifts was observed in [13] under the adjunction of a multiplicative noise. Some other results studied stochastic scalar conservation laws, stochastic Hamilton–Jacobi equations, or stochastic heat equation with multiplicative noise.

All these results were exploiting the stochastic nature of the perturbation W_t, for example through the use of tools from Itô calculus. Then Davie [9] proved a *path-wise* analogue of the results of [17, 23] for stochastic differential equations, namely that for (2.1) with a merely bounded measurable vector field b, for almost every given trajectory of the Brownian motion W_t, there is indeed a unique solution to (2.1). In particular this result relies more on the properties of sample paths of W_t and less on stochastic analysis.

Then Catellier and Gubinelli [5] observed that this latter result indeed relied only on the irregularity of W_t and not its randomness, provided that one captures the former appropriately. Indeed, the point of view of [5] was to rewrite the perturbed ODE (2.2) as

$$y(t) = x_0 + \int_0^t b\big(y(s) + W_s\big)\,\mathrm{d}s, \quad y(t) := x(t) - W_t. \tag{2.3}$$

Attempting to solve (2.3) through a Picard iteration, one finds the first order approximation

$$y(t) \approx x_0 + \int_0^t b\big(x_0 + W_s\big)\,\mathrm{d}s. \tag{2.4}$$

Then, mimicking the layer cake representation, we can rewrite (2.4) as

$$y(t) \approx x_0 + \int_{\mathbb{R}} b\big(x_0 + z\big)\mathrm{d}\mu_{[0;t]}(z) = x_0 + \big(b * \mu_{[0;t]}(-\,\cdot\,)\big)(x_0), \tag{2.5}$$

where for an interval $I \subset \mathbb{R}$, μ_I is the *occupation measure* of W_t defined as

$$\mu_I(A) := \int_I \mathbf{1}_A(W_t)\,\mathrm{d}t = \mathrm{Leb}\big(\{t \in I,\ W_t \in A\}\big),\ A \in \mathcal{B}(\mathbb{R}). \tag{2.6}$$

Going from (2.4) to (2.5) is the so-called *occupation time formula*, which is better known in the context of stochastic processes W_t, for which typically one has $\mathrm{d}\mu_I(z) \ll \mathrm{d}z$ and $\frac{\mathrm{d}\mu_I(z)}{\mathrm{d}z}$ is the *local time* in I of the stochastic process W_t.

It then becomes clear from (2.5) that if the measure $\mu_{[0;t]}$ is *regular* enough (with respect to both z and t in (2.5)), then the effective vector field $b * \mu_{[0;t]}$ appearing in (2.5) turns out to be sufficiently regular so that one can indeed control this first Picard iterate. The same estimates show that one in turn can successfully run a Picard iteration to solve (2.4), similarly as one would to deal with (2.1) for Lipschitz vector fields.

Studying the properties of stochastic or deterministic rough real-valued functions through the lens of their occupation measure has been an active field of investigation since the 70's; we refer to the very complete review [15] on this topic. In particular, there are several topologies used to measure the regularity properties of μ: at first in Hölder spaces for both z and t [15], but more recently in Fourier–Lebesgue spaces in z [5] or Besov spaces [20]. The common feature being that *regularity of μ implies irregularity of W_t*, for example when measured as local Hölder continuity [15]. In turn, sufficient conditions are known for stochastic processes W_t, in particular Gaussian stochastic processes, to satisfy almost surely a given regularity assumption on μ.

This new perspective led to a systematic study of the noiseless analogue of the earlier results on regularization by noise; we refer to the thesis of Galeati [14] for a review on the subject. In the following, we will see how this approach can lead to a regularization effect for some modulated dispersive PDEs (1.1).

3. Well-posedness of (1.3), (1.4), and (1.5)

Before addressing the effect of the modulation in (1.1) on the dispersive dynamics, we review the mechanisms used to study the well-posedness of the nonlinear dispersive PDEs (1.3)–(1.4)–(1.5).

The NLS equation

We start with the NLS equation (1.3) on $\mathcal{M} = \mathbb{R}^d$, for which we refer to the monographs [6] and [18]. In this case, the equation is invariant under the rescaling $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$, $\lambda > 0$, such that the homogeneous Sobolev norm is rescaled as $\|u_\lambda(t)\|_{\dot{H}^s} = \lambda^{s-s_c}\|u(t)\|_{\dot{H}^s}$ for the critical exponent $s_c = \frac{d}{2} - \frac{2}{p-1}$. This suggests that the equation should be well-posed for *sub-critical* and *critical* exponents $s \geq \max(s_c, 0)$, and ill-posed for *super-critical* exponents $s < s_c$.

To solve (1.3) for data $u_0 \in H^s(\mathbb{R}^d)$, $s \geq \max(s_c, 0)$, one usually tries to implement a Picard iteration on the Duhamel formula

$$u(t) = e^{it\Delta}u_0 \pm i\int_0^t e^{i(t-t')\Delta}|u|^{p-1}u(t')\,\mathrm{d}t'. \tag{3.1}$$

Naively, if one tries to control directly $|u|^{p-1}u$ in $H^s(\mathbb{R}^d)$, one faces the restriction $s > \frac{d}{2}$ for $H^s(\mathbb{R}^d)$ to be an algebra. Thus one has to exploit better integrability properties associated with the solution of the linear[1] equation. In this case, we have (global) Strichartz estimates

$$\left\|e^{it\Delta}u_0\right\|_{L^q_t L^r_x} \leq C\|u_0\|_{L^2}, \tag{3.2}$$

for *admissible pairs* $q, r \geq 2$ satisfying $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $r < \infty$ for $d = 2$ and $r < \frac{2d}{d-2}$ for $d \geq 3$. The estimate (3.2) is itself a direct consequence of the dispersion estimate

$$\left\|e^{it\Delta}u_0\right\|_{L^\infty_x} \leq C|t|^{-\frac{d}{2}}\|u_0\|_{L^1} \tag{3.3}$$

which, in the constant coefficient case, can be checked directly from the explicit expression of the kernel of $e^{it\Delta}$.

From (3.2), one finds the nonlinear estimate

$$\left\|\langle D\rangle^s \int_0^t e^{i(t-t')\Delta}|u|^{p-1}u(t')\,\mathrm{d}t'\right\|_{L^q_T L^r_x} \leq CT^{\eta_s}\|\langle D\rangle^s u\|^p_{L^q_T L^r_x} \tag{3.4}$$

for any $T > 0$, where $\eta_s > 0$ when $s > s_c$, while $\eta_s = 0$ for $s = s_c$. This allows to get local well-posedness of (1.3) on $[0;T]$ with $T \sim (1 + \|u_0\|_{H^s})^{-\theta}$ for some $\theta > 0$ in case $s > s_c$ and $s \geq 0$, while $T = T(u_0)$ depends on the profile of u_0 in case $s = s_c \geq 0$.

As for globalizing the solution, we can use the conserved quantities of the equation (1.3) to iterate the local well-posedness result: the mass

$$M(u(t)) = \frac{1}{2}\|u(t)\|^2_{L^2_x} \tag{3.5}$$

and the energy

$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|^2_{L^2_x} \mp \frac{1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}} \tag{3.6}$$

are invariant under the flow of (1.3). In the mass-subcritical case $s_c < 0$, this allows to iterate the local well-posedness in $H^s(\mathbb{R}^d)$, $s \geq 0$, since the local time T only depends on the conserved quantity $\|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x}$ (3.5). Similarly, in the defocusing (plus sign in (3.6)) energy subcritical case $s_c < 1$, one can use conservation of the energy (3.6) to iterate the local well-posedness and globalize the solution in $H^s(\mathbb{R}^d)$, $s \geq 1$. In the *mass-critical* case $s_c = 0$, however, one can only get *small-data* global well-posedness due to $\eta_{s_c} = 0$ in (3.4), and thus the bad dependence of T on u_0. One even has a net dichotomy: there is a constant C_0 (the mass of the ground state) such that if $\|u_0\|_{L^2} < C_0$ then the solution is global, while in the focusing case (minus sign in (3.6)) there are data with $\|u_0\|_{L^2} = C_0$ for which the solution blows-up in finite time.

The fractional KdV equation

The argument sketched above thus purely relies on the *linear* estimate (3.2). However, such estimates are typically not true on compact manifolds such as $\mathcal{M} = \mathbb{T}^d$ (the estimate (3.3) cannot hold for all $t \neq 0$ for constant initial data), and one has to rely instead on *multilinear* estimates. This is also suitable to deal with derivative losses in the nonlinearity such as in (1.4). To sketch how these multilinear effects can be used, instead of looking for a solution u to the mild formulation

$$u(t) = e^{t\partial_x D^\alpha}u_0 + \int_0^t e^{(t-t')\partial_x D^\alpha}\partial_x(u^2)(t')\,\mathrm{d}t'$$

of (1.4), we can make the change of unknown $v(t) := e^{-t\partial_x D^\alpha}u(t)$ which solves

$$v(t) = u_0 + \int_0^t e^{-t'\partial_x D^\alpha}\partial_x\big(e^{t'\partial_x D^\alpha}v\big)^2(t')\,\mathrm{d}t'. \tag{3.7}$$

[1] Recall that we treat the nonlinearity perturbatively via a fixed-point argument.

Decomposing $v = \sum_{k\in\mathbb{Z}} v_k(t)e^{ikx}$ as a Fourier[2] series, we can rewrite (3.7) as

$$v_k(t) = (u_0)_k + \sum_{\substack{k_1,k_2\in\mathbb{Z}\\ k_1+k_2=k}} ik \int_0^t e^{it'\Phi_\alpha(k_1,k_2)} v_{k_1}(t')v_{k_2}(t')\,\mathrm{d}t', \tag{3.8}$$

for any $k \in \mathbb{Z}$, where

$$\big|\Phi_\alpha(k_1,k_2)\big| = \big|k_1|k_1|^\alpha + k_2|k_2|^\alpha - (k_1+k_2)|k_1+k_2|^\alpha\big| \sim |k_{\max}|^\alpha|k_{\min}| \tag{3.9}$$

where $|k_{\max}| = \max(|k_1|,|k_2|,|k|)$ and $|k_{\min}| = \min(|k_1|,|k_2|,|k|)$.

An important remark here is that we can ensure $|\Phi_\alpha| \gtrsim |k_{\max}|^\alpha \geq 1$ by considering initial data with mean zero, which restricts the sum to $k_1, k_2, k \neq 0$ in (3.8). This is not restrictive since $\int_\mathbb{T} u(x)dx$ is invariant under the flow of (1.4), and the change of unknown

$$\widetilde{u}(t,x) := u(t,x+ct) - c \tag{3.10}$$

with $c = \int_\mathbb{T} u_0(x)dx$ also leaves the equation (1.4) invariant, while mapping a function with non-zero mean to one with zero mean.

The method developed by Bourgain then amounts to building a solution v to (3.8) in $H_t^{\frac12+}H_x^s \subset C_tH_x^s$ to benefit from the highly oscillatory factor $e^{it'\Phi_\alpha}$: indeed, from the boundedness properties of the time integral and a standard product estimate in Sobolev spaces, one finds[3]

$$\begin{aligned}\Big\|ik\int_0^t e^{it'\Phi_\alpha(k_1,k_2)}v_{k_1}(t')v_{k_2}(t')\,\mathrm{d}t'\Big\|_{H_t^{\frac12}} &\lesssim |k|\big\|e^{it\Phi_\alpha(k_1,k_2)}v_{k_1}v_{k_2}\big\|_{H_t^{-\frac12}}\\ &\lesssim |k|\big\|e^{it\Phi_\alpha(k_1,k_2)}\big\|_{W_t^{-\frac12,\infty}}\|v_{k_1}\|_{H_t^{\frac12}}\|v_{k_2}\|_{H_t^{\frac12}}.\end{aligned} \tag{3.11}$$

From an explicit computation for the norm of $e^{it\Phi_\alpha}$, we find

$$\begin{aligned}\Big\|ik\int_0^t e^{it'\Phi_\alpha(k_1,k_2)}v_{k_1}(t')v_{k_2}(t')\,\mathrm{d}t'\Big\|_{H_t^{\frac12}} &\lesssim \frac{|k|}{|\Phi_\alpha(k_1,k_2)|^{\frac12}}\|v_{k_1}\|_{H_t^{\frac12}}\|v_{k_2}\|_{H_t^{\frac12}}\\ &\sim \frac{|k|}{|k_{\max}|^{\frac{\alpha}{2}}|k_{\min}|^{\frac12}}\|v_{k_1}\|_{H_t^{\frac12}}\|v_{k_2}\|_{H_t^{\frac12}}.\end{aligned} \tag{3.12}$$

We see on this computation that in the case $\alpha \geq 2$, corresponding to the usual KdV equation ($\alpha = 2$), or its higher dispersion analogue ($\alpha > 2$), the phase Φ_α coming from the multilinear interaction in the nonlinearity allows to compensate for the derivative loss, in particular in the case $|k| = |k_{\max}|$, and to run a fixed point argument to build v in $H_t^{\frac12}H_x^s$. However, the case $\alpha < 2$ is radically different: the gain of the factor $|\Phi_\alpha|^{\frac12}$ in the previous computation is not enough to compensate for the derivative loss. And this is not only technical, since one can show that due to the problematic High×Low→High interaction (i.e. $|k| \sim |k_{\max}| \gg |k_{\min}|$), it is actually impossible to solve (1.4) via a fixed point argument in (a subspace of) $C_tH_x^s$ for *any* $s \in \mathbb{R}$, as the flow map fails to be locally uniformly continuous.

The fractional Wick-ordered cubic NLS

As for the periodic Wick-ordered fractional cubic NLS (1.5), a mix of the two behaviours described for (1.4) above occurs. We first explain the reason for the Wick-ordering of the nonlinearity in (1.5). Starting from the mild formulation of the equation with a genuine cubic nonlinearity

$$u(t) = e^{itD^\alpha}u_0 \mp i\int_0^t e^{i(t-t')D^\alpha}|u|^2u(t')\,\mathrm{d}t',$$

one can similarly as above rewrite the equation for $v = e^{-itD^\alpha}u$ as

$$v_k(t) = (u_0)_k \mp i \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k_1-k_2+k_3=k}} \int_0^t e^{it'\Phi_\alpha} v_{k_1}(t')\overline{v_{k_2}}(t')v_{k_3}(t')\,\mathrm{d}t' \tag{3.13}$$

[2]In the following, we will neglect various factors of 2π when decomposing in Fourier coefficients since they are irrelevant to our analysis.

[3]Pretending here for the sake of the exposition that we can work with the endpoint case of the Sobolev embedding, namely with $H_t^{\frac12}$ in place of $H_t^{\frac12+}$.

for all $k \in \mathbb{Z}$, where now

$$\left|\Phi_\alpha(k_1,k_2,k_3)\right| = \left||k_1|^\alpha - |k_2|^\alpha + |k_3|^\alpha - |k|^\alpha\right| \sim |k_{\max}|^{\alpha-2}|(k_1-k_2)(k_3-k_2)|. \tag{3.14}$$

In order to benefit as before from the multilinear interactions through a gain of $|\Phi_\alpha|^{\frac{1}{2}}$, we see that there is an issue with the contributions $k_1 = k_2$ and $k_3 = k_2$, similar to the case $k_{\min} = 0$ discussed for (1.4). By the change of unknown

$$\widetilde{u}(t) = e^{-i4M(u_0)t}u(t), \tag{3.15}$$

one can convert $|u|^2u$ into $(|u|^2 - 2\|u\|_{L^2}^2)u$ (recall that M is conserved under the flow of (1.5)), thus (3.13) into

$$v_k(t) = (u_0)_k \mp i \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k_1-k_2+k_3=k\\ k_2\notin\{k_1,k_3\}}} \int_0^t e^{it'\Phi_\alpha} v_{k_1}(t')\overline{v_{k_2}}(t')v_{k_3}(t')\,\mathrm{d}t' \pm i\int_0^t |v_k|^2 v_k(t')\,\mathrm{d}t'. \tag{3.16}$$

The point of this transformation is that, although there is still the contribution from $\int_0^t |v_k|^2v_k(t')\,\mathrm{d}t'$ in (3.16) for which $\Phi_\alpha = 0$, this last multilinear form is at least bounded on $H^s(\mathbb{T})$ for any $s \geq 0$. Therefore, the same argument as above[4] ensures well-posedness in $H^s(\mathbb{T})$ for any $s \geq 0$. Note that in this regime the cubic equation (3.13) and the renormalized one (3.16) are equivalent since the gauge transformation (3.15) is well-defined and bounded on $H^s(\mathbb{T})$, $s \geq 0$.

However, note that the scaling critical regularity for this model is $s_c = \frac{1-\alpha}{2}$, so that we could expect to be able to treat initial data of regularity H^s for some $s < 0$. In view of (3.14) and the argument above, this would be the case when $\alpha > 2$ if not for the contribution of the resonant part $|v_k|^2v_k$. A key observation is that (1.5) has a Hamiltonian structure $\partial_t u = -i\nabla_{\overline{u}}H(u)$ with $H = H_0 + \mathcal{NR} + \mathcal{R}$ where $H_0 = \frac{1}{2}\|\nabla u\|_{L^2}^2$, $\mathcal{NR}$ is the non resonant part, and $\mathcal{R}$ is the problematic resonant contribution, and that this latter Poisson commutes with H_0: $\mathrm{Re}\big\{i\langle\nabla_{\overline{u}}\mathcal{R}(u), \langle D\rangle^{2s}u\rangle_{L^2}\big\} = 0$ for any $s \in \mathbb{R}$. This allows to derive energy estimates at negative regularity since the resonant part does not contribute to the growth of the Sobolev norm. The obtained well-posedness at negative regularity only holds for the renormalized equation (3.16), as one cannot invert the gauge transform (3.15) since it is ill-defined when $s < 0$. Actually, one can exploit this to show that there can be no solution to the original equation (3.13) in $H^s(\mathbb{T})$ for $s < 0$. As for the renormalized equation (3.16), the obtained flow map for $s < 0$ is not locally Lipschitz continuous, since the solution is build by a compactness method, due to the need to use energy estimates to deal with the resonant part of the nonlinearity.

Summary

To conclude this section, we summarize the current well-posedness results for (1.3)–(1.4)–(1.5) discussed above:

- for NLS (1.3) on $\mathcal{M} = \mathbb{R}^d$, well-posedness via linear (Strichartz) estimates is known in the whole sub-critical and critical regime $s \geq s_c = \frac{d}{2} - \frac{2}{p-1} \geq 0$, with only small data global well-posedness in the mass-critical case $s = s_c = 0$;

- the well-posedness of the periodic fractional KdV equation (1.4) differs drastically depending on the value of α, and the various results in the litterature can be summarized by Figure 3.1[5], where the scaling critical regularity is $s_c = \frac{1}{2} - \alpha$.

- the best well-posedness result to date for the fractional Wick-ordered cubic NLS is, to the best of the author's knowledge, that of [3] in $H^s(\mathbb{T})$ for any $s > \frac{2-\alpha}{6}$ and $\alpha > 2$.

[4]Actually, one also has the Strichartz estimate $\|e^{itD^\alpha}u_0\|_{L^4_{t,x}(\mathbb{T}\times\mathbb{T})} \leq C\|u_0\|_{L^2(\mathbb{T})}$ for any $\alpha \geq 2$, which is enough to get well-posedness of (1.5) in $H^s(\mathbb{T})$ for any $s \geq 0$ as in the case $\mathcal{M} = \mathbb{R}$.

[5]Note that for the usual KdV equation ($\alpha = 2$) and the Benjamin–Ono equation ($\alpha = 1$), much better well-posedness results hold than in the other cases, due to the complete integrability of these equations.

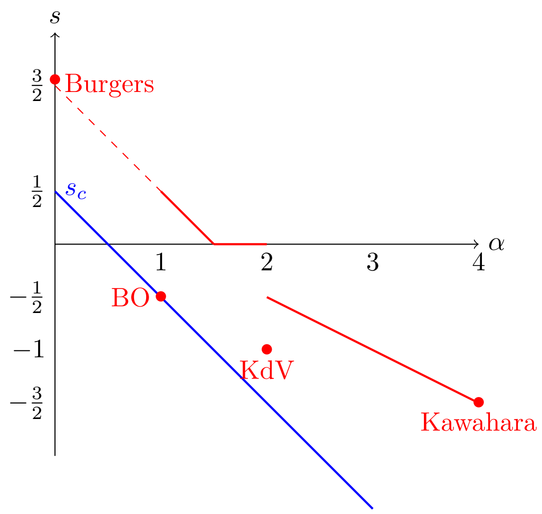

Figure 3.1: Up-to-date well-posedness in $H^s(\mathbb{T})$

4. The case of strongly non-resonant modulated dispersive PDEs

We now proceed to show how the irregular modulation term in (1.1) can be dealt with for the models corresponding to (1.3)–(1.4)–(1.5).

The modulated NLS equation

We start with (1.2). Writing the mild formulation of the equation

$$u(t) = e^{i(W_t - W_0)\Delta} \pm i \int_0^t e^{i(W_t - W_{t'})\Delta} |u|^{p-1} u(t')\, \mathrm{d}t',$$

and up to replacing u_0 by $e^{-W_0\Delta}u_0$ which has the same $H^s(\mathbb{R}^d)$ norm as u_0, we see that we can assume that $W_0 = 0$ and try to use Strichartz estimate for the semigroup $e^{iW_t\Delta}$. A direct computation using the occupation time formula gives

$$\|e^{iW_t\Delta}u_0\|^q_{L^q_T L^r_x} = \int_0^T \|e^{iW_t\Delta}u_0\|^q_{L^r_x}\, \mathrm{d}t = \int_{\mathbb{R}} \|e^{iz\Delta}u_0\|^q_{L^r_x}\, \mathrm{d}\mu_{[0;T]}(z).$$

This shows that if $\mathrm{d}\mu_{[0;T]}(z) \ll dz$ and the local time $\frac{\mathrm{d}\mu_{[0;T]}}{\mathrm{d}z}$ of W_t is bounded, then one has the same (local) Strichartz estimates in both the deterministic and the modulated case:

$$\|e^{iW_t\Delta}u_0\|_{L^q_T L^r_x} \leq C\|\mu_{[0;T]}\|^{\frac{1}{q}}_{L^\infty}\|u_0\|_{L^2_x}. \tag{4.1}$$

Thus one can get the same well-posedness result for (1.2) as that of (1.3). Since this property of μ is satisfied for a large class of Gaussian stochastic processes, including the case of W_t a (fractional) Brownian motion (of any Hurst parameter $H \in (0;1)$), this computation unifies the results of [1, 8, 10, 12]. In particular, if the local time $\frac{\mathrm{d}\mu_{[0;t]}}{\mathrm{d}z}$ is jointly continuous in (t,z) and if $\omega(\,\cdot\,, z)$ is a modulus of continuity for $t \mapsto \frac{\mathrm{d}\mu_{[0;t]}}{\mathrm{d}z}(z)$, then using (4.1), the nonlinear estimate (3.4) becomes in this case

$$\left\|\langle D\rangle^s \int_0^t e^{i(W_t - W_{t'})\Delta}|u|^{p-1}u(t')\, \mathrm{d}t'\right\|_{L^q_T L^r_x} \leq CT^{\eta_s}\|\omega(T,z)\|_{L^\infty_z}\|\langle D\rangle^s u\|^p_{L^q_T L^r_x}. \tag{4.2}$$

This shows that even in the mass-critical case $s = s_c = 0$ for which $\eta_s = 0$, one can still exploit the factor $\omega(T,z)$ to get that there is $T = T(\|u_0\|_{L^2}) > 0$, for which one can get well-posedness via a fixed point argument in Strichartz space. Namely:

Theorem 1 ([19])**.** *Assume that $\mathrm{d}\mu_{[0;t]} \ll \mathrm{d}z$ for any $t \geq 0$, and that $\frac{\mathrm{d}\mu_{[0;\cdot]}}{\mathrm{d}z}(\,\cdot\,)$ is jointly continuous, then (1.2) is locally well-posed in $H^s(\mathbb{R}^d)$ for any $s \geq s_c$. Moreover, in the mass-critical case $s = s_c = 0$, (1.2) is globally well-posed in $L^2(\mathbb{R}^d)$ for any initial data.*

As mentioned above, this result is similar to the results in [1, 8, 10, 12, 14] regarding (1.2). Compared to [1, 10, 12], the result of Theorem 1 is entirely deterministic, and covers as a special case the situation where W_t is a path of a (fractional) Brownian motion (of any Hurst parameter $H \in (0;1)$), which is the case dealt with in [1, 10, 12]. The assumption on μ is also weaker than that of [8] which assumed $\mu_{[0;\cdot]}(\,\cdot\,) \in C_t^{\frac{1}{2}+\epsilon}\mathcal{F}L^{\rho,\infty}$ for some $0 < \epsilon \ll 1$ and $\rho > 1$, where $\mathcal{F}L$ denotes the Fourier–Lebesgue space.

The last part in Theorem 1 can already be seen as a noiseless regularization by noise in view of the discussion in Section 3 regarding the finite time blow-up for large initial data for the mass-critical NLS equation (1.3). Actually, the discussion before the statement of Theorem 1 on the dependence of the local time of existence on $\|u_0\|_{L^2}$ and not on the whole profile of u_0 indicates that the modulation in (1.2) turns the mass-critical equation into a sub-critical one.

However, we see no improvement for the modulated NLS (1.2) on the range $s \geq s_c$ for which we can show local well-posedness in $H^s(\mathbb{R}^d)$ compared to the deterministic case (1.3). This is actually sharp in the following sense: since $W_0 = 0$, if[6] moreover $\frac{\mathrm{d}\mu_{[0;T]}}{\mathrm{d}z}(0) > 0$, then by continuity there is $\delta > 0$ such that $\frac{\mathrm{d}\mu_{[0;T]}}{\mathrm{d}z}(z) \geq \frac{1}{2}\frac{\mathrm{d}\mu_{[0;T]}}{\mathrm{d}z}(0) > 0$ on $[-\delta;\delta]$, and

$$\|e^{iW_t\Delta}u_0\|^q_{L^q_T L^r_x} = \int_{\mathbb{R}} \|e^{iz\Delta}u_0\|^q_{L^r_x}\mathrm{d}\mu_{[0;T]}(z) \geq \left(\frac{1}{2}\frac{\mathrm{d}\mu_{[0;T]}}{\mathrm{d}z}(0)\right)\int_{-\delta}^{\delta}\|e^{iz\Delta}u_0\|^q_{L^r_x}dz,$$

which shows that one cannot expect an improvement on the range of (q,r) in the *linear* (Strichartz) estimates. Thus any improvement (in the range of s for a given p, or in the range of p for, say, $s = 0$) on the well-posedness theory for (1.2) can only come from *multilinear* estimates. The computation above is similar to the result of Stewart [22] showing that in the case of the (mass-critical) quintic *periodic* NLS, which is well-posed in $H^s(\mathbb{T})$ for any $s > 0$ but for which the case $s = 0$ is still open, then the same obstruction as in the deterministic case holds for the modulation equation, preventing it from being well-posed in $L^2(\mathbb{T})$ by a fixed point argument. Note that the same result (4.1) above on Strichartz estimates for the modulated equation shows well-posedness for the modulated quintic NLS in $H^s(\mathbb{T})$ for any $s > 0$ as in the deterministic case [19].

The case of strongly non-resonant models

Our main result is that there is indeed an improvement in the range of regularity s for well-posedness for the modulated equation (1.1) associated with the toy-models (1.4)–(1.5), namely

$$\partial_t u + \frac{\mathrm{d}W_t}{\mathrm{d}t}\partial_x D^\alpha u + u\partial_x u = 0, \tag{4.3}$$

and

$$i\partial_t u + \frac{\mathrm{d}W_t}{\mathrm{d}t}D^\alpha u \pm \big(|u|^2 - 2\|u\|^2_{L^2_x}\big)u = 0, \tag{4.4}$$

both on $\mathcal{M} = \mathbb{T}$.

Theorem 2 ([19])**.** *Assume that $\mu_{[0;\cdot]}(\,\cdot\,) \in C_t^{\frac{1}{2}+\epsilon}\mathcal{F}L^{\rho,\infty}$ for some $0 < \epsilon \ll 1$ and $\rho > 0$.*

(i) *For any $s \in \mathbb{R}$ and $\alpha > 0$, if $\rho > \max\big(\frac{1-4s}{2\alpha};\frac{3-2s}{2(\alpha+1)}\big)$, then (4.3) is locally well-posed in $H^s_0(\mathbb{T}) = \big\{u \in H^s(\mathbb{T}),\ \int_{\mathbb{T}} u\,\mathrm{d}x = 0\big\}$. Moreover the flow map is locally Lipschitz continuous;*

(ii) *For any $s \in \mathbb{R}$ and $\alpha > 2$, if $\rho > \frac{1-3s}{\alpha-2}$, then (4.4) is locally well-posed in $H^s(\mathbb{T})$.*

In view of the well-posedness for the deterministic equations (1.4)–(1.5) presented in Section 3, we see that Theorem 2 is a manifestation of noiseless regularization by noise, actually on two aspects. The first one is obviously the range of regularity s amenable to well-posedness in Theorem 2 compared to the well-posedness results presented in Section 3 for (1.4)–(1.5). The second one is that, for (4.3), even in the case $\alpha < 2$, the flow map constructed in Theorem 2 is locally Lipschitz continuous, which is in sharp contrast with the situation for the deterministic equation (1.4), as explained in Section 3. It may be surprising that the assumption of μ having enough regularity

[6] Note that in the case where W_t is a Brownian motion, $L_T(0) := \frac{\mathrm{d}\mu_{[0;T]}}{\mathrm{d}z}(0)$ has the same law as $\frac{1}{2}\max_{[0;T]} W_t$, which is a.s. positive.

(measured in Fourier–Lebesgue spaces) implies that this regularization effect occurs when the modulation W_t is *irregular* enough, which is in sharp contrast to the case of spatially dependent coefficients as discussed in the introduction. As we will see below, the reason is that when W_t is very rough, its oscillations actually concur with the oscillations coming from multilinear interactions.

The case $\alpha = 2$ in Theorem 2 (i) corresponds to the KdV equation and is treated in [7]. The main observation in [19] is that this phenomenon actually occurs for the *strongly non-resonant* models, i.e. all those for which the multilinear phase function Φ_α as in (3.9) and (3.14) satisfies a bound

$$|\Phi_\alpha| \gtrsim |k_{\max}|^\theta \tag{4.5}$$

for some $\theta > 0$. This explains why our result deal with *periodic* equations: up to gauge transforms as in (3.10) or (3.15), one can sometimes get rid of *exact resonances* for which $\Phi_\alpha = 0$ to obtain (4.5) for the gauged equation. On the contrary, on $\mathbb{R}^d$, one faces the issue of very small frequencies, e.g. $|\Phi_\alpha| \gtrsim |\xi_{\max}|^\alpha |\xi_{\min}|$ for (1.4) on $\mathcal{M} = \mathbb{R}$, and the regime $|\xi_{\min}| \ll |\xi_{\max}|^{-\alpha}$ shows that one cannot benefit from the amplification by μ of the multilinear oscillations (see (4.9) below), contrary to the case when (4.5) holds.

The point in (ii) is that this phenomenon also occurs for models which are *not* strongly non-resonant in the sense of (4.5) due to perturbations by *completely resonant* nonlinearities which Poisson commute with the quadratic part of the Hamiltonian, as discussed for (4.4) in Section 3.

Sketch of the proof

Similarly to the computation in (3.11), the approach in [7] consists in making the change of unknown $v = e^{-W_t \partial_x D^\alpha} u$, where u solves (4.3), so that we seek v as a solution to

$$v_k(t) = (u_0)_k + ik \sum_{\substack{k_1,k_2 \in \mathbb{Z} \\ k_1+k_2=k}} \int_0^t e^{iW_{t'}\Phi_\alpha(k_1,k_2)} v_{k_1}(t') v_{k_2}(t') \,\mathrm{d}t',$$

for any $k \in \mathbb{Z}$. In order to build v in $C_t^{\frac{1}{2}+} H_x^s$, we rewrite the nonlinear part as

$$\int_0^t e^{iW_{t'}\Phi_\alpha(k_1,k_2)} v_{k_1}(t') v_{k_2}(t') \,\mathrm{d}t' = \int_0^t (v_{k_1} v_{k_2})(t') \,\mathrm{d}F^{W_t}_{k_1,k_2}(t'), \tag{4.6}$$

where

$$F^{W_t}_{k_1,k_2}(t) = \int_0^t e^{iW_{t'}\Phi_\alpha(k_1,k_2)} \,\mathrm{d}t' = \widehat{\mu_{[0;t]}}\big(\Phi_\alpha(k_1,k_2)\big).$$

With our assumption on μ, we have that F_{k_1,k_2} is not of bounded variation, but only in $C_t^{\frac{1}{2}+\epsilon}$ for some $0 < \epsilon \ll 1$. Since this is also the case for v, the last integral in (4.6) cannot be made sense of as a Stieltjes integral, but can as a Young integral.

Indeed, recall that when F is piecewise C^1 and G is piecewise C^0, then the Riemann integral

$$\int_0^t G(t') \,\mathrm{d}F(t') = \int_0^t G(t') F'(t') \,\mathrm{d}t'$$

is defined as the limit of the Riemann sum

$$\int_0^t G(t') \,\mathrm{d}F(t') = \lim_{|\pi| \to 0} \sum_{t_j \in \pi} G_{t_j} F'_{t_j} \cdot (t_{j+1} - t_j), \tag{4.7}$$

where π denotes a partition of $[0;t]$, and is bilinear and bounded $C^0 \times C^1 \to C^1$.

When F is only of bounded variation $C^{\infty\text{-var}}$, (4.7) becomes ill-defined, and instead the Riemann–Stieltjes integral

$$\int_0^t G(t') \,\mathrm{d}F(t') := \lim_{|\pi| \to 0} \sum_{t_j \in \pi} G_{t_j} \big(F_{t_{j+1}} - F_{t_j}\big) \tag{4.8}$$

is well-defined and is bilinear and bounded $C^0 \times C^{\infty\text{-var}} \to C^{\infty\text{-var}}$.

Then, if F is only of finite q variation $C^{q\text{-var}}$ for some $q < \infty$, provided that now G is of finite p variation $C^{p\text{-var}}$ with $\frac{1}{p} + \frac{1}{q} > 1$, then (4.8) still makes sense and defines the Young integral, which is now bilinear and bounded $C^{p\text{-var}} \times C^{q\text{-var}} \to C^{q\text{-var}}$. In particular this is the case if $F \in C_t^{\frac{1}{q}}$ and $G \in C_t^{\frac{1}{p}}$.

In view of the time regularity for v and the assumption on μ, we see that we can indeed make sense of (4.6) as a Young integral (4.8). This allows to close a fixed point argument for v in $C_t^{\frac{1}{2}+}H_x^s$, since due to the regularity of μ in Fourier–Lebesgue spaces and the expression for $F_{k_1,k_2}^{W_t}$ above, we get in place of (3.12) the estimate

$$\begin{aligned}\Big\| ik\int_0^t (v_{k_1}v_{k_2})(t')\mathrm{d}F_{k_1,k_2}^{W_t}(t')\Big\|_{C_t^{\frac{1}{2}+}} &\lesssim \frac{|k|}{|\Phi_\alpha(k_1,k_2)|^\rho}\|v_{k_1}\|_{C_t^{\frac{1}{2}+}}\|v_{k_2}\|_{C_t^{\frac{1}{2}+}}\\ &\sim \frac{|k|}{|k_{\max}|^{\rho\alpha}|k_{\min}|^\rho}\|v_{k_1}\|_{C_t^{\frac{1}{2}+}}\|v_{k_2}\|_{C_t^{\frac{1}{2}+}}. \end{aligned} \tag{4.9}$$

We thus see that, provided that ρ is large enough (depending on s), the estimate above allows to sum in k_1, k_2 after adding the weight $\frac{\langle k\rangle^s}{\langle k_1\rangle^s\langle k_2\rangle^s}$, showing local well-posedness for v via a fixed point argument in $C_t^{\frac{1}{2}+}H_x^s$ for any $s \in \mathbb{R}$. Note that the same approach works also in the case $W_t = t$ since an exact computation gives in this case $F_{k_1,k_2}^t(t) = \frac{e^{it\Phi_\alpha(k_1,k_2)}-1}{i\Phi_\alpha(k_1,k_2)}$. Thus (4.9) recovers (3.11) in this case.

The proof of (ii) follows similarly from a modification of the argument presented in Section 3 regarding (1.5), using the regularity of μ to amplify multilinear oscillations in the non-resonant part, and dealing with the completely resonant part by an appropriate energy estimate.

Further remarks

Let us make some final comments on the set of "good" functions W_t for which Theorem 2 holds. Catellier and Gubinelli [5] showed that for W_t a fractional Brownian motion of Hurst parameter $H \in (0;1)$, $\mu_{[0;\cdot]}(\,\cdot\,) \in C_t^{\frac{1}{2}+}\mathcal{F}L^{\rho,\infty}$ a.s. for any $\rho < \frac{1}{2H}$. This can be seen as follows: a fractional Brownian motion with Hurst parameter H is a Gaussian stochastic process whose covariance function is given by

$$\mathbb{E}\big[W_tW_s\big] = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big),$$

for any $t, s \geq 0$. Using the Gaussianity of W_t, this allows to compute

$$\begin{aligned}\mathbb{E}\big|\widehat{\mu_{[0;t]}}(\xi)\big|^2 &= \mathbb{E}\Big|\int_0^t e^{iW_{t'}\xi}\,\mathrm{d}t'\Big|^2 = \int_0^t\int_0^t e^{-\frac{1}{2}\xi^2\mathbb{E}|W_{t_1}-W_{t_2}|^2}\mathrm{d}t_1\mathrm{d}t_2 = \int_0^t\int_0^t e^{\xi^2|t_1-t_2|^{2H}}\mathrm{d}t_1\mathrm{d}t_2\\ &\sim \min\big(t^2; |\xi|^{-\frac{1}{H}}t\big).\end{aligned}$$

This indicates that indeed $\mu_{[0;\cdot]}(\,\cdot\,) \in C_t^\gamma\mathcal{F}L^{\frac{1-\gamma}{H},\infty}$ for any $\gamma \in [\frac{1}{2};1]$.

In particular, Theorems 1 and 2 are not empty, and there are "plenty" of rough functions W_t satisfying the assumptions. It is known that the fractional Brownian motion W_t as above is a.s. in C_t^γ for $\gamma < H$, and a.s. not in C_t^γ for $\gamma \geq H$. Thus we can see that regularity of μ indeed implies irregularity of W_t is this case, through the lower bound on ρ giving an upper bound on H.

5. Perspectives

To conclude this note, we present two further directions of investigation that we believe are worth digging into.

Failure of the strong non-resonance condition

As emphasized above, the noiseless regularization phenomenon presented in this note relies on the strongly non-resonant character (4.5) of the deterministic models (1.4)–(1.5). However, as soon as the degree of the algebraic nonlinearity $\mathcal{N}$ is too big, or the dimension increases, (4.5) has lesser chances to hold. In particular, this does not hold for the NLS equation (1.2) on $\mathcal{M} = \mathbb{R}$, which is the physically relevant model. For this model, there is another mechanism yielding Strichartz estimates (3.2), which is *transversality* of the resonant interactions. Namely, in the bilinear interaction for NLS, when the corresponding phase satisfies $|\Phi| \lesssim 1$, it also holds $|\nabla\Phi| \gtrsim |\xi_{\max}|$, which can be exploited by integrations by parts in x rather than in t. As a first step towards understanding if this phenomenon can be exploited in the context of resonant modulated equations, it would be interesting to study the case of the modulated KP-I equation, which is a prototype of nonlinear dispersive PDE with a large set of resonances, but with a nonlinearity of low degree ($u\partial_x u$).

Long time behaviour of the solutions

Another direction of interest is to study the long time behaviour of the local solutions built in Theorem 2. Even if the deterministic equation is Hamiltonian, the modulation in (1.1) destroys the conservation of the energy. One often still has the invariance of the L^2 norm, allowing to globalize solutions in L^2. Due to the low regularity where one can show well-posedness, this can also be used to prove invariance of the white noise for (1.1), which is a Gaussian measure supported on $H^s(\mathbb{T}^d)$ for any $s < -\frac{d}{2}$, and formally given by "$\mathrm{d}\nu = e^{-\frac{1}{2}\|u\|^2_{L^2}}\mathrm{d}u$". We see on this formal expression that invariance of the L^2 norm should in principle give invariance of ν. This latter property allows to use Poincaré's recurrence theorem to get a qualitative description of the long time behaviour of the flow of (4.3) and (4.4) for initial data in the support of ν.

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