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Propagating fronts and terraces in multistable reaction-diffusion equations

Thomas Giletti

Abstract

This short paper will be devoted to propagation phenomena for a general reaction-diffusion equation, i.e. when it may admit an arbitrarily large number of stationary states. Large time propagation can no longer be described by a single front, but by a family of several stacked fronts (or "propagating terrace") involving intermediate transient equilibria. We will review several strategies, differing in their range of application (homo- or heterogeneous, one- or multi-dimensional, semi- or non-linear equations…), to handle such dynamics.

1. Introduction

We first consider the scalar reaction-diffusion equation

$$\partial_t u = \Delta u + f(u), \quad t > 0, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where the function f is C^1 and admits at least two equilibria. More precisely, up to some normalization, we have that

$$f(0) = f(1) = 0.$$

Further below we will also deal with a few extensions of (1.1). For now, it is supplemented together with an initial datum

$$u(t = 0, \cdot\,) \equiv u_0 : \mathbb{R} \to [0, 1]. \tag{1.2}$$

The Cauchy problem is then well-posed and, by the parabolic comparison principle, the solution remains bounded between 0 and 1 for all positive times. However, so far we have made no assumption on the behavior of f between these two extremal steady states.

In many typical cases, some of which we will briefly review below, either steady state invades the other and the solution u converges to a (planar) travelling front connecting 0 and 1; see the seminal papers [3, 4, 9, 10, 18] for related results. Here a travelling front refers to a special entire in time solution of the form

$$u(t, x) = U(x \cdot e - ct), \tag{1.3}$$

where $e \in S^{d-1}$ is the travelling front direction, $c \in \mathbb{R}$ its speed, and its profile U also satisfies the following asymptotics

$$U(-\infty) = 1 > U(\,\cdot\,) > U(+\infty) = 0. \tag{1.4}$$

We point out that even in higher dimensions, the solution typically develops symmetries and asymptotically becomes one-dimensional [4, 16, 17].

This type of behavior has motivated in the past century a lot of mathematical activity and the use of reaction-diffusion models in a wide range of applications, from physics and the life sciences [21, 22, 25]: combustion, genetics, ecology and epidemiology, tumor growth… and wherever transition phenomena from one steady state to another are expected.

As we have only hinted at so far, dynamics strongly depend on the choice of the reaction term. Obviously none of the above occurs in the heat equation case $f \equiv 0$. In this short review, we will mainly follow this author's works [7, 12, 13, 14] (drawing also some inspiration from earlier mathematical literature [3, 9, 18, 29]). While the aforementioned references deal with slightly different set of assumptions, a common denominator is the following "multistable" assumption:

Assumption 1.1. Equation (1.1) is said to be of the multistable type if:

- there exist $K \geq 2$ and a finite sequence of linearly stable constant steady states
$$q_1 = 1 > q_2 > \cdots > q_{K-1} > q_K = 0,$$
i.e. such that
$$\forall\, 1 \leq k \leq K, \quad f(q_k) = 0 \text{ and } f'(q_k) < 0;$$
- any other constant steady state is linearly unstable, i.e.
$$\forall\, s \in [0,1] \setminus \{(q_k)_{1 \leq i \leq K}\}, \quad f(s) = 0 \Rightarrow f'(s) > 0.$$

Remark 1.2. At this stage, the reader may rightfully consider the above assumption unnecessary convoluted. Indeed, in the homogeneous context of (1.1), it simply means that f has a finite number of zeros between 0 and 1, all of which are simple. This wording is justified by the fact that several arguments described hereafter apply to more general situations, as we will try to make more explicit throughout this review. Assumption 1.1 is designed to better reflect the key features needed in the proofs.

Assumption 1.1 includes the more typical bistable case when $K = 2$. The prototype example is
$$f_{bistable}(u) = u(1-u)(u-\theta), \tag{1.5}$$
where 0 and 1 are the stable and $\theta \in (0,1)$ the unstable steady state. Another classical case (arguably simpler but not covered by Assumption 1.1 as we wrote it) is the monostable one, of which a prototype example is
$$f_{monostable} = u(1-u)(1+au), \tag{1.6}$$
where $a \in (0,+\infty)$, 0 is the unstable and 1 the stable steady state. When $a = 0$, one recovers the logistic map.

We sum up a few celebrated results in the next theorem:

Theorem 1.3 (Monostable and bistable cases)**.** *The following two statements hold true.*

- *If the reaction term is of the bistable type* (1.5)*, then* (1.1) *admits a unique (up to shift) travelling front connecting* 0 *and* 1 *with speed* $c^* \in \mathbb{R}$.
- *If the reaction term is of the monostable type* (1.5)*, then there exists* $c^* > 0$ *such that* (1.1) *admits a travelling front connecting* 0 *and* 1 *with speed* c *if and only if* $c \geq c^*$.

Furthermore, in both cases, the solution of the Cauchy-problem (1.1)–(1.2) *with dimension* $d = 1$ *and* u_0 *the Heaviside type initial datum*
$$u_0 = \mathbb{1}_{(-\infty,0)},$$
converges to the travelling front with speed c^*.

The existence of travelling fronts has been dealt with in [3] using ODE technics which we will sketch in Section 2. The celebrated work of Kolmogorov, Petrovsky and Piskunov included a proof of convergence for Heaviside type initial data but only in the particular logistic case [18]; we instead refer to [26, 27] for more general monostable nonlinearities. The bistable case has been addressed in particular in [9].

Remark 1.4. The last part of Theorem 1.3 is a blanket statement which hides various behaviors depending on the choice of the function f. In the bistable case, the initial datum must lie above the threshold θ on a large enough set, e.g.
$$u_0(-\infty) > \theta > u_0(+\infty).$$
In the monostable case, a hair-trigger effect occurs, that is a travellong front appears as soon as u_0 is non-trivial. On the other hand, different travelling fronts may be selected depending on the initial decay in the space variable. The minimal speed c^* is selected provided that the initial

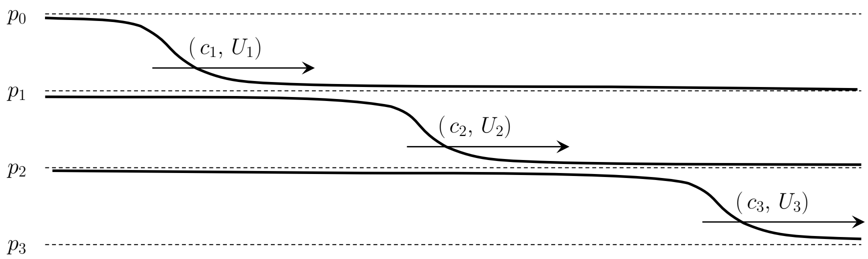

Figure 1.1: Example of a propagating terrace consisting of three fronts connecting the steady states p_0, p_1, p_2 and p_3.

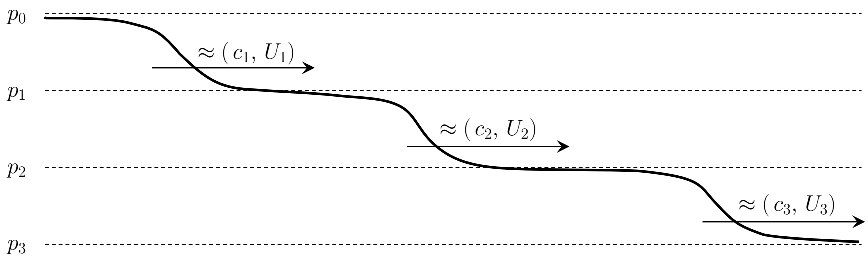

Figure 1.2: Asymptotic profile of a solution which approaches a propagating terrace.

datum's support is finite in the right direction, but the convergence then occurs in an appropriate moving frame possibly involving a logarithmic drift [6, 15]. We do not go into further details as the focus here will mostly be on the existence of travelling fronts, and their stability will only be considered as a by-product.

Under the more general Assumption 1.1, it should no longer be expected that a single travelling front connects directly 0 and 1. The transition between these two extremal equlibria may now occur into several steps involving intermediate steady states. This leads us to introduce the notion of a propagating terrace, which roughly refers to a family of stacked fronts with ordered speeds.

Définition 1.5. A propagating terrace $\mathcal{T}$ connecting 1 and 0 consists of:

- a finite sequence of steady states
$$p_1 = 1 > p_2 > \cdots > p_{I+1} = 0;$$
- a finite sequence of travelling fronts $(U_i)_{1\leq i\leq I}$ connecting respectively p_i and p_{i+1} with speeds c_i, i.e. solutions of the form $U_i(x - c_i t)$ with
$$U_i(-\infty) = p_i > U_i(\,\cdot\,) > U_i(+\infty) = p_{i+1},$$
such that also
$$c_1 < c_2 < \cdots < c_I.$$

Figures 1.1–1.2 illustrate this notion and are certainly more telling that the mathematical definition. We highlight that, even though a propagating terrace involves several different special solutions of (1.1), what we have in mind is to describe the large time behaviour of solutions of the Cauchy problem. In particular, Figure 1.2 hopefully clarifies why we impose that the travelling front speeds are ordered. Indeed this is consistent with the fact that lower level sets should spread faster than higher level sets, e.g. when the solution starts from Heaviside type initial data.

In this short review we will present three approaches to construct such propagating terraces under the multistable Assumption 1.1. There are of course other known methods to construct travelling fronts, such as fixed point theorems or topological degree arguments. However, in our opinion the three methods outlined here are better suited for a situation where more than a single front appears.

- ODE technics [12]: this is the most classical method and it was used as early as in the seminal papers [10, 18], and more extensively in [3, 9]. It is also arguably the most natural one as it simply relies on plugging the ansatz $U(x \cdot e - ct)$ in (1.1) and solving the resulting ordinary differential equation, together with the wanted asymptotics (1.4). While this approach can be used for some nonlinear equations or even reaction-diffusion systems, here we will benefit from an analogy with mechanics which is only available in the scalar homogeneous case.

- A steepness argument [7, 13]: this relies on a so-called intersection-number argument (or parabolic Sturm–Liouville theory), of which some ideas can already be found in [18] and which has been more recently developed in [2]. The starting point is the sharp steepness of Heaviside type initial data, which must be monotonically damped by diffusion. In its full generality, this argument applies to a wide range of (possibly nonlinear, heterogeneous, even degenerate) equations, but remains restricted to the scalar and spatially one-dimensional framework.

- Monotone iteration method [14]: our last argument draws inspiration from [8, 29] on monotone dynamical systems. It somehow again relies on the expectation that a travelling front or terrace appears in large time. The main idea is to force the time monotonicity of the solution in a moving frame by cancelling the time derivative when negative. It applies to a typically wider range of problems than the previous methods (though some non-degeneracy assumption is needed), and whenever a comparison principle is available.

The next three sections will each deal with one of those three methods and include more accurate statements on the existence of propagating terraces in a few contexts. The interested reader can of course check our related references [7, 12, 13, 14], where they were applied once to a discontinuous but mostly to spatially periodic heterogeneous equations. Here we will give slightly new results (including nonlinear diffusion or cooperative systems) which we hope will better illustrate the advantages of each method.

2. By ODE technics

By plugging the ansatz $u(t,x) = U(x \cdot e - ct)$ in the scalar equation (1.1), one immediately finds (regardless of spatial dimension) that U solves the second-order ODE

$$U'' + cU' + f(U) = 0. \tag{2.1}$$

However one should keep in mind that the front speed c is also unknown here. Therefore one's goal is typically to find (by some shooting method) the correct value of the parameter c such that (2.1) admits a solution satisying also

$$U(-\infty) = p_1 > U(\,\cdot\,) > U(+\infty) = p_2.$$

Here p_1 and p_2 are steady states of (1.1). In the monostable and bistable cases, $(p_1, p_2) = (0, 1)$. In the general multistable situation, a suitable combination of such travelling fronts must be constructed to meet the criteria of Definition 1.5.

In this section and in the footsteps of pioneering references [3, 18, 9], we will use this approach to prove the next theorem.

Theorem 2.1. *Consider* (1.1) *under the multistable Assumption 1.1. Then:*

- *there exists a (unique up to shifts) propagating terrace* $\mathcal{T} = \{(p_i)_{1\leq i\leq I+1}, (U_i)_{1\leq i\leq I}\}$ *in the sense of Definition 1.5;*

- *furthermore, all the steady states* p_i *are stable. In particular, in the bistable case* $K = 2$ *the propagating terrace consists of a single front.*

One recovers the very classical bistable result, but we omitted the monostable case in Theorem 2.1 for the sake of convenience. Indeed the latter does not fit into Assumption 1.1, and more importantly the outcome is slightly different as the admissible front speed is no longer unique (see Theorem 1.3 above). Regardless, we will try to include all these cases in the sketch of proof below.

A sketch of the mathematical proof. Rewriting 2.1 as a first-order system, we must solve

$$\begin{cases} p' = q \quad \text{and} \quad q' = -cq - f(p) \\ (p,q)(-\infty) = (p_1, 0), \quad (p,q)(+\infty) = (p_2, 0), \quad \text{and} \quad p_1 < p < p_2. \end{cases}$$

where $f(p_i) = 0$ for $i = 1, 2$. Let us start from $p_1 = 1$ which is linearly stable in all cases, and p_2 is to be determined later as it may or may not be 0 depending on how many fronts a terrace contains.

In terms of the ODE system, the point $(1, 0)$ in the phase plane is a saddle point as illustrated on Figure 2.1. In particular, there is a unique trajectory originating from $(1, 0)$ in the negative half-plane, which we will denote by $\tau_{1,c}$. The above problem is solved if and only if this trajectory remains in the region $\{p_2 < p < 1\}$ and goes to the point $(p_2, 0)$.

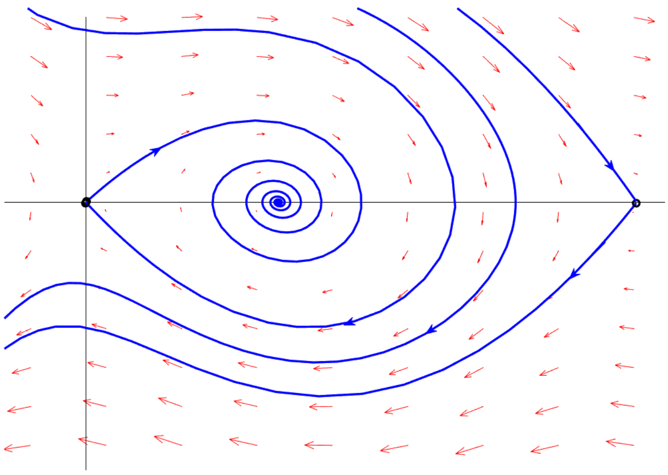

Figure 2.1: Phase plane, including a few special trajectories, for a bistable nonlinearity $f(u) = u(1-u)(u-0.3)$ with $c = 0.1$. We point out that both $(0, 0)$ and $(1, 0)$ are saddle points, while $(0.3, 0)$ is a stable spiral point.

The key point is the following monotonicity lemma with respect to c.

Lemma 2.2. *Let any $c_1 < c_2$. Then the corresponding trajectory τ_{1,c_1} lies below the trajectory τ_{1,c_2} as long as both remains in the region $\{q < 0\}$.*

Proof. On a neighborhood of $(1, 0)$, it relies on a phase plane analysis of the linearized ODE system

$$\begin{cases} p' = q \\ q' = -cq - f'(1)p. \end{cases}$$

Indeed, after a straightforward computation, one finds that the trajectory τ_c is tangential to the vector $(-1, -\lambda(c))$ where $\lambda(c) = \frac{-c+\sqrt{c^2-f'(1)}}{2}$ is a decreasing function of c.

Further away from $(1, 0)$, the trajectories of τ_{1,c_1} and τ_{1,c_2} may not interesect due to the fact that

$$-c_1 q - f(p) < -c_2 q - f(p),$$

as long as $q < 0$. □

This leads us to define

$$c^* := \sup\{c \,|\, \tau_{1,c} \text{ leaves the region } (0,1) \times (-\infty, 0) \text{ through } \{0\} \times (-\infty, 0)\}. \tag{2.2}$$

Provided this is a well-defined real number, one recovers in the trajectory τ_{1,c^*} a travelling front connecting 1 and some other steady state which may not necessarily be 0.

For further details on this last part of the argument, which varies drastically depending on the choice of the function f, we again refer to e.g. [3]. To keep it short, we sum it up in the next two points.

- A monostable function f is positive in $(0,1)$, which prevents any trajectory $\tau_{1,c}$ to cross the horizontal axis. Therefore by Lemma 2.2 the only possibilty is that $\tau_{1,c}$ connects $(1,0)$ and $(0,0)$ for all $c \geq c^*$, resulting in a continuum of admissible speeds as stated in Theorem 1.3.

- For a bistable f, the point $(0,0)$ is also a saddle point and a symmetrical argument could be made to find a single (monotonic in c) trajectory going to $(0,0)$. Then c^* is the single value such that both trajectories originating from $(1,0)$ and going to $(0,0)$ coincide.

- Finally, in the less standard multistable case, we only find that the trajectory τ_{1,c^*} must touch the horizontal axis, and by a continuity argument it must do so at some equilibrium point. This results in the slowest travelling front converging to 1 at $-\infty$. The rest of the terrace is constructed by induction.

A convenient analogy with classical mechanics. Interestingly, the mathematical argument above has a very clear and intuitive interpretation. The reader may have already recognized in

$$U'' + cU' + f(U),$$

a typical ODE arising from an application of Newton's second law. Under this new point of view, we identify:

- U as the position (on a horizontal axis) of a ball, depending on a new time variable;

- c as a friction parameter;

- $f(U)$ as the gravity force projected on the horizontal axis, if the ball rolls on the potential curve $F(U) = \int_0^U f$.

The existence of a travelling front can then be reformulated as follows: given any two equilibrium points, can the ball roll from one to another under the correct amount of friction?

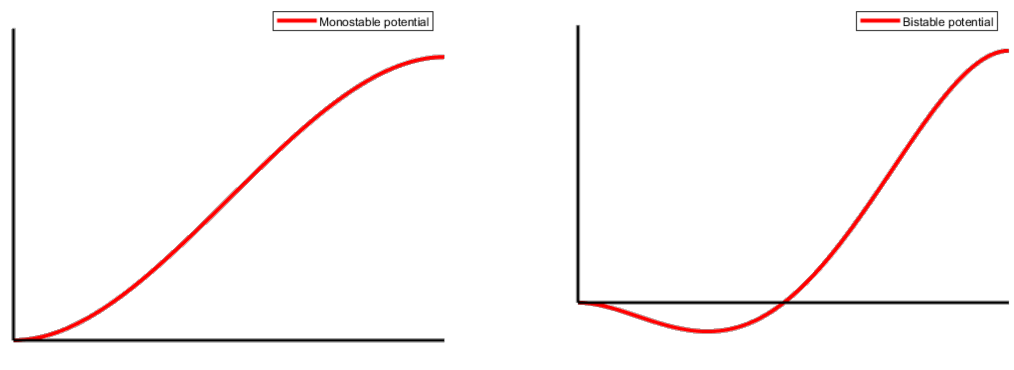

Figure 2.2: Monostable (left) and bistable (right) potentials.

In Figure 2.2 we draw the potential curves in both classical monostable and bistable cases. Physical intuition tells us that in the monostable case, a ball will move from the rightmost point 1 to the leftmost point 0 if and only if the friction parameter c exceeds some critical positive value; for lower values of c, the ball instead passes through 0 without stopping. On the other hand, in the bistable case the ball will pass 0 if friction is too weak, and go to the potential minimum (which is θ if reaction f is chosen as in (1.5)) if friction is too strong; the ball only has the wanted trajectory for a single value.

This short and intuitive argument immediately explains Theorem 1.3, that is the existence of infinitely many travelling fronts above a minimal speed in the monostable case, and the uniqueness of the admissible front speed in the bistable case. Furthermore, a second reading of the shooting

ODE argument, with this analogy in mind, will make it obvious that it can entirely be understood now in terms of this classical mechanics interpretation.

In particular, the monotonicity Lemma 2.2 basically means that the stronger the friction, the slower the ball will move, which will certainly come as no surprise. The fact that the set in the right-hand side of (2.2) is bounded is physically obvious.

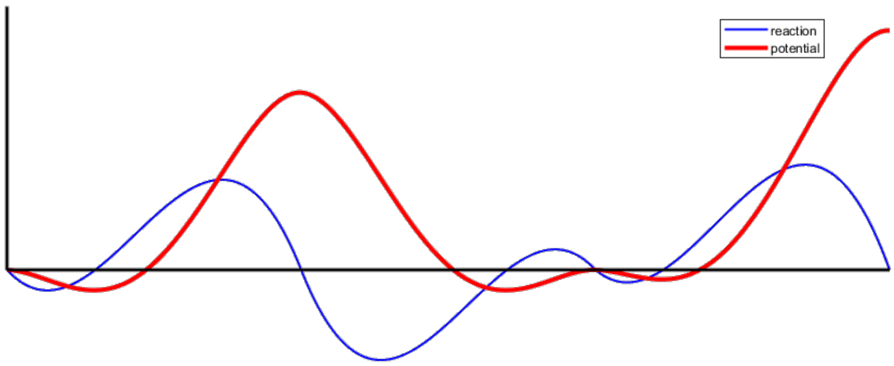

Figure 2.3: Multistable nonlinearity (blue) and the corresponding potential (red).

This analogy remains helpful in the multistable case, as we can illustrate on the example provided in Figure 2.3. The physical interpretation suggests that there cannot be a travelling front connecting directly the rightmost point 1 to 0. Recalling (2.2), we instead expect to find a travelling front with speed $c^* = c_1$ connecting 1 to some intermediate equilbrium which here is the second highest point of the potential. Another travelling front then connects this intermediate point to 0 with a (faster) speed c_2.

The fact that the speeds of those two travelling fronts are ordered correctly may not be as obvious as the physical intuition makes it seem. Roughly speaking, if friction is slightly less than c_1 and if a first ball starts from 1, then it crosses the intermediate equlibrium with a very slow speed and its trajectory is then close to that of a second ball starting from the intermediate equilibrum. If by contradiction the friction c_1 is larger than c_2, then the ball shall not cross 0 which contradicts the definition of c_1.

Remark 2.3. We mention that the travelling wave with speed c^* may act as a supersolution. Moreover, as a by-product of the above arguments, for any c less than the minimal travelling front speed c^*, the trajectory $\tau_{1,c}$ defines a sign-changing solution. Cutting it by 0 on a right half-line, we find a subsolution. Therefore, for any initial data in-between, the solution should spread with the speed c^*, which is also a celebrated result of e.g. [3]. As a matter of fact, these sub- and supersolutions are not readily ordered, but some perturbation argument makes this argument rigorous.

Finally, we end this section by briefly pointing out that ODE technics can be used in a few more general cases, including some reaction-diffusion systems, equations with delay in time, or some special heterogeneities e.g. piecewise constant.

3. By steepness arguments

The strategy in this section will be to find the travelling front or propagating terrace at the large time limit of the solution of the Cauchy problem, in dimension $d = 1$, starting from a Heaviside type initial data

$$u_0 = \mathbb{1}_{(-\infty,0)}. \tag{3.1}$$

The key step is to prove that the "steepness" of this solution is decreasing in time. A more rigorous meaning of steepness will be given below. This will be done in two different ways, which we will refer to as "analytical" and "topological".

The analytical way has an application range similar to the ODE technics presented in the previous section, and as a matter of fact can be interpreted as an evolution problem in the phase plane of (2.1). It can be traced back to the celebrated works [9, 18]. We should also mention a much more recent paper [1] which introduces a new notion of "steepness defect" to obtain very refined results of convergence to a travelling front.

The topological way relies on the non-increase of the number of zeros of solutions of a linear parabolic equation, or parabolic Sturm–Liouville theory [2, 11, 20]. We used it in [7, 13] to get the existence of propagating terraces in the spatially periodic framework. Indeed this method is very well-suited for spatial or time-heteregeneity since the parabolic Sturm–Liouville theory applies regardless.

Both ways are restricted to the spatially one-dimensional scalar equation, for reasons that will be made clearer below. This may seem like a moot point since in the homogeneous case, a planar travelling front of form (1.3) is always a solution of the one-dimensional problem. Still we want to highlight again that the steepness argument (at least in the topological way) remains valid in many heterogeneous cases. We refer to [23] where it is used to construct so-called "critical fronts". Since there are many different notions of travelling fronts for heterogeneous equations, to avoid this discussion and for the sake of brevity we instead choose to apply this argument to the nonlinear reaction-diffusion equation

$$\partial_t u = \partial_x^2(D(u)) + f(u). \tag{3.2}$$

We will prove the following theorem.

Theorem 3.1. *If D is smooth, $D' > 0$ and Assumption 1.1 holds, then equation* (3.2) *admits a propagating terrace connecting* 1 *and* 0*.*

3.1. Analytical way

Recall that we start from a Heaviside type initial datum. Then by the comparison principle, the solution is decreasing in space for all positive times. In particular, for each $t > 0$ and thanks to the spatial one dimensionality,

$$x \mapsto u(t,x),$$

is a bijection from $\mathbb{R}$ to $(0,1)$. This allows us to define a function $p : \mathbb{R}_+ \times (0,1) \to \mathbb{R}$ by

$$p(t, u(t,x)) := \partial_x u(t,x). \tag{3.3}$$

We call this the steepness function since it returns the slope of the solution at each level set. Moreover, for each time $t > 0$, the graph of $p(t,\cdot)$ is the trajectory of $u(t,\cdot)$ in the phase plane of (2.1).

After a straightforward computation taking advantage of the homogeneity of (3.2), we find that

$$\partial_t p = p^2 \partial_u^2 (D'(u)p) - \partial_u p f(u) + p f'(u). \tag{3.4}$$

Though it is degenerate, this is a parabolic equation and it satisfies a comparison principle. In particular, since we started from a Heaviside initial datum which has infinite slope, we deduce that p is negative and increasing in time. Therefore it converges to some function p_∞ as $t \to +\infty$. A priori this convergence is only pointwise, but parabolic estimates are available by possibly going back to the original equation.

Eventually, we find that p_∞ is the (constant in time) steepness function of a (entire in time) solution u_∞ of (3.2). Reverting transformation (3.3) back to the original variables, it follows that

$$u_\infty(t,x) = U_\infty(x - m(t)),$$

i.e. it has the same profile for all times and only varies by some time dependent shift $m(t)$. Plugging this into (3.2), we get that $m'(t) \equiv c \in \mathbb{R}$. It follows that U_∞ is either a constant solution or a travelling front, which by spatial monotonicity has to connect two steady states of (3.2).

The above sketch ignored some subtlety. Indeed, if p_∞ takes the value 0, then we no longer have a bijection between $\mathbb{R}$ and $(0,1)$. Reverting back in the original variable we now find a collection of travelling front solutions of (3.2) connecting consecutive steady states. Finally, for this collection to form a propagating terrace, it remains to check that their speeds are correctly ordered. This follows from an intuititve contradiction argument: since they have been obtained as the large time limit of a decreasing in space solution, lower travelling fronts have to spread faster than the upper ones. This last step also implies that the steady states must be stable: if otherwise there is an

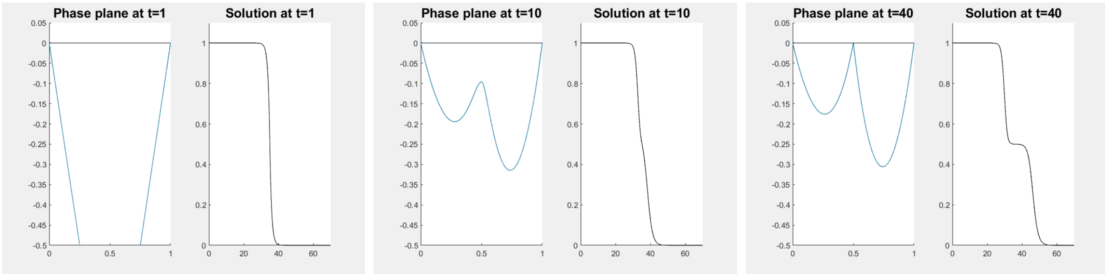

Figure 3.1: Numerical simulation of the steepness function (left) and the solution profile (right) for $t = 1, 10, 40$ when starting from a Heaviside type initial datum, and the reaction is tristable.

unstable state, then some upper travelling front must have positive speed and some lower front would have negative speed. Therefore by Assumption 1.1 this collection of travelling fronts is finite.

This method is illustrated in Figure 3.1 where we see that the steepness function approaches the value 0 once in the interval $(0, 1)$, which means that the solution converges to a propagating terrace consisting of two fronts. One advantage is that, compared to the ODE technics of the previous section where an induction was needed in general, here all the parts of the terrace are obtained "simultanenously" as the limit of steepness. It is also possible to adapt this argument in the case when f has a finite number of (not necessarily simple) zeros. One may even go further and consider more degenerate situations where the collection of limiting fronts is infinite, as in [24] where the notion of a system of waves was introduced.

3.2. Topological way

As we outlined before, there is a more general way to characterize steepness. The so-called parabolic Sturm–Liouville principle refers to the non-increase in time of the number of zeros (or rather sign changes) of solutions of linear parabolic equations. It is related to the more famous elliptic Sturm–Liouville theory which typically states that a second-order elliptic operator $\mathcal{L}$ with

$$-\mathcal{L}u := a(x)\partial_x^2 u + b(x)\partial_x u + c(x)u,$$

in a bounded interval supplemented together with boundary conditions, admits a Hilbertian basis of eigenfunctions

$$(\varphi_k)_{k\geq 0}, \qquad \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots$$

such that also the number of zeros of φ_k is a decreasing function of the rank k. Translating this in the parabolic problem, the solution can be written as

$$u(t,x) = \sum_k a_k e^{-\lambda_k t}\varphi_k(x). \tag{3.5}$$

We recover that higher frequencies dissipate faster, just like by Fourier analysis for the heat equation. This suggests that the number of zeros of the solution should be a non-increasing function of time.

This is confirmed by the following property, obtained in [2, 20].

Lemma 3.2. *Let u be a solution of the following linear parabolic equation in dimension* 1*,*

$$\partial_t u = a(t,x)\partial_x^2 u + b(t,x)\partial_x u + c(t,x)u, \qquad t > 0,\ x \in \mathbb{R},$$

where the coefficients a, b and c are smooth and bounded. Define also

$$\mathcal{Z}[w] = \sup\{k \in \mathbb{N} \mid \exists\, x_1 < x_2 < \cdots < x_{k+1}, \quad \forall\, 1 \leq i \leq k, \quad w(x_i).w(x_{i+1}) < 0\},$$

the number of sign changes of any function w.

Then

$$t \mapsto \mathcal{Z}[u(t,\cdot)]$$

is a non-increasing function.

Due to the unboundedness of the domain and the time dependance of the coefficients, clearly the above spectral argument does not work. As a matter of fact, even if it did it is actually very difficult to compute the number of zeros of the solution using (3.5). Interestingly, the number of zeros of such a series can actually be estimated thanks to Lemma 3.2 in a converse way. We refer to [11] for historical notes on this Sturm–Liouville theory, and to [2, 20] for full proofs of Lemma 3.2.

Sketch of the proof of Lemma 3.2. The actual argument relies on the comparison principle. For simplicity (and because this is actually enough for our purpose of constructing a propagating terrace), we assume that u changes sign once at time 0. We now consider $t_1 > 0$ and aim to show that $\mathcal{Z}[u(t_1, \cdot\,)] \leq 1$.

First, let x_+ be any point such that $u(t_1, x_+) > 0$, and ω_+ be the connected component of the positive set of u containing (t_1, x_+). Then ω_+ must touch the parabolic boundary, i.e. $t = 0$. Otherwise, by the comparison principle we would get that $u \equiv 0$ in ω_+, a contradiction. Therefore there exists a continuous path in ω_+ connecting (t_1, x_+) to some $(0, x_{0,+})$.

Proceeding in the same way, for any x_- such that $u(t_1, x_-) < 0$, there exists a continuous path connecting (t_1, x_-) to some $(0, x_{0,-})$, and such that u is negative along this path. In particular, these two paths cannot intersect. This means that $x_+ - x_-$ has the same sign than $x_{0,+} - x_{0,-}$, which does not depend on the choice of x_+ and x_- due to the fact that $\mathcal{Z}[u(0, \cdot\,)] = 1$. We conclude that $u(t_1, \cdot\,)$ changes sign at most once. ☐

Remark 3.3. Before we proceed, we highlight the following two points.

- Let us denote by $SGN[w(\,\cdot\,)]$ the word consisting of $+$ and $-$ that describe the sign of $w(x_1), \ldots, w(x_{k+1})$ in the definition of $\mathcal{Z}$ with maximal k. Then, as a by-product of the proof of Lemma 3.2, we find that
$$t \mapsto SGN[u(t, \cdot\,)]$$
is also a non-increasing function, with respect to the subword order.

- One may naively wonder if the number of connected components of the zero set of a solution is still non-increasing in time in higher dimensions. This is false in general and one can easily construct a counter-example by looking at the heat equation in $\mathbb{R}^2$, with an initial datum positive on a dumbbell shaped subdomain and negative outside.

  This is consistent with the crucial role which the two-dimensionality of the parabolic domain played in the proof of Lemma 3.2. By some spectral arguments, one may only find some decreasing in time estimates of the zero set of the solution [19].

With this lemma in hand, the proof of existence of a terrace proceeds similarly to the previous subsection. Indeed, notice that the initial datum $u_0 = \mathbb{1}_{(-\infty,0)}$ is steeper than any function w taking values in $(0, 1)$, in the sense that
$$\mathcal{Z}[u_0 - w] = 1, \qquad SGN[u_0 - w] = [+, -].$$
Denote now by u the solution associated with the Heaviside initial datum u_0, and by v any other solution of (3.2). Then $u - v$ solves
$$\begin{aligned}
\partial_t(u - v) &= \partial_x^2(D(u) - D(v)) + f(u) - f(v) \\
&= D'(u)\partial_x^2 u - D'(v)\partial_x^2 v + D''(u)(\partial_x u)^2 - D''(v)(\partial_x v)^2 + f(u) - f(v) \\
&= D'(u) \times \partial_x^2(u - v) + \frac{D'(u) - D'(v)}{u - v}\partial_x^2 v \times (u - v) \\
&\qquad + D''(u)(\partial_x u + \partial_x v) \times \partial_x(u - v) + \frac{D''(u) - D''(v)}{u - v}(\partial_x v)^2 \times (u - v) \\
&\qquad + \frac{f(u) - f(v)}{u - v} \times (u - v),
\end{aligned}$$
which we may see as a linear equation with bounded coefficients. Therefore, it follows from Lemma 3.2 and Remark 3.3 that u is steeper than v for all times, in the sense that
$$\forall\, t > 0, \quad \mathcal{Z}[u(t, \cdot\,) - v(t, \cdot\,)] \leq 1, \tag{3.6}$$

and $SGN[u(t,\cdot) - v(t,\cdot)]$ is a subword of $[+,-]$. As a side note, it is equivalent to say that $p^u(t,\cdot) \leq p^v(t\cdot,)$, where p^u and p^v denote the steepness functions (as defined in subsection 3.1) associated respectively with u and v. Thus, the comparison principle for (3.4) is in fact a special case of this new steepness argument.

For any $\lambda \in (0,1)$ define $m_\lambda(t)$ as the (unique) point such that $u(t, m_\lambda(t)) = \lambda$. By parabolic estimates and compact embeddings, up to extraction of a subsequence $t_n \to +\infty$ we have as $n \to +\infty$ that

$$u(t + t_n, x + m_\lambda(t_n)) \to u_\infty(t,x).$$

From (3.6) and a continuity argument, $SGN[u_\infty(t,\cdot) - v(t,\cdot)]$ is a subword of $[+,-]$ for any $t \in \mathbb{R}$ and any entire in time solution v of (3.2). In particular, u_∞ has to be steeper than itself and any of its time shift. In a similar fashion as in Subsection 3.1, this is only possible if u_∞ is of the form

$$u_\infty(t,x) = U_\infty(x - m(t)),$$

and plugging this into (3.2) this is a travelling front.

By taking different values of λ in the interval $(0,1)$, we again find a collection of travelling fronts with ordered speeds, which under Assumption 1.1 can be shown to be finite. This ends the construction of a propagating terrace of (3.2).

Remark 3.4. The steepness property also implies that the limit u_∞ does not depend on the choice of the subsequence $t_n \to +\infty$. In particular, we recover as in the analytical way the convergence to a travelling front of the solution with Heaviside type initial datum. It also follows that the limiting travelling fronts are steeper than any other solutions, and by redoing the argument this convergence can be extended to any steep enough initial data.

4. By an iterative method

We end this review with a third proof which almost purely relies on the comparison principle. It again uses the a posteriori knowledge that travelling fronts appear in the large time behaviour of solutions of the Cauchy problem. Instead of taking advantage of the decrease in time of the solution's steepness, here we will force the solution itself to become monotonic in time in any moving frame. Since such monotonicity obviously does not hold, this is done through some well-chosen perturbation, spatially localized enough to disappear in the large time limit.

To highlight a situation where this third method works and not the previous two, we will consider here a cooperative system in an abstract form, i.e.

$$\begin{cases} \partial_t u = d_u \Delta u + F(u,v), \\ \partial_t v = d_v \Delta v + G(u,v), \end{cases} \tag{4.1}$$

where

$$v \mapsto F(u,v),\ u \mapsto G(u,v) \ \text{ are non-decreasing.}$$

In particular this generates a monotonic semiflow and the parabolic comparison principle is available. On the other hand, even in spatial dimension 1 the parabolic Sturm–Liouville theory does not apply here.

The notions of travelling fronts and propagating terraces are the same as before, the only difference being that solutions are now valued in $\mathbb{R}^2$. We will also make the multistable Assumption 1.1, tailored in a straightforward way to a reaction-diffusion system. For convenience we will equip $\mathbb{R}^2$ with the partial order

$$(x_1, y_1) \preceq (x_2, y_2) \iff [x_1 \leq x_2 \text{ and } y_1 \leq y_2].$$

Then Assumption 1.1 becomes:

- $(0,0)$ and $(1,1)$ are the extremal steady states, i.e.,

$$F(0,0) = G(0,0) = 0, \qquad F(1,1) = G(1,1) = (0,0);$$

- there exists a finite and ordered sequence of linearly stable constant steady states

$$q_1 = (1,1) \succ q_2 \succ \cdots \succ q_K = (0,0);$$

- any other constant steady state is linearly unstable.

With a slight abuse of notation, we will still refer to this set of hypotheses as Assumption 1.1.

Following the strategy of [14, 29], in this section we will sketch the proof of the theorem below.

Theorem 4.1. *Under Assumption 1.1, system* (4.1) *admits a propagating terrace connecting* $(1,1)$ *and* $(0,0)$.

Sketch of the proof of Theorem 4.1. As for the scalar equation, planar travelling fronts are one-dimensional and therefore we can put ourselves in this framework (though this is not necessary for this proof). We will also denote by

$$(u,v)\,(\cdot,\cdot\,;u_0,v_0),$$

the solution of (4.1) associated with the initial pair (u_0,v_0).

From now on, we consider the initial datum

$$u_0 = v_0 = (1-\delta) \times \mathbb{1}_{(-\infty,0)},$$

where $\delta > 0$ is small enough so that $(1-\delta, 1-\delta)$ lies in the basin of attraction of $(1,1)$. Then, for any $c \in \mathbb{R}$ we define a sequence by the induction:

$$\begin{cases} u_{c,n+1} : x \in \mathbb{R} \mapsto \max\{u_0(x), u(1, x+c; u_{c,n}, v_{c,n})\}, \\ v_{c,n+1} : x \in \mathbb{R} \mapsto \max\{v_0(x), v(1, x+c; u_{c,n}, v_{c,n})\}. \end{cases} \tag{4.2}$$

To explain this, let us point out that when discarding the maximum in (4.2), then the sequence is merely the solution of (4.1) in the moving frame with speed c, taken at integer times. Moreover, u_0 and v_0 have a bounded support in the right direction, so in that sense they are spatially "localized" and we hope that they should not have too much impact on the large time behavior. The role of the maximum in (4.2) is to make this sequence monotonic.

Indeed, we have by definition that

$$(u_{c,1}, v_{c,1}) \succeq (u_0, v_0).$$

Then by the comparison principle

$$(u,v)\,(1, x+c; u_{c,1}, v_{c,1}) \succeq (u,v)\,(1, x+c; u_0, v_0),$$

hence

$$(u_{c,2}, v_{c,2}) \succeq (u_{c,1}, v_{c,1}).$$

The monotonicity of the whole sequence follows from a straightforward induction.

Therefore, for each $c \in \mathbb{R}$ the sequence converges to some

$$(u_{c,\infty}, v_{c,\infty}).$$

Since this may be understood as some large time limit in the moving frame with speed c, our hope now should be to choose the correct c so that $(u_{c,\infty}, v_{c,\infty})$ is a travelling front. Unfortunately, it will turn out below that things are not so simple.

For now, we only know from standard parabolic estimates that

$$\begin{cases} u_{c,\infty}(x) = \max\{u_0(x), u(1, x+c; u_{c,\infty}, v_{c,\infty})\}, \\ v_{c,\infty}(x) = \max\{v_0(x), v(1, x+c; u_{c,\infty}, v_{c,\infty})\}. \end{cases} \tag{4.3}$$

After another use of the comparison principle and passing to the limit, we also get that $u_{c,\infty}$ and $v_{c,\infty}$ are non-increasing in space, hence with respect to c as well. In particular,

$$(u_{c,\infty}, v_{c,\infty})\,(-\infty) = q_1 \succeq (u_{c,\infty} v_{c,\infty})\,(+\infty) = q_2.$$

Since $(1-\delta, 1-\delta)$ lies in the basin of attraction of $(1,1)$, one may check that $q_1 = (1,1)$. On the other hand, by the strong comparison principle, we have that $u_{c,\infty} > 0 = u_0$ and $v_{c,\infty} > 0 = v_0$ on $(0,+\infty)$. Passing to the limit in (4.3) as $x \to +\infty$, we get that

$$q_2 = (u,v)\,(1, \cdot + c; q_2),$$

i.e. q_2 can be extended into a 1-periodic in time and constant in space solution of (4.1). Due to its cooperative nature, the underlying ODE system does not admit any cycle and this implies that q_2 is actually a steady state.

Remark 4.2. If also $(u_{c,\infty}, v_{c,\infty}) \succ (u_0, v_0)$, then the pair $(u_{c,\infty}, v_{c,\infty})$ is (almost) a stationary solution of (4.1) in the moving frame with speed c. More precisely, it is stationary along integer times:

$$\forall\, n \in \mathbb{Z}, \quad (u_{c,\infty}, v_{c,\infty})(n, \cdot\,) \equiv (u_{c,\infty}, v_{c,\infty})(0, \cdot\,).$$

By a sliding argument in the spirit of [5], it is possible to infer that $(u_{c,\infty}, v_{c,\infty})$ is either a steady state (if $q_2 = 1$) or a travelling front. Alternatively, one may replace 1 in induction (4.2) by an arbitrarily small sequence of time steps.

According to Remark 4.2, we would like to pick c such that both $(u_{c,\infty}, v_{c,\infty}) \succ (u_0, v_0)$ and $q_2 < 1$. It turns out that both critera cannot be met simultaneously and another limiting argument will be needed. Regardless, this put us on the right track to choose the correct value of the travelling front speed c. That is, we define

$$c_1^* := \sup\{c \in \mathbb{R} \,|\, (u_{c,\infty}, v_{c,\infty}) = (1,1)\}. \tag{4.4}$$

Without going into the details, it is not so hard to check that such a c_1^* is a well-defined real number, for instance by the comparison principle and framing solutions between that of two linearized systems, whose speeds can be computed more or less explicitely [28, 30].

This choice of c_1^* relies on the a posteriori knowledge that the travelling front speed is also the large time spreading speed of solutions. More precisely, the solution with Heaviside type initial datum converges to $(1,1)$ in a moving frame with speed c if and only if c is strictly less than the travelling front speed. This is consistent with (4.4), provided of course that the induction (4.2) is close enough to the actual solution.

We are now in a position to conclude this (sketch of a) proof. Thanks to the spatial monotonicity, for any $c < c_1^*$ there is n_c large enough such that

$$(u_{c,n_c}, v_{c,n_c}) \succ (u_0, v_0),$$

and then x_c such that

$$x_c = \sup\{x \,|\, (u_{c,n_c}, v_{c,n_c}) \succ (1-\delta, 1-\delta)\}.$$

Then after a few complications that we will skip here, one eventually finds that $(u_{c,n_c}, v_{c,n_c})(\,\cdot + x_c)$ converges as $c \to c_1^*$ to a travelling front U_1 connecting $(1,1)$ to another steady state q_2.

It still remains to reiterate, when $q_2 \neq (0,0)$, until we construct a whole propagating terrace connecting $(1,1)$ and $(0,0)$. Naturally, we want to do the same argument again, replacing $(1,1)$ by q_2, to get a travelling front connecting q_2 to some q_3 with speed c_2^*. However, we need:

- q_2 to be stable in order for this argument to work;
- to check that the speeds of travelling fronts are ordered, i.e. $c_2^* \geq c_1^*$.

Both points follow from a similar argument. Proceed by contradiction and assume that q_2 is unstable, and even linearly unstable according to Assumption 1.1. Intuitively, and keeping in mind that we are considering spatially decreasing solutions, this means that any solution close to but above q_2 should move with positive speed (i.e. rightward), and any solution close to but below q_2 should move with negative speed (i.e. leftward). Therefore, going back to our construction the upper level sets of the sequence (u_{c,n_c}, v_{c,n_c}) should move faster than the lower level sets, a contradiction.

More precisely, by standard arguments [28, 30], we have on the one hand that

$$c_1^* \geq c_{lin} > 0,$$

where c_{lin} is the speed of the linearized system around q_2. On the other hand, going back to the family $(u_{c,n_c}, v_{c,n_c})\,(\,\cdot + x_c)$ which converges to U_1 as $c \nearrow c_1^*$, we have that

$$(u_{c,n_c}, v_{c,n_c})\,(\,\cdot + x_c) \prec q_2 + (\delta, \delta),$$

on a right half line $(x^\delta, +\infty)$, where $\delta > 0$ is arbitrarily small, independently of c close to c_1^*. Due to also $(u_{c,n_c}, v_{c,n_c})(+\infty) = (0,0)$ for each c, we can then compare it with another solution of the linearized problem around q_2, propagating with speed arbitrarily close to c_{lin}, but in the left direction. It ultimately follows that $\lim_{x\to+\infty} \limsup_{c \nearrow c_1^*} (u_{c,n_c}, v_{c,n_c})(x + x_c) \prec q_2$. This contradicts the fact that it converges as $c \nearrow c_1^*$ to a travelling front connecting $(1,1)$ and q_2.

We deduce that q_2 is (linearly) stable and by reiterating the previous argument we get a second travelling front U_2 with speed c_2^*, connecting q_2 to some lower steady state q_3. Finally, let us exclude by contradiction the situation when $c_2^* < c_1^*$. Then $U_2(x - c_1^* t)$ is a supersolution of (4.1). Due also to

$$U_2(-\infty) = U_1(+\infty) = q_2,$$

and using some classical perturbation in the spirit of [9], we can glue $U_1(x - c_1^* t)$ and $U_2(x - c_2^* t)$ into a single supersolution, still moving with speed c_1^*, and connecting $(1,1)$ and q_3. By yet another comparison with the family (u_{c,n_c}, v_{c,n_c}), one eventually finds that

$$\lim_{x \to +\infty} \lim_{c \nearrow c_1^*} (u_{c,n_c}, v_{c,n_c})(x + x_c) \prec q_3,$$

a contradiction with the fact that its limit as $c \nearrow c_1^*$ is the travelling front U_1 such that $U_1(+\infty) = q_2$.

The announced induction finally ends this sketch of the proof of Theorem 4.1 for cooperative systems, as well as this review of a few methods to construct propagating terraces. □

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