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Abstract

This expository article, written for the proceedings of the Journées EDP 2023, presents mainly joint works with Dolbeault and Laflèche [1] and Mouhot [3]. We will review some results about long time behaviour of linear kinetic equations for which the microscopic equilibrium (that is, the kernel of the reorientation operator) is typically a density with polynomial decay. There will be no space confinement and the reorientation operator could be of scattering, Fokker–Planck or Levy–Fokker–Planck types. We will first present a spectral approach a la Ellis and Pinsky that yields to a unified treatment of the macroscopic limits for this kind of equations and then focus on re-shaping the Dolbeault–Mouhot–Schmeiser L^2-hypocoercivity method to get explicit rates of decay to zero in suitable weighted norms.

In this note, we consider solutions to the kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad f(0,x,v) = f^{\rm in}(x,v), \tag{0.1}$$

when the *local equilibrium* $\mathcal{M}$ has a *fat tail* given for some $\gamma > 0$ by

$$\forall\, v \in \mathbb{R}^d, \quad \mathcal{M}(v) = \frac{c_\alpha}{\lfloor v \rceil^{d+\alpha}} \quad \text{where} \quad \lfloor v \rceil := \sqrt{1+|v|^2}\,. \tag{0.2}$$

The relevance of such equilibria comes from a full literature in biology and physics, that we shall not review here, but which is presented in [1, 3].

In (0.1), the distribution function $f(t,x,v)$ depends on a *position* variable $x \in \mathbb{R}^d$, on a *velocity* variable $v \in \mathbb{R}^d$, and on *time* $t \geq 0$. The *collision operator* L acts only on the v variable and, by assumption, its null space is spanned by $\mathcal{M}$. In (0.2) the normalization constant is $c_\alpha = \pi^{-d/2}\,\Gamma\big((d+\alpha)/2\big)/\Gamma(\alpha/2)$ and associated to the measure

$$\mathrm{d}\mu = \mathcal{M}^{-1}(v)\,\mathrm{d}v\,,$$

we define a scalar product and a norm respectively by

$$\langle f, g \rangle := \int_{\mathbb{R}^d} \bar{f}\, g\, \mathrm{d}\mu \quad \text{and} \quad \|f\|^2 := \int_{\mathbb{R}^d} |f|^2\, \mathrm{d}\mu \tag{0.3}$$

for functions f and g of the variable $v \in \mathbb{R}^d$. Here, $\bar{f}$ denotes the complex conjugate of f, as we shall later allow for complex valued functions. For any $k \in \mathbb{R}$, we define

$$\|f\|_k := \|f\|_{\mathrm{L}^2(\lfloor v \rceil^k\, \mathrm{d}x\, \mathrm{d}\mu)}\,,$$

and

$$|||f|||_k := \|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v) \cap \mathrm{L}^2(\lfloor v \rceil^k\, \mathrm{d}x\, \mathrm{d}\mu)} := \Big(\|f\|^2_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)} + \|f\|^2_{\mathrm{L}^2(\lfloor v \rceil^k\, \mathrm{d}x\, \mathrm{d}\mu)} \Big)^{1/2}\,.$$

We consider three examples of linear collision operators, that cover a wide class of operators that conserve mass, used in the literature, and define for each of them an associated parameter β.

1. the *Fokker–Planck* operator with $\beta := 2$ and local equilibrium $\mathcal{M}$

$$\mathsf{L}_1 f := \nabla_v \cdot \big(\mathcal{M}\, \nabla_v \big(\mathcal{M}^{-1} f \big) \big)\,.$$

2. the *linear Boltzmann* operator, or *scattering* collision operator

$$\mathsf{L}_2 f := \int_{\mathbb{R}^d} \mathrm{b}(\,\cdot\,, v')\,\Big(f(v')\,\mathcal{M}(\,\cdot\,) - f(\,\cdot\,)\,\mathcal{M}(v')\Big)\,\mathrm{d}v'\,,$$

with positive, locally bounded *collision frequency*

$$\nu(v) := \int_{\mathbb{R}^d} \mathrm{b}(v, v')\,\mathcal{M}(v')\,\mathrm{d}v' \underset{|v|\to+\infty}{\sim} |v|^{-\beta} \tag{0.4}$$

3. the *fractional Fokker–Planck* operator

$$\mathsf{L}_3 f := \Delta_v^{\sigma/2} f + \nabla_v \cdot (E\,f)$$

with $0 < \sigma < 2$, $\beta := \sigma - \alpha$ and a radial *friction force* $E = E(v)$ as a solution of

$$\mathsf{L}_3 \mathcal{M} = \Delta_v^{\sigma/2} \mathcal{M} + \nabla_v \cdot (E\,\mathcal{M}) = 0\,. \tag{0.5}$$

It turns out from a technical result that such a friction force then behaves like $\lfloor v \rceil^{-\beta} v$ at infinity.

Actually, this parameter β can be defined, such that the following microscopic coercivity holds.

Hypothesis 1 (Weighted coercivity)**.** *The operator L is linear, independent of time t and space x, commutes with rotations in v, is closed densely defined on* $\mathrm{Dom}(\mathsf{L}) \subset L^2_v(\mathcal{M})$ *and satisfies* $\mathsf{L}(1) = \mathsf{L}^*(1) = 0$*, where L^* is the $L^2_v(\mathcal{M})$-adjoint. Finally $\widetilde{\mathsf{L}} := \lfloor\cdot\rceil^{\frac{\beta}{2}} \mathsf{L}(\lfloor\cdot\rceil^{\frac{\beta}{2}}\cdot)$ is closed densely defined on* $\mathrm{Dom}(\widetilde{\mathsf{L}}) \subset L^2_v(\mathcal{M})$*, with the spectral gap estimate*

$$\forall\, g \in \mathrm{Dom}(\widetilde{\mathsf{L}}), \quad g \perp \lfloor\cdot\rceil^{-\frac{\beta}{2}}, \quad -\mathrm{Re}\big\langle \widetilde{\mathsf{L}} g, g\big\rangle \geq \lambda\,\|g\|^2\,.$$

This means, translating back to L*,*

$$\forall\, h \in \mathrm{Dom}(\mathsf{L}), \quad -\mathrm{Re}\big\langle \mathsf{L}h, h\big\rangle \geq \lambda\,\|h - \mathsf{P}h\|^2_{-\beta} \quad \textit{with} \quad \mathsf{P}h := \left(\int_{\mathbb{R}^d} h(v')\lfloor v'\rceil^{-\beta}\mathcal{M}(v')\mathrm{d}v'\right).$$

Observe that due to total mass conservation, an initial data with finite total mass will necessary go to zero as t goes to infinity since there is no global equilibrium state with finite mass except from zero. Our first concern is to derive rates of convergence to zero. We shall use the notation $\beta_+ = \max(0, \beta)$ and the convention $1/0_+ = +\infty$. Let

$$\zeta = \zeta(\alpha, \beta) := \begin{cases} 2 & \text{when } \alpha \geq 2 + \beta \\ \dfrac{\alpha + \beta}{1 + \beta} & \text{when } \alpha < 2 + \beta, \end{cases} \tag{0.6}$$

Theorem 2. *Let $d \geq 2$, $\beta \in \mathbb{R}$, $\alpha > \max\{0, -\beta\}$, ζ given by* (0.6) *and $k \in [0, \zeta)$. Under Assumption* (H)*, let us consider a solution f to* (0.1) *with initial condition* $f^{\mathrm{in}} \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v) \cap \mathrm{L}^2\big(\lfloor v\rceil^k \mathrm{d}x\,\mathrm{d}\mu\big)$*. If $\alpha \neq 2 + \beta$ or if $\alpha = 2 + \beta$ and $\frac{k}{\beta_+} > \frac{d}{2}$, then*

$$\forall\, t \geq 0\,, \quad \|f(t, \cdot\,, \cdot\,)\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \leq \frac{C}{(1+t)^\tau}\, |||f^{\mathrm{in}}|||^2_k \quad \textit{with} \quad \tau = \min\left\{\tfrac{d}{\zeta}, \tfrac{k}{\beta_+}\right\}.$$

In the critical case $\gamma = 2 + \beta$, and with either $k = 0$ if $\beta < 0$, or $k > 0$ if $\beta \geq 0$, and under the additional condition $\frac{k}{\beta_+} \leq \frac{d}{2}$ if $d \geq 3$,

$$\forall\, t \geq 2\,, \quad \|f(t, \cdot\,, \cdot\,)\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \leq \frac{C}{(t\,\log t)^{d/2}}\, |||f^{\mathrm{in}}|||^2_k\,.$$

In the above estimates, $C > 0$ is a constant which does not depend on f^{in}.

This result is in the spirit of theorems proved by the Dolbeault–Mouhot–Schmeiser method [5]. However major difficulties arise compare to the original matter of [5]. Indeed, in the present context, no microscopic and macroscopic coercivities hold. We shall present some elements of the strategy of the proof. Thanks to the translational invariance, we may perform a mode-by-mode analysis taking the Fourier transform in space.

$$\partial_t \widehat{f} + i\,(v \cdot \xi)\widehat{f} = \mathsf{L}\widehat{f}\,,$$

We define a new mode-by-mode hypocoercivity functional by

$$\mathsf{H}_\xi[f] := \frac{1}{2}\|\widehat{f}\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}_\xi \widehat{f}, \widehat{f}\rangle, \qquad \mathsf{A}_\xi := \frac{1}{\lfloor v\rceil^2}\,\mathsf{\Pi}\,\frac{(-\,i\,v\cdot\xi)\lfloor v\rceil^{-\beta}}{1+\lfloor v\rceil^{2\,|1-\beta|}\,|\xi|^2}.$$

One key step of the proof is to obtain in a general fashion for all models,

$$\mathrm{Re}\langle \mathsf{A}_\xi \widehat{f}, \widehat{f}\rangle \geq k_1(\xi)\,\|\mathsf{\Pi}\widehat{f}\|^2 - k_2(\xi)\,\|(1-\mathsf{\Pi})\widehat{f}\|_\eta^2\,.$$

To come back to the space variable, one may integrate over ξ,

$$\mathsf{H}[f] = \int_{\mathbb{R}^d} \mathsf{H}_\xi[f]\,\mathrm{d}\xi.$$

Then, $\mathsf{H}[f]$ is equivalent to $\|\cdot\|$ when δ is small and Plancherel identities give

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] = \mathrm{Re}\langle f\,, \mathsf{L}f\rangle + \delta\int_{\mathbb{R}^d}\mathrm{Re}\langle \mathsf{A}_\xi \widehat{f}, \widehat{f}\rangle\,\mathrm{d}\xi.$$

A crucial ingredient then is to relate norms with losses of weights $\|(1-\mathsf{\Pi})f\|_\eta$ to $\|(1-\mathsf{\Pi})f\|$, for this we may prove and use conservation of weighted L^2 norms,

$$\forall\,t\geq 0\,,\quad \|f(t,\cdot\,,\cdot)\|_{\mathrm{L}^2(\lfloor v\rceil^k \mathrm{d}x\,\mathrm{d}\mu)} \leq \mathcal{C}_k\,\|f^{\mathrm{in}}\|_{L^2(\lfloor v\rceil^k\mathrm{d}x\,\mathrm{d}\mu)}\,.$$

All in all, yields an ODE of the form:

$$H' \leq -\phi(t)H^r,$$

which can be integrated to give the result of the theorem. The full details of the proofs are in [1].

We shall end this exposition with a comment about the stability of the rates in the diffusive scaling. In the range of parameters for which we prove decay at rate $t^{-\frac{d}{\alpha}}$, our previous computations give that the rate is uniform in the rescaling $t\to\frac{t}{\varepsilon^\alpha}$ and $x\to\frac{x}{\varepsilon}$. When the rate is given by $t^{-\frac{k}{\beta}}$, the way that the rate degenerates into $t^{-\frac{d}{\alpha}}$ (the rate of the macroscopic limit, see e.g. [3]) is a bit more intricate, and we leave this issue for future work.

The second result of this discussion is about scaling limits in such a framework. Consider a solution f in $L^\infty_t([0,+\infty);L^2_{x,v}(\mathcal{M}^{-1}))$ to (0.1) with initial data $f^{(\varepsilon)}_{\mathrm{in}}$. Note that the initial data $f^{(\varepsilon)}_{\mathrm{in}}$, before the rescaling, is allowed to depend on ε. Given $\varepsilon>0$ and $\theta(\varepsilon)$, we rescale the solution and define a weighted rescaled spatial density:

$$f_\varepsilon(t,x,v) := \frac{1}{\varepsilon^d} f\left(\frac{t}{\theta(\varepsilon)},\frac{x}{\varepsilon},v\right) \in L^\infty_t\left([0,+\infty);L^2_{x,v}(\mathcal{M}^{-1})\right)$$

$$r_\varepsilon(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\lfloor v\rceil^{-\beta}\mathrm{d}v.$$

The equation satisfied by f_ε is

$$\theta(\varepsilon)\partial_t f_\varepsilon + \varepsilon v\cdot\nabla_x f_\varepsilon = \mathsf{L}f_\varepsilon. \tag{0.7}$$

The *rescaled* initial data is then $f_\varepsilon(0,x,v) = \varepsilon^{-d} f^{(\varepsilon)}_{\mathrm{in}}(\varepsilon^{-1}x,v)$, and in the following theorem we assume the original initial data $f^{(\varepsilon)}_{\mathrm{in}}$ to be *well-prepared* (see (0.9)–(0.10)–(0.11)): this means that the fluid limit holds at time zero with $f_\varepsilon(0,\cdot)\sim r_\varepsilon(0,\cdot)\mathcal{M}$ and $r_\varepsilon\sim r(0,\cdot)$ which provides the initial data for the limit equation; this is standard in the literature. We however note that when (0.9) is satisfied but (0.10)–(0.11) are not imposed at $t=0$, the energy estimate and compactness arguments on r_ε would imply that (0.10)–(0.11) are satisfied at any later positive time $\tau>0$ (without information on the rate though), and our method would prove the fluid approximation for $t\geq\tau$. This would allow us for instance to choose $f^{(\varepsilon)}_{\mathrm{in}} = r\mathcal{M}$ independent of ε. We however kept the assumptions (0.10)–(0.11) in order to precisely track the rate of convergence and the initial data of the limit equation.

It turns out from a deep understanding that the *scaling function* should be

$$\theta(\varepsilon) := \begin{cases} \varepsilon^\zeta & \text{when } \alpha\in(0,+\infty]\setminus\{2+\beta\},\\[1ex] \varepsilon^2|\ln\varepsilon| & \text{when } \alpha = 2+\beta,\\[1ex] \dfrac{\varepsilon^{\frac{\beta}{1+\beta}}}{|\ln\varepsilon|} & \text{when } \alpha = 0.\end{cases} \tag{0.8}$$

Note that the threshold $\alpha = 2+\beta$ between standard and fractional diffusion corresponds to whether or not $\lfloor v \rceil^{-\beta}\mathcal{M}$ has finite variance.

The main result of [3] is as follows.

Theorem 3 (Unified second fluid approximation)**.** *Assume structural hypothesis detailed precisely in* [3]*, have $\alpha \geq 0$, and consider $f_\varepsilon \in L^\infty_t([0,+\infty); L^2_{x,v}(\mathcal{M}^{-1}))$ solving* (0.7) *in the weak sense with initially*

$$\left\| \frac{f_\varepsilon(0,\cdot,\cdot)}{\mathcal{M}} \right\|_0 = o\left(\begin{cases} \theta(\varepsilon)^{-\frac12}, & \text{when } \alpha > \beta, \\ \varepsilon^{-\frac{\beta}{1+\beta}} \left|\ln(\varepsilon)\right|^{-\frac12} & \text{when } \alpha = \beta, \\ \varepsilon^{-\frac{\alpha}{1+\beta}} & \text{when } \alpha \in (0,\beta), \\ |\ln(\varepsilon)|^{\frac32} & \text{when } \alpha = 0, \end{cases} \right) \tag{0.9}$$

and

$$\left\| \frac{f_\varepsilon}{\mathcal{M}}(0,\cdot,\cdot) - r_\varepsilon(0,\cdot) \right\|_{-\beta} = o\left(\begin{cases} 1, & \text{when } \alpha > \beta, \\ \left|\ln(\varepsilon)\right|^{-\frac12} & \text{when } \alpha = \beta, \\ \varepsilon^{\frac{\beta-\alpha}{2(1+\beta)}} & \text{when } \alpha \in (0,\beta), \\ \varepsilon^{\frac{\beta}{2(1+\beta)}} |\ln(\varepsilon)| & \text{when } \alpha = 0, \end{cases} \right), \tag{0.10}$$

and (recalling the definition of ζ in (0.6)*)*

$$r_\varepsilon(0,\cdot) \xrightarrow[\varepsilon\to 0]{H^{-\zeta}(\mathbb{R}^d)} r(0,\cdot). \tag{0.11}$$

Then for any $T>0$

$$\left\| \frac{f_\varepsilon}{\mathcal{M}} - r \right\|_{L^2_t([0,T];H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon\to 0]{} 0 \tag{0.12}$$

when $\alpha > \beta$ and

$$\left\| \left| \ln \frac{2|\nabla_x|}{1+|\nabla_x|} \right| \left(\frac{f_\varepsilon}{\mathcal{M}} - r \right) \right\|_{L^2_t([0,T];H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon\to 0]{} 0$$

when $\alpha = \beta$ and

$$\left\| |\nabla_x|^{\frac{\beta-|\alpha|}{2(1+\beta)}} \lfloor \nabla_x \rceil^{-\frac{\beta-|\alpha|}{2(1+\beta)}} \left(\frac{f_\varepsilon}{\mathcal{M}} - r \right) \right\|_{L^2_t([0,T];H^{-\zeta}_x L^2_v(\mathcal{M}_\beta))} \xrightarrow[\varepsilon\to 0]{} 0$$

when $\alpha \in [0,\beta)$, where $r = r(t,x)$ solves

$$\partial_t r = \kappa\, \Delta_x^{\frac{\zeta}{2}} r, \quad t>0, \quad \text{with initial data } r(0,\cdot) \text{ defined in } (0.11).$$

The coefficient κ is explicit and its derivation is explained below. The rates of convergence are estimated and are explicit in terms of parameters and known initial error terms. Apart from (0.11)*, the errors we obtain are polynomial in ε for $\alpha \in (-\beta,+\infty) \setminus \{0, 2+\beta\}$ and logarithmic for $\alpha \in \{0, 2+\beta\}$.*

The main ingredient we need to derive this result is a quantitative construction of a branch of "fluid eigenmode" in the asymptotic of large time and small spatial frequencies, i.e. a unique eigenvalue branching from zero for $\widetilde{L}^* + i\eta\lfloor v \rceil^\beta (v\cdot\sigma)$ for small η.

Lemma 4 (Construction of the fluid mode)**.** *Given some structural hypotheses on L as detailed in* [3]*, there are $\eta_0 > 0$ and $r_0 \in (0,\lambda)$, explicit in terms of the constants in these hypotheses, such that for any $\eta \in (0,\eta_0)$ and any $\sigma \in \mathbb{S}^{d-1}$, there is a unique solution $\phi_\eta = \phi_\eta(v) \in L^2_v(\lfloor\cdot\rceil^{-\beta}\mathcal{M})$ and $\mu(\eta) \in B(0,r_0)$ to*

$$-L^*\phi_\eta - i\eta(v\cdot\sigma)\phi_\eta = \mu(\eta)\lfloor v \rceil^{-\beta}\phi_\eta \quad \text{with} \quad \int_{\mathbb{R}^d} \phi_\eta(v)\, \mathcal{M}(v) \lfloor v \rceil^{-\beta} \mathrm{d}v = 1.$$

Moreover, the branch $(\phi_\eta, \mu(\eta))$ connects to $(1,0)$ as $\eta \to 0$, with $\mu(\eta) > 0$ and the asymptotics

$$\|\phi_\eta - 1\|_{-\beta} \lesssim \mu(\eta)^{\frac12} \quad \text{and} \quad \mu(\eta) \in (R_0\Theta(\eta), R_1\Theta(\eta)) \tag{0.13}$$

for some $0 < \mathrm{R}_0 < \mathrm{R}_1$*, where the function* Θ *is defined by*

$$\Theta(\eta) := \begin{cases} \eta^2 & \textit{when } \alpha > 2+\beta, \\ \eta^2 |\ln(\eta)| & \textit{when } \alpha = 2+\beta, \\ \eta^{\frac{\alpha+\beta}{1+\beta}} & \textit{when } -\beta < \alpha < 2+\beta. \end{cases} \tag{0.14}$$

Note that Θ is well-defined in the case $\alpha \in (-\beta, 2+\beta)$ since $(1+\beta) > (\alpha+\beta)/2 > 0$. In this lemma and in the rest of the paper the dependency in σ is kept implicit rather than explicit in order to lighten notation. In fact, ϕ_η also depends on σ, but $\mu(\eta)$ does not if L is invariant by rotations in v. To identify the macroscopic limit with quantitative rates and constants, it is necessary to estimate the leading order of $\mu(\eta)$. This requires estimates on the eigenvector, that are quantitative and obtained for each typical operator L. For the sake of brevity, we refer directly to [3, Hypothesis 4] and state the important result directly.

With these four hypotheses we can characterise the precise scaling of the fluid eigenvalue:

Lemma 5 (Rescaled limit of the fluid eigenvalue)**.** *The eigenvalue* $\mu(\eta)$ *constructed in Lemma 4 satisfies (with convergence rate explicit in terms of the constants, error terms and convergence rates in the hypotheses)*

$$\mu(\eta) \sim_{\eta\to 0} \mu_0 \Theta(\eta), \tag{0.15}$$

where the constant $\mu_0 \in (\mathrm{R}_0, \mathrm{R}_1)$ *is positive and determined as follows:*

$$\begin{cases} \mu_0 := \displaystyle\int_{\mathbb{R}^d} (v\cdot\sigma)\, F(v) \mathcal{M}(v) \mathrm{d}v \quad \textit{when } \alpha > 2+\beta, \\ \qquad \textit{where } \; F = \displaystyle\lim_{\eta\to 0} \frac{\operatorname{Im}\phi_\eta}{\eta} \; \textit{is solution to } LF = -(v\cdot\sigma) \;\; \textit{and} \;\; \displaystyle\int_{\mathbb{R}^d} F(v)\, \mathcal{M}_\beta(v) \mathrm{d}v = 0, \\ \mu_0 := \dfrac{c_{2+\beta,\beta}}{1+\beta} \displaystyle\int_{\mathbb{S}^{d-1}} (\sigma\cdot\sigma')\Omega(\sigma')\mathrm{d}\sigma' \quad \textit{when } \alpha = 2+\beta, \\ \qquad \textit{where } \; \Omega(u) = \displaystyle\lim_{\lambda\to 0,\; \lambda\neq 0} \frac{\operatorname{Im}\Phi(\lambda u)}{\lambda^{1+\beta}} \;\; \textit{and} \;\; \Phi = \lim_{\eta\to 0} \Phi_\eta = \lim_{\eta\to 0} \phi_\eta\left(\eta^{-\frac{1}{1+\beta}} \cdot\right), \\ \mu_0 := c_{\alpha,\beta} \displaystyle\int_{\mathbb{R}^d} (u\cdot\sigma) \operatorname{Im}\Phi(u) |u|^{-d-\alpha} \mathrm{d}u \quad \textit{when } \alpha \in (-\beta, 2+\beta). \end{cases}$$

Note how, when $\alpha > 2+\beta$, the function F used in the previous works on standard diffusive limit (usually with $\beta = 0$) is recovered here as a limit of our fluid mode; this allows our proof to track the convergence rate.

Lemma 6 (Diffusion coefficient)**.** *Assume structural hypotheses detailed in* [3] *hold true and* $\alpha \geq 0$*. Then the following limit holds true with convergence rate explicit in terms of the constants, error terms and convergence rates in the hypotheses: for any* $\xi \in \mathbb{R}^d \setminus \{0\}$*,*

$$\kappa := \lim_{\varepsilon\to 0} \left(\frac{\mu(\varepsilon|\xi|)|\xi|^{-\zeta}}{\theta(\varepsilon)\left\langle 1, \phi_{\varepsilon|\xi|}\right\rangle} \right) = \mu_0 \times \begin{cases} \|\mathcal{M}\|^{-1}_{L^1(\mathbb{R}^d)} & \textit{when } \alpha > 0, \\ \dfrac{1+\beta}{|\mathbb{S}^{d-1}|} & \textit{when } \alpha = 0. \end{cases} \tag{0.16}$$

The diffusion coefficient thus emerges from ratios between (rescaled) integrals as follows:

$$\kappa := \begin{cases} \dfrac{\int_{\mathbb{R}^d} (v\cdot\sigma)\, F(v)\mathcal{M}(v)\mathrm{d}v}{\|\mathcal{M}\|_{L^1(\mathbb{R}^d)}} & \text{when } \alpha > 2+\beta \\ \dfrac{1}{1+\beta} \dfrac{\int_{\mathbb{S}^{d-1}} (\sigma\cdot\sigma')\Omega(\sigma')\mathrm{d}\sigma'}{\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha} \mathrm{d}v} & \text{when } \alpha = 2+\beta \\ \dfrac{\int_{\mathbb{R}^d} (u\cdot\sigma) \operatorname{Im}\Phi(u)|u|^{-d-\alpha}\mathrm{d}u}{\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha} \mathrm{d}v} & \text{when } \alpha \in (0, 2+\beta) \\ \dfrac{1+\beta}{|\mathbb{S}^{d-1}|} \dfrac{\int_{\mathbb{R}^d} (u\cdot\sigma) \operatorname{Im}\Phi(u)|u|^{-d-\alpha}\mathrm{d}u}{\int_{\mathbb{R}^d} \lfloor v \rceil^{-d-\alpha-\beta} \mathrm{d}v} & \text{when } \alpha = 0 \end{cases} \tag{0.17}$$

where we recall

$$F=\lim_{\eta\to0}\frac{\operatorname{Im}\phi_\eta}{\eta},\qquad \Phi=\lim_{\eta\to0}\Phi_\eta=\lim_{\eta\to0}\phi_\eta\left(\eta^{-\frac{1}{1+\beta}}\cdot\right),\qquad \Omega(u)=\lim_{\lambda\to0,\ \lambda\neq0}\frac{\operatorname{Im}\Phi(\lambda u)}{\lambda^{1+\beta}},$$

and (when $\alpha>2+\beta$) F is also the unique solution to $LF=-(v\cdot\sigma)$ with $\int_{\mathbb{R}^d}F(v)\lfloor v\rceil^{-d-\alpha-\beta}\mathrm{d}v=0$.

The method presented here extends to the fractional diffusive limit the approach pioneered in [6, 11] of constructing exact dispersion laws in the regime of parabolic time-space scaling and small eigenvalues; this extension is inspired by the recent one-dimensional result [9] and in particular we use and generalise the idea of rescaling velocities to obtain a non-trivial dispersion law in [9]. In comparison with [9], the main novelty of the present paper is a quantitative spectral method for constructing the branch of fluid eigenvalue: in [9] it was done by a one-dimensional argument connecting two infinite series on $\mathbb{R}_-$ and $\mathbb{R}_+$ (and it was done by fixed points in the simpler case of classical diffusive limit in the older works [6, 11]). In terms of results, this recovers and unifies the fractional diffusive limit results in [4, 7, 8, 9, 10]. Novel contributions include formulas for the diffusion coefficient in dimension higher than 1, and the quantitative argument providing a convergence rate.

We shall conclude with perspectives and extensions. An interesting work in progress with Mouhot and Kanzler [2] is the use of the spectral method in a context where the operator L will have more than one invariant, namely three invariants that are mass, momentum and energy. Several models from the literature for which not much is known fall down in this context.

References

[1] E. Bouin, J. Dolbeault & L. Lafleche, "Fractional hypocoercivity", *Commun. Math. Phys.* **390** (2022), no. 3, p. 1369-1411.

[2] E. Bouin, L. Kanzler & C. Mouhot, "Quantitative fluid approximation with more invariants", in progress, 2023.

[3] E. Bouin & C. Mouhot, "Quantitative fluid approximation in transport theory: a unified approach", *Probability and Mathematical Physics* **3** (2022), no. 3, p. 491-542.

[4] P. Cattiaux, E. Nasreddine & M. Puel, "Diffusion limit for kinetic Fokker-Planck equation with heavy tails equilibria: the critical case", *Kinet. Relat. Models* **12** (2019), no. 4, p. 727-748.

[5] J. Dolbeault, C. Mouhot & C. Schmeiser, "Hypocoercivity for Linear Kinetic Equations Conserving Mass", *Trans. Am. Math. Soc.* **367** (2015), no. 6, p. 3807-3828.

[6] R. S. Ellis & M. A. Pinsky, "The first and second fluid approximations to the linearized Boltzmann equation", *J. Math. Pures Appl.* **54** (1975), p. 125-156.

[7] N. Fournier & C. Tardif, "Anomalous diffusion for multi-dimensional critical kinetic Fokker–Planck equations", *Ann. Probab.* **48** (2020), no. 5, p. 2359-2403.

[8] ———, "One dimensional critical kinetic Fokker-Planck equations, Bessel and stable processes", *Commun. Math. Phys.* **381** (2021), no. 1, p. 143-173.

[9] G. Lebeau & M. Puel, "Diffusion approximation for Fokker Planck with heavy tail equilibria: a spectral method in dimension 1", *Commun. Math. Phys.* **366** (2019), no. 2, p. 709-735.

[10] E. Nasreddine & M. Puel, "Diffusion limit of Fokker-Planck equation with heavy tail equilibria", *ESAIM, Math. Model. Numer. Anal.* **49** (2015), no. 1, p. 1-17.

[11] B. Nicolaenko, "Dispersion Laws for plane wave propagation", in *The Boltzmann Equation* (A. Grünbaum, ed.), Courant Institute, 1971.

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