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Toan T. Nguyen

Amortissement Landau linéaire en $\mathbb{R}^3$

Résumé

Cet article donne un aperçu de l'amortissement Landau linéaire pour les modèles cinétiques sans collision tels que les systèmes non relativistes de Vlasov–Poisson et relativistes de Vlasov–Maxwell proches d'états stationnaires radiaux spatialement homogènes sur l'espace des phases $\mathbb{R}^3_x \times \mathbb{R}^3_v$.

Abstract

This article gives an overview on linear Landau damping for collisionless kinetic models such as the non-relativistic Vlasov–Poisson and relativistic Vlasov–Maxwell systems near spatially homogenous radial steady states on the phase space $\mathbb{R}^3_x \times \mathbb{R}^3_v$.

1. Introduction

We are interested in the large time behavior of solutions to the linearized non-relativistic Vlasov–Poisson and relativistic Vlasov–Maxwell systems near spatially homogenous steady states. These are classical collisionless kinetic models that are used to describe the dynamics of charged particles with a self-consistent electromagnetic field in a uniform ions background. Specifically, letting $\mu(v)$ be a spatially homogenous fixed background profile for electrons, the linearized Vlasov–Maxwell system reads

$$\begin{cases} \partial_t f + \widehat{v} \cdot \nabla_x f + \left(E + \dfrac{1}{c}\widehat{v} \times B\right) \cdot \nabla_v \mu = 0, \\ \qquad \dfrac{1}{c}\partial_t B + \nabla_x \times E = 0, \qquad \nabla_x \cdot E = \rho[f], \\ -\dfrac{1}{c}\partial_t E + \nabla_x \times B = \dfrac{1}{c}\mathbf{j}[f], \qquad \nabla_x \cdot B = 0, \end{cases} \tag{1.1}$$

in the phase space $(x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$, in which c is the speed of light and $\widehat{v} = v/\sqrt{1+|v|^2/c^2}$ denotes the relativistic velocity. The system (1.1) describes the linearized dynamics of density distribution for electrons near the background profile $\mu(v)$, having the electromagnetic fields E, B being generated self-consistently through the Maxwell equations by the charge and current densities

$$\rho[f] = \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v, \qquad \mathbf{j}[f] = \int_{\mathbb{R}^3} \widehat{v} f(t,x,v)\,\mathrm{d}v. \tag{1.2}$$

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In the non-relativistic limit of $c \to \infty$, the system (1.1) reduces to the classical Vlasov–Poisson system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu = 0, \\ \nabla_x \times E = 0, \qquad \nabla_x \cdot E = \rho[f], \end{cases} \tag{1.3}$$

in the phase space $(x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$.

Of great interest is to identify decaying mechanism and the large time behavior of solutions to both the linearized systems (1.1) and (1.3). The study will play an important role in understanding the large time dynamics of charged particles in a nonlinear setting. In absence of the background profile $\mu(v) = 0$, the particles follow the free transport dynamics $\partial_t f^0 + \widehat{v} \cdot \nabla_x f^0 = 0$, whose charge and current densities are computed by

$$\rho^0(t, x) = \int_{\mathbb{R}^3} f_{\text{in}}(x - \widehat{v}t, v)\, \mathrm{d}v, \qquad \mathbf{j}^0(t, x) = \int_{\mathbb{R}^3} \widehat{v} f_{\text{in}}(x - \widehat{v}t, v)\, \mathrm{d}v \tag{1.4}$$

for initial data $f^0(0, x, v) = f_{\text{in}}(x, v)$. Namely, the particles travel along the free transport characteristics $x(t) = x + \widehat{v}t$. Upon introducing the change of variable $y = x - \widehat{v}t$, the densities $\rho^0(t, x), \mathbf{j}^0(t, x)$ thus decay at rate of order t^{-3} in the large time. As a result, the electromagnetic fields disperse in space like a free wave. Such a dispersion in fact persists at the nonlinear level near vacuum [1, 2, 14, 16] in the three-dimensional case; see also recent works [9, 10].

The main objective of this article is to review recent results concerning decay of electromagnetic fields in presence of nontrivial background profiles $\mu(v)$, namely solutions to the linear systems (1.1) and (1.3). Observe that the dynamics of $f(t, x, v)$ is no longer decoupled from the fields, but solving the transport equation in (1.1), which gives the charge and current densities $\rho[f]$ and $\mathbf{j}[f]$

$$\begin{aligned} \rho[f] &= \rho^0(t, x) - \int_0^t \int_{\mathbb{R}^3} \mathcal{K}(s, x - (t - s)\widehat{v}, v) \cdot \nabla_v \mu\, \mathrm{d}v \mathrm{d}s \\ \mathbf{j}[f] &= \mathbf{j}^0(t, x) - \int_0^t \int_{\mathbb{R}^3} \widehat{v} \mathcal{K}(s, x - (t - s)\widehat{v}, v) \cdot \nabla_v \mu\, \mathrm{d}v \mathrm{d}s \end{aligned} \tag{1.5}$$

in which $\mathcal{K} = E + \frac{1}{c}\widehat{v} \times B$ and $\rho^0, \mathbf{j}^0$ are the densities generated by the free dynamics as in (1.4). Putting these into the Maxwell equations, one can easily obtain a closed system for the electromagnetic fields E, B, involving the nonlocal operators as in (1.5), cf. [27, 28] where the orbital stability and instability of inhomogenous equilibria was studied. The system is however delicate to analyze due to the nonlocal nature of the integral terms in (1.5). As a matter of facts, it has been elusive to determine if there were any spectrally stable equilibria $\mu(v)$ (in the presence of magnetic fields), not to mention the apparent lack of study on boundedness and decay of the fields. The question remains open for general equilibria.

Most recently, in a joint work with D. Han-Kwan and F. Rousset [22], we resolve the linear stability problem completely for radial equilibria. To proceed, we shall work with the Coulomb gauge for the electromagnetic potentials $\phi(t, x) \in \mathbb{R}$ and $A(t, x) \in \mathbb{R}^3$ defined through $E = -\nabla_x \phi - \frac{1}{c}\partial_t A$ and $B = \nabla_x \times A$, together with $\nabla_x \cdot A = 0$. The Maxwell equations then become

$$-\Delta_x \phi = \rho[f], \qquad (\partial_t^2 - c^2 \Delta_x) A = c\mathbb{P}\mathbf{j}[f] \tag{1.6}$$

where $\mathbb{P}$ denotes the classical Leray projector, that is $\mathbb{P}\mathbf{j} = \mathbf{j} - \nabla_x \Delta_x^{-1} \nabla_x \cdot \mathbf{j}$ for $\mathbf{j} \in \mathbb{R}^3$. Using (1.5), we now obtain the following closed system for the potential functions

$$\mathcal{M} \begin{pmatrix} \phi \\ A \end{pmatrix} = \begin{pmatrix} \rho^0 \\ c\mathbb{P}\mathbf{j}^0 \end{pmatrix} \tag{1.7}$$

for some matrix operator $\mathcal{M}$, which can be derived explicitly. The decoupling of the electric and magnetic potentials in (1.6) appears to be the advantage of employing the Coulomb gauge for the electromagnetic fields. However, the matrix operator $\mathcal{M}$ remains complicated to study, again due to the presence of the nonlocal operators as in (1.5).

As discovered in [22], in the case when $\mu(v)$ is radial in v, the matrix operator $\mathcal{M}$ turns out to be diagonal, and therefore the electric and magnetic potentials are completely decoupled (for the linearized system (1.1)), which we now describe, following the classical Laplace–Fourier approach. Indeed, let $\widetilde{\phi}_k(\lambda), \widetilde{A}_k(\lambda)$ be the Laplace–Fourier transform of the potentials $\phi(t, x), A(t, x)$, respectively. It follows that for each $\lambda \in \mathbb{C}$ and $k \in \mathbb{R}^3$, we obtain

$$\begin{aligned} |k|^2 D(\lambda, k)\widetilde{\phi}_k &= \widetilde{\rho}^0_k \\ M(\lambda, k)\widetilde{A}_k &= c\mathbb{P}_k \widetilde{\mathbf{j}}^0_k + \lambda \widehat{A}^0_k + \widehat{A}^1_k \end{aligned} \tag{1.8}$$

for initial data $\widehat{A}^0_k, \widehat{A}^1_k$, where $D(\lambda, k)$ and $M(\lambda, k)$ are scalar functions defined by

$$\begin{aligned} D(\lambda, k) &= 1 - \frac{1}{|k|^2} \int_{\mathbb{R}^3} \frac{ik \cdot \nabla_v \mu}{\lambda + ik \cdot \widehat{v}} \, \mathrm{d}v, \\ M(\lambda, k) &= \lambda^2 + c^2|k|^2 - \frac{c\lambda}{2} \int_{\mathbb{R}^3} \frac{|\mathbb{P}_k \widehat{v}|^2}{ik \cdot \widehat{v}} \frac{ik \cdot \nabla_v \mu}{\lambda + ik \cdot \widehat{v}} \, \mathrm{d}v, \end{aligned} \tag{1.9}$$

in which $\mathbb{P}_k = (\mathbb{I} - \frac{k \otimes k}{|k|^2})$ is the Fourier symbol for the Leray projector. Namely, in the case of radial equilibria, the resolvent solutions are completely decoupled for the electric and magnetic potentials. In (1.8), $\widetilde{\rho}^0, \widetilde{\mathbf{j}}^0$ are the Laplace–Fourier transform of the densities propagated by the free dynamics, see (1.4). It thus reduces to study the spacetime scalar symbols (1.9) independently, which we shall refer to them as the electric and magnetic dispersion functions, respectively. In view of the resolvent equations (1.8), we obtain the electric and magnetic potentials in the physical space via a spacetime convolution with the corresponding electric and magnetic Green functions $G(t, x)$ and $H(t, x)$ against the charge and current densities $\rho^0(t, x), \mathbf{j}^0(t, x)$ by the free dynamics (1.4), plus initial data contributions. Here, the Green functions are constructed by

$$\begin{aligned} G(t, x) &= \frac{1}{2\pi i} \int_{\{\Re\lambda = \gamma_0\}} \int_{\mathbb{R}^3} e^{\lambda t + ik \cdot x} \frac{1}{D(\lambda, k)} \, \mathrm{d}k \mathrm{d}\lambda, \\ H(t, x) &= \frac{1}{2\pi i} \int_{\{\Re\lambda = \gamma_0\}} \int_{\mathbb{R}^3} e^{\lambda t + ik \cdot x} \frac{1}{M(\lambda, k)} \, \mathrm{d}k \mathrm{d}\lambda, \end{aligned} \tag{1.10}$$

which are well-defined as oscillatory integrals for $\gamma_0 > 0$. The main results established in [22] are pointwise decay estimates on the Green functions.

Observe that for each $k \in \mathbb{R}^3$, the zeros $\lambda^{\text{elec}}(k)$ of $D(\lambda, k) = 0$ yield mode solutions of the form $\phi(t, x) = e^{\lambda^{\text{elec}}(k)t + ik \cdot x} a_k$ and $A(t, x) = 0$ for any constant a_k, while the zeros $\lambda^{\text{mag}}(k)$ of $M(\lambda, k) = 0$ yields mode solutions of the form $\phi(t, x) = 0$ and $A(t, x) = e^{\lambda^{\text{mag}}(k)t + ik \cdot x} \mathbb{P}_k A_k$ for any vector constant A_k in $\mathbb{R}^3$. Therefore, growing solutions correspond to those with $\Re\lambda^{\text{elec}}(k) > 0$ or $\lambda^{\text{mag}}(k) > 0$. In this article, we focus precisely on radial equilibria, for which such an unstable mode solution does not exist. However, there are purely oscillating solutions both for the electric dispersion relation $D(\lambda, k) = 0$ and the magnetic dispersion relation $M(\lambda, k) = 0$, which we shall now present.

Finally, we note that in the case of the linearized Vlasov–Poisson system (1.3), the resolvent equation (1.8) simply reduces to $|k|^2 D(\lambda, k)\widetilde{\phi}_k = \widetilde{\rho}^0_k$, for the same symbol $D(\lambda, k)$ in (1.9) with $\widehat{v}$ replaced by v.

1.1. Landau damping

In this section, we shall focus on the electric dispersion function $D(\lambda, k)$ for the linearized Vlasov–Poisson system (1.3), namely

$$D(\lambda, k) = 1 - \frac{1}{|k|^2} \int_{\mathbb{R}^3} \frac{ik \cdot \nabla_v \mu}{\lambda + ik \cdot v} \, \mathrm{d}v.$$

Three regimes follow.

- $|k| \gg 1$: *free transport regime.* In this case the electric field is negligible with respect to the transport part, since $D(\lambda, k) \to 1$. As a consequence, the linearized electric field is a perturbation of that generated by the free transport dynamics, which decays rapidly fast to 0, with a speed proportional to k, exponentially if data are analytic and polynomially if data are Sobolev. This exponential damping is at the heart of Mouhot–Villani's celebrated proof of the asymptotic behavior of solutions to the nonlinear Vlasov–Poisson system in the periodic case, see [4, 17, 29].

- $|k| \sim 1$: *Penrose's stable regime.* In this regime, the electric field and the free transport are of the same magnitude, and the plasma may or may not be stable, depending on the background profile $\mu(\cdot)$. It is spectrally stable if and only if $D(\lambda, k)$ never vanishes on $\Re\lambda > 0$, which holds for a large class of positive radial equilibria [29]. Under a stronger, quantitative Penrose stability condition: namely,

$$\inf_k \inf_{\Re\lambda \geq 0} |D(\lambda, k)| \geq \theta_0 > 0, \tag{1.11}$$

the dynamics can again be approximated by that of the free transport and therefore the main damping mechanism in this regime is again *phase mixing*, which was also justified for the nonlinear problem with analytic or Gevrey data on the torus, see [4, 17, 29]. See also [5, 20] for the screened Vlasov–Poisson system on the whole space, for which (1.11) holds for $k \in \mathbb{R}^3$, and the free transport dynamics remains dominant. Specifically, we establish in [17, 20] that the linearized electric field $\widehat{E}_k(t)$ in this Penrose's stable regime can be written as

$$\widehat{E}_k(t) = \widehat{E}_k^0(t) + \widehat{G}_k \star_t \widehat{E}_k^0(t) \tag{1.12}$$

for each wave number k, where $\star_t$ denotes the convolution in time, $\widehat{E}_k^0(t)$ is again the free transport electric field and $\widehat{G}_k(t)$ is exponentially localized $|\widehat{G}_k(t)| \lesssim e^{-\langle kt \rangle}$, leading to a much simplified proof of the nonlinear Landau damping [17] and a construction of echoes solutions for a large class of Sobolev data [18]. We mention that such a representation of the electric field was also established for the weakly collisional regime [12].

- $|k| \ll 1$: *Landau's damping regime.* It turns out that in this regime, the strong Penrose stability condition (1.11) never holds for *any equilibria*! In fact, it is classical in the physical literature that at the very low frequency, plasmas oscillate and disperse with a Schrödinger type dispersion relation

$$\Im\lambda_\pm(k) = \pm\Big(\tau_0 + \frac{\tau_1^2}{2\tau_0^3}|k|^2 + \mathcal{O}(|k|^4)\Big) \tag{1.13}$$

for $|k| \ll 1$, where $\tau_j^2 = \int_{\mathbb{R}^3} |v|^{2j}\mu(v)\,\mathrm{d}v$, $j = 0, 1$. These oscillations are classically known as Langmuir's waves in plasma physics [33]. Naturally, the central question is that whether such oscillations are damped. Landau in his 1946 seminal paper [26] addressed this very issue, and managed to compute the dispersion relation $\lambda = \lambda_\pm(k)$ (i.e. solutions of $D(\lambda_\pm(k), k) = 0$) for Gaussians $\mu = e^{-\frac{1}{2}|v|^2}$, yielding

$$\Re\lambda_\pm(k) \approx \frac{1}{4|k|^2}\partial_v\mu(v)_{|_{v=\nu_*(k)}} \tag{1.14}$$

for sufficiently small $|k|$, where $\nu_*(k) \sim \frac{\tau_0}{|k|}$ denotes the phase velocity of the oscillatory Langmuir's waves (1.13). Note in particular that (1.11) fails as $|k| \to 0$, since $\Re\lambda_\pm(k) \to 0$ super exponentially fast. The same damping law (1.14) holds for any positive radial equilibria [33]. Physically, this leads to a transfer of energy from the electric energy to the kinetic energy of these particles (i.e. damping in the L^2 energy norm). This transfer of energy at the resonant velocity defines the classical notion of Landau damping. In the other words, Landau damping occurs due to the resonant interaction between particles and the oscillatory waves.

  The faster the profile $\mu(v)$ decays, the weaker Landau damping is. In particular, it is polynomially small for power-law equilibria and super exponentially small for Gaussian equilibria. The main mechanism is therefore the dispersion of the electric field, which is seen on the *imaginary* part of Landau's dispersion relation (1.14), whereas the Landau damping rate is seen on its *real* part of (1.14). As a consequence, the electric field is *not* exponentially decreasing at the very low frequency regime, but *oscillatory* like a Schrödinger type equation.

The Schrödinger type dispersion (1.13) leads to a dispersive decay of the electric field of order $t^{-3/2}$ as was proven recently in [6, 21] for general radial analytic equilibria. The Landau damping rate and its sensitivity to the decay of $\mu(v)$ were also seen in the pioneering works by Glassey and Schaeffer [13, 15], where the authors proved that for the linearized Vlasov–Poisson system near a Maxwellian on the whole line, the electric field cannot in general decay faster than $1/(\log t)^{13/2}$ in

L^2 norm, while near polynomially decaying equilibria at rate $\langle v\rangle^{-\alpha}$, $\alpha > 1$, it cannot decay faster than $t^{-\frac{1}{2(\alpha-1)}}$. In addition, it was also shown in [13, 15] that there is no Landau damping (i.e. no decay for L^2 norm of the electric field) near compactly supported equilibria.

1.2. The survival threshold

As a matter of facts, the three regimes described in the previous section apply precisely to the case when equilibria are *positive* for all $v \in \mathbb{R}^3$. For compactly supported equilibria $\mu(v)$, we established in [30] for the non-relativistic Vlasov–Poisson system (1.3) that there is a survival threshold of wave numbers $\kappa_0 > 0$, which may not be small, below which the Penrose stability condition (1.11) fails. See Figure 1.1 for an illustration of the threshold. The survival threshold κ_0 depends on the maximal speed of the particle velocities present in the background profile $\mu(v)$: $\kappa_0 = 0$ for positive equilibria (e.g., for Gaussians), while $\kappa_0 > 0$ for compactly supported equilibria. Similar results are also established for quantum meanfield models [31].

We now focus on the relativistic Vlasov–Maxwell system (1.1), for which the particle velocities are always bounded by the speed of light, and therefore the threshold κ_0 exists and is positive. Let us detail this point. In what follows, we shall consider the radial and rapidly decaying equilibria of the form $\mu = \mu(\langle v\rangle)$ with $\langle v\rangle = \sqrt{1+|v|^2/c^2}$, and set

$$\Upsilon := \sup\Big\{|\widehat{v}|, \quad \mu(\langle v\rangle) \neq 0\Big\} \tag{1.15}$$

to be the maximal speed of particle velocities, which is finite and bounded by the speed of light c, recalling that $\widehat{v} = v/\langle v\rangle$. We then introduce the survival threshold of wave numbers κ_0 defined by

$$\kappa_0^2 = 2\int_0^\Upsilon \frac{u^2\kappa(u)}{\Upsilon^2 - u^2}\,\mathrm{d}u \tag{1.16}$$

in which $\kappa(u) = -2\pi c^2 \int_{1/\sqrt{1-u^2/c^2}}^\infty \mu'(s)s^2\,\mathrm{d}s$. For the derivation of the survival threshold κ_0, see Section C. Since $\mu(s)$ is non-negative and decays rapidly to zero as $s \to \infty$, $\kappa(u) \geq 0$, upon integrating by parts in s. As a result, κ_0 is well-defined and finite. Note that $\kappa_0 > 0$ for any non-negative radial equilibria $\mu(v)$, since $\Upsilon \leq c < \infty$. In the non-relativistic limit of $c \to \infty$, we study radial equilibria of the form $\mu = \mu(\frac{1}{2}|v|^2)$, and so introduce the survival threshold κ_0 as in (1.16) with $\kappa(u) = 2\pi\mu(\frac{1}{2}u^2)$, for which $\kappa_0 = 0$ in the case of positive equilibria, since $\Upsilon = \infty$.

Our main results established in [22, 30] are as follows.

- *Plasma oscillations:* for $0 \leq |k| \leq \kappa_0$, there are exactly two pure imaginary solutions $\lambda_\pm^{\mathrm{elec}}(k) = \pm i\tau_*(k)$ of the dispersion relation $D(\lambda, k) = 0$, which obey a Klein–Gordon type dispersion relation: namely, for $0 < |k| < \kappa_0$,

$$\tau_0 < \tau_*(|k|) < \kappa_0, \qquad |k| < \tau_*(|k|) < \sqrt{\tau_0^2 + |k|^2}, \tag{1.17}$$

  and for some constants $c_0, c_1, C_0 > 0$,

$$c_0|k| \leq \tau_*'(|k|) \leq C_0|k|, \qquad c_1 \leq \tau_*''(|k|), \tag{1.18}$$

  for all $0 \leq |k| \leq \kappa_0$. These oscillatory modes experience no Landau damping $\Re\lambda_\pm^{\mathrm{elec}}(k) = 0$, but disperse in space, since the group velocity $\tau_*'(k)$ is strictly increasing in $|k|$. This dispersion leads to a $t^{-3/2}$ decay of the electric field in the physical space. These oscillations are known as Langmuir's waves in plasma physics [33]. In addition, the phase velocity of these oscillatory waves $\nu_*(k) = \tau_*(k)/|k|$ is a decreasing function in $|k|$ with $\nu_*(0) = \infty$ and $\nu_*(\kappa_0) = \Upsilon$ (the maximal speed of particle velocities).

- *Landau damping:* as $|k|$ increases past the critical wave number κ_0, the phase velocity of Langmuir's oscillatory waves enters the range of admissible particle velocities, namely $|\nu_*(k)| < \Upsilon$. That is, there are particles that move at the same propagation speed of the waves. This resonant interaction causes the dispersion functions $\lambda_\pm^{\mathrm{elec}}(k)$ to leave the imaginary axis, and thus the purely oscillatory modes get damped. Landau [26] computed this law of damping for Gaussians in the non-relativistic case (and hence, $\kappa_0 = 0$) as reported

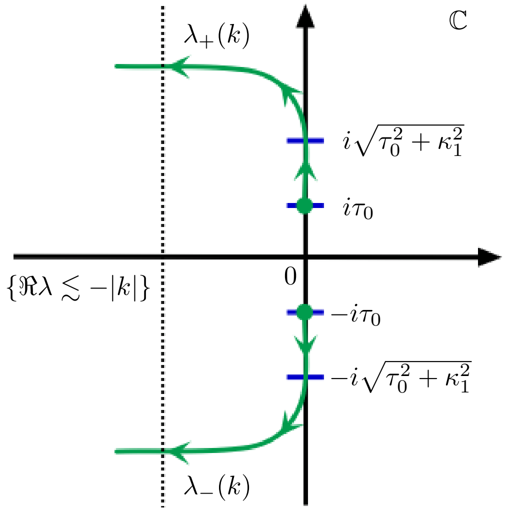

Figure 1.1: Depicted are the solutions $\lambda_\pm(k)$ to the electric dispersion relation $D(\lambda,k)=0$ that start from $\lambda_\pm(0)=\pm i\tau_0$, remain on the imaginary axis and obey a Klein–Gordon's dispersion relation $\tau_*(k)\sim\sqrt{\tau_0^2+|k|^2}$ for $0\le|k|\le\kappa_0$, and then depart from the imaginary axis at $|k|=\kappa_0^+$ due to Landau damping towards the phase mixing regime $\{\Re\lambda\lesssim-|k|\}$, as the wave numbers $|k|$ increase. The group velocity $\tau_*'(k)$ is strictly increasing, while the phase velocity $\nu_*(k)=\tau_*(k)/|k|$ is strictly decreasing in $|k|$, with $\nu_*(0)=\infty$ and $\nu_*(\kappa_0)=\Upsilon$.

in (1.14). For the case of relativistic equilibria $\Upsilon\le c<\infty$, we have $\kappa_0>0$, and the Landau's law of decay can be explicitly computed[1], giving

$$\Re\lambda_\pm^{\mathrm{elec}}(k)\approx-\kappa_1^2\Big[u\mu\Big(\frac{u}{\sqrt{1-u^2/c^2}}\Big)\Big]_{u=\nu_*(k)} \tag{1.19}$$

as $|k|\to\kappa_0^+$, where $\nu_*(k)=\Upsilon-2\kappa_0\kappa_1^2(|k|-\kappa_0)$ for some positive constant κ_1. That is, the vanishing rate of equilibria at the maximal velocity dictates the Landau damping rate of the oscillations at the critical wave number. The faster $\mu(v)$ vanishes, the weaker Landau damping is.

- *Penrose's stable regime:* for $|k|>\kappa_0^+$, the strong Penrose stability condition (1.11) holds, and therefore the behavior of the electric field is governed by the free transport dynamics as discussed in the previous section.

The above results confirm the existence of plasma oscillations or Langmuir's oscillatory waves known in the physical literature. In particular, it is shown that there is a survival threshold of wave numbers κ_0, below which Langmuir's plasma oscillations survive Landau damping, while at the threshold, Landau's law of damping is present and explicitly computed. Beyond κ_0, the strong Penrose stability condition is ensured, and the free transport dynamics is a good approximation for the large time behavior of solutions to the linearized electric potential. In particular, we note that oscillations obey a Klein–Gordon's dispersion relation: namely $\tau_*(k)\sim\sqrt{1+|k|^2}$, which coincides with the dispersion of a Schrödinger's type at the very low frequency (1.13).

1.3. Magnetic dispersion relation

In this section, we study the magnetic dispersion relation $M(\lambda,k)=0$, which can be written as

$$M(\lambda,k)=\lambda^2+c^2|k|^2-\frac{ic\lambda}{2|k|}\int_{-\Upsilon}^{\Upsilon}\frac{q(u)}{-i\lambda/|k|+u}\,\mathrm{d}u,$$

[1] Note that this is computed for radial equilibria in dimension three, cf. (1.14) in dimension one.

for some non-negative and even function $q(u)$ (depending on the equilibrium $\mu(\langle v\rangle)$). See Section C for the details. At vacuum $\mu = 0$, the function $M(\lambda, k) = \lambda^2 + c^2|k|^2$ is the spacetime symbol of the free wave operator $\partial_t^2 - c^2\Delta_x$. At nontrivial equilibria $\mu(\langle v\rangle)$, the symbol $M(\lambda, k)$ turns out to resemble a Klein–Gordon type dispersion, see Section D. It is remarkable that the nonlocal effect has some monotonicity in the temporal frequency, which contributes a positive mass into the spacetime symbol $M(\lambda, k)$, yielding the two unique zeros $\lambda_\pm^{\mathrm{mag}}(k) = \pm i\nu_*(k)$ of the magnetic dispersion relation $M(\lambda, k) = 0$. In addition, $\nu_*(k)$ obeys a Klein–Gordon type dispersion: namely,

$$c_0\sqrt{1+|k|^2} \le \nu_*(|k|) \le C_0\sqrt{1+|k|^2}, \tag{1.20}$$

$$c_0\frac{|k|}{\sqrt{1+|k|^2}} \le \nu_*'(|k|) \le C_0\frac{|k|}{\sqrt{1+|k|^2}}, \tag{1.21}$$

$$c_0(1+|k|^2)^{-3/2} \le \nu_*''(|k|) \le C_0. \tag{1.22}$$

uniformly with respect to $k \in \mathbb{R}^3$, for some positive constants c_0, C_0.

Unlike the case near vacuum, the symbol $M(\lambda, k)$ is not holomorphic in λ for $\Re\lambda < 0$. This causes a fundamental issue in calculating the residual at the poles of the resolvent kernel $\frac{1}{M(\lambda,k)}$. More precisely, we may write $M(\lambda, k) = \lambda^2 + c^2|k|^2 - \frac{ic\lambda}{2|k|}\mathcal{H}(-i\lambda/|k|)$, where

$$\mathcal{H}(z) = \int_{-\Upsilon}^{\Upsilon} \frac{q(u)}{z+u}\,\mathrm{d}u = \int_0^\infty e^{-izt}\int_{-\Upsilon}^{\Upsilon} e^{-iut}q(u)\,\mathrm{d}u\mathrm{d}t = \int_0^\infty e^{-izt}\widehat{q}(t)\,\mathrm{d}t$$

in which $\widehat{q}(t)$ denotes the Fourier transform of $q(u)$ in u. The regularity of $q(u)$ yields a rapid decay of $\widehat{q}(t)$ in t. Therefore, the function $H(z)$ is analytic in $\Im z < 0$ and well-defined up to the real axis $\Im z = 0$, or equivalently $M(\lambda, k)$ is analytic in $\Re\lambda > 0$ and well-defined up to the imaginary axis $\Re\lambda = 0$. However, as $q(u)$ is compactly supported on $[-\Upsilon, \Upsilon]$, its Fourier transform $\widehat{q}(t)$ does not decay exponentially fast. As a result, $\mathcal{H}(z)$ cannot be meromorphically extended past the real axis. This causes a serious issue in applying the standard Cauchy's theory to isolate the oscillatory modes (for instance, as done in [6, 21]). We bypass this issue via a sufficiently accurate approximation by rational functions with appropriate poles, for which we can use Cauchy's residue theorem, leaving the errors acceptable [22, 30].

Finally, after extracting the oscillatory modes (i.e. poles of the resolvent kernel $\frac{1}{M(\lambda,k)}$), we'd expect to obtain phase mixing type estimates on the regular part of the kernel (e.g., similar to those obtained for $\widehat{G}_k(t)$ in (1.12)). This turns out to be delicate, and in fact, false at the very low frequency. A new phenomenon arises. Precisely, in order to gain a factor of decay in time at order t^{-1} from the representation (1.10), we are obliged to bound the ratio

$$\frac{\partial_\lambda M(\lambda, k)}{M(\lambda, k)}. \tag{1.23}$$

In the low frequency regime $|k| \ll 1$ and $\lambda = i\widetilde{\tau}|k|$ with $|\widetilde{\tau}| \ll 1$, a direct calculation yields $|M(i\widetilde{\tau}|k|, k)| \gtrsim |k|^2$, see (D.1) below, while $|\partial_\lambda M(\lambda, k)| \lesssim 1 + |k|^{-1}$, where the $|k|^{-1}$ term is due to the integral term in $M(\lambda, k)$. This proves that the above ratio is bounded by $|k|^{-3}$, yielding that the regular part of the Green function $\widehat{H}_k(t)$ decays at rate of order $\langle |k|^3 t\rangle^{-N}$ in the very low frequency regime $|k| \ll 1$.

2. Landau damping results

In this section, we report the main damping results established in [22] for the relativistic Vlasov–Maxwell system (1.1). Throughout the section, we assume that equilibria are sufficiently regular and rapidly decaying in v, and are of the form $\mu(\langle v\rangle)$.

Theorem 2.1. *Let $G(t, x)$ and $H(t, x)$ be the electric and magnetic Green functions defined as in* (1.10)*, and let $\lambda_\pm^{\mathrm{elec}}(k)$ and $\lambda_\pm^{\mathrm{mag}}(k)$ be the solutions to the electric and magnetic dispersion relations as described in Section 1.2 and Section 1.3, respectively. Then, there hold*

$$\begin{aligned} G(t, x) &= \delta_{t=0} + \sum_\pm G_\pm^{\mathrm{osc}}(t, x) + G^r(t, x) \\ H(t, x) &= \sum_\pm H_\pm^{\mathrm{osc}}(t, x) + H^r(t, x) \end{aligned} \tag{2.1}$$

where $G^{\rm osc}_\pm(t,x)$ *and* $H^{\rm osc}_\pm(t,x)$ *are oscillatory kernels whose Fourier transforms are*

$$\widehat{G}^{\rm osc}_\pm(t,k) = e^{\lambda^{\rm elec}_\pm(k)t}a_\pm(k), \qquad \widehat{H}^{\rm osc}_\pm(t,k) = e^{\lambda^{\rm mag}_\pm(k)t}b_\pm(k),$$

for some smooth functions $a_\pm(k) = \alpha_\pm(|k|^2)$ *whose support is contained in* $B(0,\kappa_0^+)$ *and* $b_\pm(k)$ *satisfying* $|b_\pm(k)| \lesssim \langle k\rangle^{-1}$ *for all* $k \in \mathbb{R}^3$*. In particular, we obtain the following dispersive estimates: for* $p \in [2,\infty)$*,*

$$\begin{aligned} \|G^{\rm osc}_\pm \star_x f\|_{L^p_x} &\lesssim \langle t\rangle^{-3\left(\frac12-\frac1p\right)}\|f\|_{L^{p'}_x}, \\ \|H^{\rm osc}_\pm \star_x f\|_{L^p_x} &\lesssim \langle t\rangle^{-3\left(\frac12-\frac1p\right)}\|b_\pm(i\partial_x)f\|_{B^{3\left(1-\frac2p\right)}_{p',2}}, \end{aligned} \tag{2.2}$$

with $\frac1p + \frac1{p'} = 1$*, with* $\|\cdot\|_{B^s_{p,q}}$ *denoting the standard Besov spaces. In addition, letting* $\chi_0(k)$ *be a smooth cutoff function whose support is contained in* $\{|k| \le 1\}$*, for* $n \ge 0$ *and* $p \in [1,\infty]$*, there hold*

$$\begin{aligned} \|\chi_0(i\partial_x)\partial_x^n\Delta_x^{-1}G^r(t)\|_{L^p_x} &\lesssim \langle t\rangle^{-4+3/p-n}, \\ \|\chi_0(i\partial_x)\partial_x^n H^r(t)\|_{L^p_x} &\lesssim \langle t\rangle^{-4/3+1/p-n/3}, \end{aligned} \tag{2.3}$$

and

$$\|(1-\chi_0(i\partial_x))\partial_x^n G^r(t)\|_{L^p_x} + \|(1-\chi_0(i\partial_x))\partial_x^n H^r(t)\|_{L^p_x} \lesssim \langle t\rangle^{-N}, \tag{2.4}$$

for some large N*.*

Having introduced the Green functions as in (1.10), we obtain the electric and magnetic potentials in the physical space by inverting the Laplace–Fourier transform of the resolvent equations (1.8). Namely,

$$\phi = (-\Delta_x)^{-1}G \star_{t,x} \rho^0 \tag{2.5}$$

and

$$A = \partial_t H(t) \star_x A^0 + H(t) \star_x A^1 + H \star_{t,x} \mathbb{P}\mathbf{j}^0. \tag{2.6}$$

for initial data A^0, A^1, where $\rho^0, \mathbf{j}^0$ denote the charge and current densities by the free dynamics (1.4). Next, using the representations in (2.1) on the Green functions, we may decompose the potentials into oscillatory and regular components. To proceed, we shall solve the Vlasov–Maxwell system (1.1), together with initial data

$$f_{|t=0} = f_{\rm in}(x,v), \qquad E_{|t=0} = E_{\rm in}(x), \qquad B_{|t=0} = B_{\rm in}(x), \tag{2.7}$$

satisfying the following compatibility conditions

$$\nabla_x\cdot E_{\rm in} = \int_{\mathbb{R}^3} f_{\rm in}(x,v)\,{\rm d}v, \qquad \nabla_x\cdot B_{\rm in} = 0, \qquad \iint f_{\rm in}(x,v)\,{\rm d}x{\rm d}v = 0. \tag{2.8}$$

The initial data A^0, A^1 in (2.6) are computed by $A^0 = -\Delta_x^{-1}\nabla_x \times B_{\rm in}$ and $A^1 = -E_{\rm in} + \nabla_x\Delta_x^{-1}\rho[f_{\rm in}]$.

The second main result established in [22] reads as follows.

Theorem 2.2. *Let* $\mu(\langle v\rangle)$ *be any sufficiently smooth and rapidly decaying equilibrium, and let* $\rho^0, \mathbf{j}^0$ *be defined as in* (1.4)*. Then, the electric and magnetic potentials* ϕ, A *to the linearized Vlasov–Maxwell system* (1.1) *with sufficiently regular initial data* (2.7)*–*(2.8) *can be expressed as*

$$\begin{aligned} \phi &= \sum_\pm \phi^{\rm osc}_\pm(t,x) + \phi^r(t,x), \\ A &= \sum_\pm A^{\rm osc}_\pm(t,x) + A^r(t,x). \end{aligned} \tag{2.9}$$

In addition, there hold the following decay estimates:

$$\begin{aligned} \|\nabla_x\phi^{\rm osc}(t)\|_{L^p_x} &\lesssim \langle t\rangle^{-3(1/2-1/p)}, \qquad p\in[2,\infty), \\ \|\partial_x^\alpha\nabla_x\phi^r(t)\|_{L^p_x} &\lesssim \langle t\rangle^{-3(1-1/p)-|\alpha|}, \quad p\in[1,\infty], \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \|\partial_t A^{\rm osc}(t)\|_{L^p_x} + \|\nabla_x\times A^{\rm osc}(t)\|_{L^p_x} &\lesssim \langle t\rangle^{-3(1/2-1/p)}, \qquad p\in[2,\infty), \\ \|\partial_x^\alpha\partial_t A^r(t)\|_{L^p_x} + \|\partial_x^\alpha\nabla_x\times A^r(t)\|_{L^p_x} &\lesssim \langle t\rangle^{-4/3-|\alpha|/3+1/p+\delta}, \quad p\in[1,\infty]. \end{aligned} \tag{2.11}$$

From the decay estimates on the potential functions established in Theorem 2.2, we immediately obtain decay estimates on the electromagnetic fields through $E = -\nabla_x \phi - \partial_t A$ and $B = \nabla_x \times A$. Observe that the leading dynamics is oscillatory and dispersive like a Klein–Gordon wave of the form $e^{\pm i\sqrt{1-\Delta_x}t}$. On the other hand, the remainder has the property that derivatives in x gain extra decay. Interestingly, the extra decay rate for the magnetic potential A is only of $t^{-1/3}$ per extra derivative, not t^{-1} as would be expected from the dispersion for the transport dynamics. This is due to the emergence of a wave structure in the magnetic part due to the long range interaction, see (1.23). We refer the interested readers to [22] for the complete proof of the above damping results.

We shall end the article with some brief remarks on the literature, starting with the celebrated work by Mouhot and Villani [29] who proved the nonlinear Landau damping on the torus $\mathbb{T}^d$ for analytic or some high Gevrey data. This was extended and the proof was simplified in [4] and more recently in [17, 18]. These results were also adapted to the *relativistic* Vlasov–Poisson system in [34, 35] on the torus. Let us also mention the works [3, 12, 32] which are concerned with the regime of weak collisions, described by a Fokker–Planck operator for the former twos, and by a Landau collisional operator for the latter.

The stability problem on the whole space is proven to be extremely rich both physically and mathematically, including plasma oscillations or Langmuir's oscillatory waves, survival threshold, and Landau damping at the resonant frequency. For the screened Vlasov–Poisson system, plasma oscillations and survival threshold are absent, and the large time dynamics is proved to be well approximated by that of the free dynamics. This was first established by [5] via the Fourier approach and latter by [20] via the Lagrangian approach and dispersive estimates in the physical space. The results were sharpened in [23, 24] that include low dimensions. We also mention the work [25] which treats a special equilibrium, for which the Landau damping and the phase mixing incidentally occur at the same time scaling, see (1.14).

Systems including magnetic fields were less studied in the mathematical literature. We mention the works [7, 11] which are concerned with the linearized Vlasov–Poisson on the torus, in the presence of a constant magnetic field, in relation with the findings of [8]. For the study of the linear stability of general equilibria, we refer to [27, 28]. In [19], we have considered the relativistic Vlasov–Maxwell on the torus, and provided long (finite) time stability estimates, for well-prepared data; long time has to be understood in terms of powers of the speed of light c in the regime $c \to +\infty$.

Appendix A. Reformulation of dispersion functions

In this section, we reformulate the dispersion functions $D(\lambda, k)$ and $M(\lambda, k)$ defined as in (1.9). Precisely, we obtain the following.

Lemma A.1. *Let Υ be the maximal particle speed as in* (1.15)*. For each $k \neq 0$ and $\Re\lambda > 0$, we may write the electric and magnetic dispersion functions in the following form*

$$\begin{aligned} D(\lambda, k) &= 1 + \frac{1}{|k|^2} \int_{-\Upsilon}^{\Upsilon} \frac{u\kappa(u)}{-i\lambda/|k| + u} \,\mathrm{d}u, \\ M(\lambda, k) &= \lambda^2 + c^2|k|^2 - \frac{ic\lambda}{2|k|} \int_{-\Upsilon}^{\Upsilon} \frac{q(u)}{-i\lambda/|k| + u} \,\mathrm{d}u, \end{aligned} \tag{A.1}$$

for some functions $\kappa(u)$ and $q(u)$ that are even and non-negative on $[-\Upsilon, \Upsilon]$*. Explicitly, we have*

$$\begin{aligned} \kappa(u) &= -2\pi c^2 \int_{1/\sqrt{1-u^2/c^2}}^{\infty} \mu'(s) s^2 \,\mathrm{d}s, \\ q(u) &= 4\pi c^4 (1 - u^2/c^2) \int_{1/\sqrt{1-u^2/c^2}}^{\infty} \mu(s) s \,\mathrm{d}s. \end{aligned} \tag{A.2}$$

Proof. Indeed, in view of (1.9) with $\nabla_v \mu = \widehat{v}\mu'(\langle v \rangle)$, we write the electric dispersion function as

$$D(\lambda, k) = 1 - \frac{1}{|k|^2} \int_{\mathbb{R}^3} \frac{ik \cdot \widehat{v}}{\lambda + ik \cdot \widehat{v}} \mu'(\langle v \rangle) dv.$$

For $k \neq 0$, we introduce the change of variables $v \mapsto (u, w)$ defined by

$$u := \frac{k \cdot v}{|k|\langle v\rangle}, \qquad w := v - \frac{(k \cdot v)k}{|k|^2}, \tag{A.3}$$

with $u \in [-\Upsilon, \Upsilon]$ and $w \in k^\perp$, the hyperplane orthogonal to k. We note that the Jacobian determinant is

$$J_{u,w} = \langle v\rangle(1 - u^2/c^2)^{-1},$$

with $\langle v\rangle = \langle w\rangle/\sqrt{1 - u^2/c^2}$. Therefore, we have

$$D(\lambda, k) = 1 + \frac{1}{|k|^2}\int_{-\Upsilon}^{\Upsilon} \frac{u}{-i\lambda/|k| + u}\Big(-\int_{w\in k^\perp} \mu'\Big(\frac{\langle w\rangle}{\sqrt{1 - u^2/c^2}}\Big)\frac{\langle w\rangle}{(1 - u^2/c^2)^{3/2}}\,\mathrm{d}w\Big)\,\mathrm{d}u.$$

Letting $\kappa(u)$ be the integral term in $w \in k^\perp$, we obtain (A.1) for $D(\lambda, k)$. In addition, we note that we may parametrize the hyperplane $k^\perp = \mathbb{R}^2$ via polar coordinates with radius $r = |w|$, and set $s = \sqrt{1 + r^2/c^2}/\sqrt{1 - u^2/c^2}$, giving

$$\begin{aligned}\kappa(u) &= -\int_{\mathbb{R}^2} \mu'\Big(\frac{\langle w\rangle}{\sqrt{1 - u^2/c^2}}\Big)\frac{\langle w\rangle}{(1 - u^2/c^2)^{3/2}}\,\mathrm{d}w\\ &= -2\pi\int_0^\infty \mu'\Big(\frac{\sqrt{1 + r^2/c^2}}{\sqrt{1 - u^2/c^2}}\Big)\frac{\sqrt{1 + r^2/c^2}}{(1 - u^2/c^2)^{3/2}}\, rdr\\ &= -2\pi c^2\int_{1/\sqrt{1-u^2/c^2}}^\infty \mu'(s)s^2\,\mathrm{d}s,\end{aligned}$$

as defined in (A.2). Similarly, note that $\mathbb{P}_k\widehat{v} = \langle v\rangle^{-1}\mathbb{P}_k v = \langle v\rangle^{-1}w$ and so $|\mathbb{P}_k\widehat{v}|^2 = |w|^2\langle v\rangle^{-2} = (1 - u^2)|w|^2\langle w\rangle^{-2}$. Therefore, we may write

$$M(\lambda, k) = \lambda^2 + c^2|k|^2 - \frac{ic\lambda}{2|k|}\int_{-1}^1 \frac{1}{-i\lambda/|k| + u}\Big(-\int_{\mathbb{R}^2} \mu'\Big(\frac{\langle w\rangle}{\sqrt{1 - u^2/c^2}}\Big)\frac{|w|^2}{\langle w\rangle\sqrt{1 - u^2/c^2}}\,\mathrm{d}w\Big)\,\mathrm{d}u.$$

Letting $q(u)$ be the integral term in w, we obtain (A.1) for $M(\lambda, k)$. In addition, we compute

$$\begin{aligned}q(u) &= -\int_{\mathbb{R}^2} \mu'\Big(\frac{\langle w\rangle}{\sqrt{1 - u^2/c^2}}\Big)\frac{|w|^2}{\langle w\rangle\sqrt{1 - u^2/c^2}}\,\mathrm{d}w\\ &= -2\pi\int_0^\infty \mu'\Big(\frac{\sqrt{1 + r^2/c^2}}{\sqrt{1 - u^2/c^2}}\Big)\frac{r^2}{\sqrt{1 + r^2/c^2}\sqrt{1 - u^2/c^2}}\, rdr\\ &= -2\pi c^4\int_{1/\sqrt{1-u^2/c^2}}^\infty \mu'(s)\big(s^2(1 - u^2/c^2) - 1\big)\,\mathrm{d}s.\end{aligned}$$

Thus, integrating by parts in s gives (A.2), upon noting that the boundary terms vanish. The lemma follows. □

Appendix B. Spectral stability

In this section, we prove that there is no exponential growing mode of the linearized Vlasov–Maxwell system (1.1). Namely, we obtain the following.

Proposition B.1. *For any non-negative radial equilibria $\mu(\langle v\rangle)$ in $\mathbb{R}^3$, the linearized system* (1.1) *has no nontrivial solution of the form $e^{\lambda t + ik\cdot x}(\widetilde{f}_k, \widetilde{\phi}_k, \widetilde{A}_k)$ with $\Re\lambda \neq 0$ for any nonzero triple $(\widetilde{f}_k, \widetilde{\phi}_k, \widetilde{A}_k)$.*

Proof. In view of the resolvent equations (1.8), it suffices to prove that for each $k \in \mathbb{R}^3$, there are no zeros of the electric or magnetic dispersion relation: $D(\lambda, k) = 0$ or $M(\lambda, k) = 0$ with $\Re\lambda \neq 0$. Indeed, starting with the dispersion function $D(\lambda, k)$, we use (A.1) for $\lambda = \gamma + i\tau$ to write

$$\begin{aligned}D(\gamma + i\tau, k) &= 1 + \frac{1}{|k|^2}\int_{-\Upsilon}^{\Upsilon} \frac{u\kappa(u)}{-i\gamma/|k| + \tau/|k| + u}\,\mathrm{d}u\\ &= 1 + \frac{1}{|k|^2}\int_{-\Upsilon}^{\Upsilon} \frac{u(\tau/|k| + u)\kappa(u)}{|-i\gamma/|k| + \tau/|k| + u|^2}\,\mathrm{d}u + \frac{i\gamma}{|k|^3}\int_{-\Upsilon}^{\Upsilon} \frac{u\kappa(u)}{|-i\gamma/|k| + \tau/|k| + u|^2}\,\mathrm{d}u.\end{aligned}$$

Now fix $k \in \mathbb{R}^3$, and suppose that $D(\gamma + i\tau, k) = 0$ for some $\gamma \neq 0$. Taking the imaginary part of the above identity yields

$$\int_{-1}^{1} \frac{u\kappa(u)}{|-i\gamma/|k| + \tau/|k| + u|^2} \,\mathrm{d}u = 0.$$

Plugging this identity into $D(\gamma + i\tau, k)$, we get

$$D(\gamma + i\tau, k) = 1 + \frac{1}{|k|^2} \int_{-\Upsilon}^{\Upsilon} \frac{u^2\kappa(u)}{|-i\gamma/|k| + \tau/|k| + u|^2} \,\mathrm{d}u,$$

which never vanishes, since $\kappa(u) \geq 0$ by Lemma A.1. That is, $D(\lambda, k)$ never vanishes for $\Re\lambda \neq 0$. Similarly, we study the magnetic dispersion function $M(\lambda, k)$. Using (A.1) for $\lambda = \gamma + i\tau$, we compute

$$M(\gamma + i\tau, k) = \gamma^2 - \tau^2 + 2i\gamma\tau + c^2|k|^2 + \frac{c|\gamma + i\tau|^2}{2|k|^2} \int_{-1}^{1} \frac{q(u)}{|-i\gamma/|k| + \tau/|k| + u|^2} \,\mathrm{d}u$$
$$- \frac{ic(\gamma + i\tau)}{2|k|} \int_{-\Upsilon}^{\Upsilon} \frac{uq(u)}{|-i\gamma/|k| + \tau/|k| + u|^2} \,\mathrm{d}u.$$

Now suppose that $M(\gamma + i\tau, k) = 0$ for some $\gamma \neq 0$. The vanishing of the imaginary part gives

$$2\tau - \frac{c}{2|k|} \int_{-\Upsilon}^{\Upsilon} \frac{uq(u)}{|-i\gamma/|k| + \tau/|k| + u|^2} \,\mathrm{d}u = 0.$$

Plugging this identity into $M(\gamma + i\tau, k)$, we get

$$M(\gamma + i\tau, k) = \gamma^2 + \tau^2 + c^2|k|^2 + \frac{c|\gamma + i\tau|^2}{2|k|^2} \int_{-\Upsilon}^{\Upsilon} \frac{q(u)}{|-i\gamma/|k| + \tau/|k| + u|^2} \,\mathrm{d}u,$$

which again never vanishes, since $q(u) \geq 0$ by Lemma A.1. That is, $M(\lambda, k)$ never vanishes for $\Re\lambda \neq 0$. The proposition follows. □

Remark B.2. In fact, the results in Proposition B.1 hold for any non-negative radial equilibria $\mu(\langle v \rangle)$ without any monotonicity in dimensions $d \geq 3$, while monotonicity is required for equilibria in dimensions $d = 1, 2$.

Appendix C. Derivation of the survival threshold

In this section, we derive the survival threshold of wave numbers as introduced in Section 1.2, using the new formulation in Lemma A.1. We first observe that at $\lambda = i\tau$ with $|\tau| < |k|\Upsilon$ (namely, λ in the continuous spectrum of the transport operator $\partial_t + \widehat{v} \cdot \nabla_x$ with v in the support of $\mu(\langle \cdot \rangle)$), the electric dispersion function can be computed from (A.1), using the Plemelj's formula as $\Re\lambda \to 0^+$, giving

$$D(i\tau, k) = 1 + P.V. \int_{-\Upsilon}^{\Upsilon} \frac{u\kappa(u)}{u + \tau/|k|} \,\mathrm{d}u - \frac{i\pi\tau}{|k|} \kappa\Big(\frac{\tau}{|k|}\Big). \tag{C.1}$$

This proves that $D(i\tau, k)$ is nonzero for $0 < |\tau| < |k|\Upsilon$, since its imaginary part is nonzero. In addition, $D(0, k) = 1 + \frac{1}{|k|^2} \int_{-\Upsilon}^{\Upsilon} \kappa(u) \,\mathrm{d}u$, which is again nonzero, since $\kappa(u) \geq 0$. Next, we study the case when $|\tau| \geq |k|\Upsilon$, for which the integrand in (A.1) has no singularity. Using the fact that $\kappa(u)$ is even in u, for $\lambda = i\tau$, we write

$$D(i\tau, k) = 1 - \frac{1}{|k|^2} \int_{-\Upsilon}^{\Upsilon} \frac{u^2\kappa(u)}{\tau^2/|k|^2 - u^2} \,\mathrm{d}u, \tag{C.2}$$

for $|\tau| \geq |k|\Upsilon$. Clearly, $D(i\tau, k)$ is a strictly increasing function in $\tau^2 \in [|k|^2\Upsilon^2, \infty]$ with $D(\pm i\infty, k) = 1$ and

$$D(\pm i|k|\Upsilon, k) = 1 - \frac{1}{|k|^2} \int_{-\Upsilon}^{\Upsilon} \frac{u^2\kappa(u)}{\Upsilon^2 - u^2} \,\mathrm{d}u = 1 - \frac{\kappa_0^2}{|k|^2},$$

recalling the definition of κ_0 in (1.16). That is, for $|k| \leq \kappa_0$, there is a unique $\tau_*(k)$ in $[|k|\Upsilon, \infty]$ so that $D(\pm i\tau_*(k), k) = 0$, while for $|k| > \kappa_0$, $D(i\tau, k)$ never vanishes for all $\tau \in \mathbb{R}$. The zeros $\lambda_\pm = \pm i\tau_*(k)$ for $|k| \leq \kappa_0$ gives rise to purely oscillating modes, known as Langmuir's oscillatory waves in the physical literature [33], which obey a Klein–Gordon type dispersion relation. See [22, 30] for the rigorous proof of these results.

Appendix D. Klein–Gordon dispersion for magnetic potentials

In this section, we derive the Klein–Gordon type dispersion for the magnetic potential, using the new formulation in Lemma A.1. Indeed, as in the previous section, we first consider the range of $\lambda = i\tau$ with $|\tau| < |k|\Upsilon$. Using the Plemelj's formula as $\Re\lambda \to 0^+$, we obtain from (A.1),

$$M(i\tau, k) = -\tau^2 + c^2|k|^2 - \frac{c\tau}{|k|} P.V. \int_{-\Upsilon}^{\Upsilon} \frac{1}{u + \tau/|k|} q(u)\,\mathrm{d}u - \frac{i\pi\tau}{|k|} q\Big(\frac{\tau}{|k|}\Big) \tag{D.1}$$

which is nonzero for $0 < |\tau| < |k|\Upsilon$, since its imaginary part is nonzero. In addition, $M(0,k) = c^2|k|^2 \neq 0$ for $k \neq 0$. On the other hand, for $|\tau| \geq |k|\Upsilon$, the integrand in (A.1) is no longer singular. We can thus write

$$M(i\tau, k) = -\tau^2 + c^2|k|^2 + \frac{c\tau^2}{2|k|^2} \int_{-\Upsilon}^{\Upsilon} \frac{q(u)}{\tau^2/|k|^2 - u^2}\,\mathrm{d}u,$$

upon using the fact that $q(u)$ is even in u. Clearly, $M(i\tau, k)$ is a strictly decreasing function in $\tau^2 \in [|k|^2\Upsilon^2, \infty]$ with $M(\pm i\infty, k) = -\infty$ and

$$M(\pm i|k|\Upsilon, k) = (c^2 - \Upsilon^2)|k|^2 + \frac{c\Upsilon^2}{2} \int_{-\Upsilon}^{\Upsilon} \frac{q(u)}{\Upsilon^2 - u^2}\,\mathrm{d}u,$$

which is strictly positive, since $\Upsilon \leq c$ and $q(u) \geq 0$. That is, for all $k \in \mathbb{R}^3$, there is a unique $\nu_*(k)$ in $[|k|\Upsilon, \infty]$ so that $M(\pm i\tau_*(k), k) = 0$. In addition, $\nu_*(k)$ satisfies Klein–Gordon dispersion properties for all $k \in \mathbb{R}^3$. See [22] for the rigorous proof of these results.

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