Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Aussois, 19–23 juin 2023

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J. É. D. P. (2023), Exposé n° II, 15 p.

<https://doi.org/10.5802/jedp.673>

Article mis en ligne dans le cadre du
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http://www.centre-mersenne.org/

RÉSEAU THÉMATIQUE AEDP DU CNRS

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Léo Bigorgne

Abstract

In this note, we first present the scattering problem for the Vlasov–Maxwell system. Then, by studying the linearised system, we explain why the distribution function merely exhibits, in general, a modified scattering dynamic.

1. Introduction

This note is concerned with the Vlasov–Maxwell system, which is a classical model in collisionless plasma physics, in the perturbative regime. More precisely, we will be interested in the dynamics of its small data solutions. This set of equations is given by

$$\partial_t f + \widehat{v} \cdot \nabla_x f + (E + \widehat{v} \times B) \cdot \nabla_v f = 0, \tag{1.1}$$

$$\nabla_x \cdot E = \int_{\mathbb{R}^3_v} f \mathrm{d}v, \qquad \partial_t E = \nabla_x \times B - \int_{\mathbb{R}^3_v} \widehat{v} f \mathrm{d}v, \tag{1.2}$$

$$\nabla_x \cdot B = 0, \qquad \partial_t B = -\nabla_x \times E, \tag{1.3}$$

where

- $f : \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}_+$ is the density distribution function of the particles.
- $\widehat{v} = \frac{v}{\langle v \rangle}$, with $\langle v \rangle := \sqrt{1 + |v|^2}$, is the relativistic speed of a particle of momentum $v \in \mathbb{R}^3_v$.
- $\int_{\mathbb{R}^3_v} f \mathrm{d}v$ and $\int_{\mathbb{R}^3_v} \widehat{v} f \mathrm{d}v$ are the total charge density and the total current density.
- $E, B : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}^3$ are respectively the electric and the magnetic field.

In order to lighten the notations, we consider, as it is usually done in the mathematical community, plasmas composed by one species of particles of charge $q = 1$ and mass $m = 1$. Note then that the results presented here can be extended without any additional difficulty to the multispecies case, with particles of different charges and strictly positive masses[1].

Remark 1. This simplification of the notations has a drawback. Indeed, the only electrically neutral plasmas that one can study does not contain any particle, that is $f(t, \cdot, \cdot) = 0$. Note however than we could relax the condition $f \geq 0$ and allow f to take negative values. In this non-physical setting, non-trivial neutral plasmas can be considered.

Remark 2. For convenience, we set in this article the speed of light c to one, $c = 1$, but in certain context it is important to keep it as a parameter in (1.1)–(1.3). In the non-relativistic limit $c \to +\infty$, as it is proved for instance in [7], the solutions to the Vlasov–Maxwell equations converge to the ones of the repulsive Vlasov–Poisson system

$$\partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = 0, \qquad \Delta \phi = \int_{\mathbb{R}^3_v} f \mathrm{d}v. \tag{VP}$$

A detailed introduction to kinetic equations can be found in [12].

[1] In contrast, the dynamics of the massless case is different [3].

1.1. Local and global well-posedness for the Vlasov–Maxwell system

An initial data set (f_0, E_0, B_0) for (1.1)–(1.3) is composed by a function $f_0 : \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}_+$ and two fields $E_0, B_0 : \mathbb{R}^3_x \to \mathbb{R}^3$ which satisfy the constraint equations

$$\nabla_x \cdot E_0 = \int_{\mathbb{R}^3_v} f_0 \, \mathrm{d}v, \qquad \nabla_x \cdot B_0 = 0. \tag{1.4}$$

The local existence and uniqueness problem for the solutions to the Vlasov–Maxwell system has been adressed by Wollman and Glassey–Strauss [17, 37]. While global weak solutions were constructed by DiPerna–Lions [10] (see also [32]), the existence of global-in-time classical solutions to the Vlasov–Maxwell system is only known in the pertubative regime [2, 4, 13, 18, 31, 33, 35, 36] or under certain symmetry assumptions [14, 15, 16, 24, 34]. Although the problem remains open besides these specific cases, various continuation criteria have been obtained during the past decades (see for instance [17, 23]).

In contrast, global existence holds for the classical solution to the Vlasov–Poisson system [22, 29].

1.2. Asymptotic stability of vacuum for the Vlasov–Maxwell system

The solutions arising from sufficiently small and regular initial data are known, in addition to be global in time, to decay with a rate corresponding to the solutions of the linearised equations around 0,

$$\partial_t f^{\mathrm{lin}} + \widehat{v} \cdot \nabla_x f^{\mathrm{lin}} = 0, \tag{1.5}$$

$$\nabla_x \cdot E^{\mathrm{lin}} = \int_{\mathbb{R}^3_v} f^{\mathrm{lin}} \mathrm{d}v, \qquad \partial_t E^{\mathrm{lin}} = \nabla_x \times B^{\mathrm{lin}} - \int_{\mathbb{R}^3_v} \widehat{v} f^{\mathrm{lin}} \mathrm{d}v, \tag{1.6}$$

$$\nabla_x \cdot B^{\mathrm{lin}} = 0, \qquad \partial_t B^{\mathrm{lin}} = -\nabla_x \times E^{\mathrm{lin}}. \tag{1.7}$$

Thus, for any such solution (f, E, B) to (1.1)–(1.3), we have for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$\int_{\mathbb{R}^3_v} f(t,x,v) \mathrm{d}v \lesssim t^{-3}, \qquad |E|(t,x) + |B|(t,x) \lesssim \langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-1}. \tag{1.8}$$

Remark 3. For (VP), the analagous result was first obtained by Bardos–Degond [1]. Note that the force field enjoys a stronger decay estimate near the light cone $\{t = |x|\}$ since $|\nabla \phi|(t,x) \lesssim \langle t \rangle^{-2}$.

This result was first obtained for compactly supported initial data by Glassey–Strauss [18], where estimates on the first order derivatives of the electromagnetic field were derived as well. Soon after, in the multi-species case, the smallness assumptions on the individual density distribution functions has been relaxed [13]. Later, Schaeffer [33] removed the support restriction on the velocity variable.

Recently, all the compact support assumptions on the initial data were removed independently by [2, 35] using robust approaches. More precisely, they rely on vector field methods as well as, for the latter, Fourier analysis. It allowed them to obtain (almost) optimal pointwise decay estimates, similar to (1.8), on the solutions and their higher order derivatives. Furthermore, in [2], the initial decay assumption in v is optimal and improved estimates on certain *null* components of the electromagnetic field are derived. Thereafter, Wei–Yang proved a global existence result allowing for large initial Maxwell fields [36]. Thus, their result implies the asymptotic stability of electromagnetic fields in vacuum, of regularity C^2 and decaying fast enough, for the Vlasov–Maxwell system. Their work, based on the framework of Glassey–Strauss, does not require any compact support restriction on the data and provides the optimal decay rates (1.8).

In [4], we provided a shorter proof of the main results of [2, 35] and we allowed, as [36], the electromagnetic field to be large. Moreover, we investigate further the asymptotic dynamics of the solutions.

Once the optimal decay estimates (1.8) are proved, the next question one may ask is whether or not f and (E, B) can be approached by a linear solution.

2. Scattering theory for the Vlasov–Maxwell system

The purpose of scattering theory is to compare the dynamics of a perturbed system, here the Vlasov–Maxwell equations, with a free dynamics, which is meant to be much simpler. At first glance, we can consider the free dynamics of the linear Vlasov equation and the homogeneous Maxwell equations. However, in order to achieve the three requirements of a satisfactory scattering theory presented below, a more involved free dynamics such as the one of the linearised system (1.5)–(1.7) must be considered. We refer to [30] for a general introduction on this subject.

2.1. Scattering states

Ideally, one would like the asymptotic behavior of the solutions to be captured by past and future *scattering states*. In this note, we will mainly focus on the future ones. For the free relativistic transport equation (1.5), the distribution function f^{lin} is constant along the characteristics, which are *timelike straight lines*, the trajectories of isolated massive bodies. More precisely,

$$\exists\, f_\infty : \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}_+, \;\; \forall\,(t,x,v) \in \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v, \qquad f^{\mathrm{lin}}(t, x+t\widehat{v}, v) = f_\infty(x,v),$$

so that the asymptotic dynamics of f^{lin} can be fully described by the knowledge of f_∞. In that case, the (future) scattering state f_∞ turns out to also correspond to the initial data $f^{\mathrm{lin}}(0,\cdot,\cdot)$. Consider an electromagnetic field $(E^{\mathrm{hom}}, B^{\mathrm{hom}})$ in vacuum, that is a solution to the homogeneous Maxwell equations

$$\nabla_x \cdot E^{\mathrm{hom}} = 0, \qquad \nabla_x \cdot B^{\mathrm{hom}} = 0, \qquad \partial_t E^{\mathrm{hom}} = \nabla_x \times B^{\mathrm{hom}}, \qquad \partial_t B^{\mathrm{hom}} = -\nabla_x \times E^{\mathrm{hom}}.$$

Then, under suitable assumptions on the data, $(rE^{\mathrm{hom}}, rB^{\mathrm{hom}})$ converges along *null rays*, the trajectories of photons. There exists E_∞, $B_\infty : \mathbb{R}_u \times \mathbb{S}^2_\omega \to \mathbb{R}^3$, the radiation fields of E^{hom} and B^{hom} along *future null infinity* $\mathcal{I}^+ \simeq \mathbb{R}_u \times \mathbb{S}^2_\omega$, such that

$$\lim_{r\to+\infty} rE^{\mathrm{hom}}(r+u, r\omega) = E_\infty(u,\omega), \qquad \lim_{r\to+\infty} rB^{\mathrm{hom}}(r+u, r\omega) = B_\infty(u,\omega), \tag{2.1}$$

for all $(u,\omega) \in \mathbb{R}_u \times \mathbb{S}^2_\omega$. Moreover, the future scattering state (E_∞, B_∞) satisfies constraint equations on $\mathcal{I}^+$ inherited from the Maxwell equations. By denoting abusively by ω the function $\omega \mapsto \omega \in \mathbb{S}^2$, they can be written as

$$\omega \cdot E_\infty = 0, \qquad \omega \cdot B_\infty = 0, \qquad E_\infty + \omega \times B_\infty = 0, \qquad B_\infty - \omega \times E_\infty = 0. \tag{2.2}$$

Conversally, we proved in [4, Theorem 7.6] that for any $L^2_{u,\omega}$ scattering state (E_∞, B_∞) satisfying the constraint equations (2.2), there exists a unique $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3_x))$ solution $(E^{\mathrm{hom}}, B^{\mathrm{hom}})$ to the vacuum Maxwell equations such that (2.1) holds, at least in a weak sense.

Thus, a global-in-time solution (f,E,B) to the Vlasov–Maxwell system would satisfy a *linear scattering* dynamic if the following statement hold. There exists $f_\infty \in L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ and $(E_\infty, B_\infty) \in L^2(\mathbb{R}_u \times \mathbb{S}^2_\omega, \mathbb{R}^6)$ such that

$$\lim_{t\to+\infty} \big\| \langle v \rangle f(t, x+t\widehat{v}, v) - \langle v \rangle f_\infty(x,v) \big\|_{L^1_{x,v}} = 0, \tag{2.3}$$

$$\lim_{r\to+\infty} \big\| \big(E,B\big)(r+u, r\omega) - \big(E_\infty, B_\infty\big)(u,\omega) \big\|_{L^2_{u,\omega}} = 0, \tag{2.4}$$

and (2.2) holds.

Remark 4. Of course, we could expect these convergences to hold for a different topology. This one is natural in view of the conservation of the total energy. If (f,E,B) is sufficiently regular,

$$\mathbb{E}_{\mathrm{tot}}[f,E,B](t) := \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_v} \langle v \rangle\, f(t,x,v) \mathrm{d}v \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3_x} |E(t,x)|^2 + |B(t,x)|^2 \mathrm{d}x \tag{2.5}$$

is preserved over time. Then, we would have

$$\forall\, t \in \mathbb{R}_+, \qquad \mathbb{E}_{\mathrm{tot}}[f,E,B](t) := \mathbb{E}_{\mathrm{tot}}[f_\infty, E_\infty, B_\infty]. \tag{2.6}$$

Thus, the linear approximation of

- the distribution function f would be the solution to the free Vlasov equation (1.5) with asymptotic data f_∞, that is $f^{\mathrm{lin}}(t,x,v) = f_\infty(x - t\widehat{v}, v)$.
- The one of the electromagnetic field (E,B) would be the solution $(E^{\mathrm{hom}}, B^{\mathrm{hom}})$ to the homogeneous Maxwell equations with scattering data (E_∞, B_∞).

As we will see, the small data solutions to the Vlasov–Maxwell system exhibit a more complicated dynamic.

2.2. The three properties of a satisfactory scattering theory

As formulated in [30] in a general setting, a satisfactory scattering theory for the Vlasov–Maxwell system must verify the following properties.

(1) *Existence of scattering states*: for any scattering state $(f_\infty, E_\infty, B_\infty)$ belonging to a suitable functional space, there exists a finite-energy solution (f, E, B) to the Vlasov–Maxwell system evolving to this given scattering state.

(2) *Uniqueness of scattering states*: two finite-energy solutions to (1.1)–(1.3) corresponding to the same scattering state must coincide.

(3) *Asymptotic completeness*: the global finite-energy solutions to the Vlasov–Maxwell system are either bound states or evolve to such a scattering state.

Of course, in order to make these statements rigorous, well-chosen functional spaces have to be identified.

Remark 5. We do not know if bound states of finite energy exist for the Vlasov–Maxwell system. However, according to (1.8), there is no bound states among the small data solutions.

Definition 6. The first two properties would constitute a global well-posedness statement for the Vlasov–Maxwell system with asymptotic data. They allow to construct the wave operator

$$\mathscr{W} : (f_\infty, E_\infty, B_\infty) \mapsto (f_0, E_0, B_0),$$

mapping any scattering state to the initial data of the corresponding solution to (1.1)–(1.3). Note that, if we impose $f_\infty \geq 0$, then $\mathscr{W}$ is an isometry for the norm $|\mathbb{E}_{\mathrm{tot}}[\,\cdot\,]|^{1/2}$.

Remark 7. Similarly, one could be interested in the past asymptotics of the solutions, as $t \to -\infty$. Then, if (1), (2) and (3) are verified for forward as well as backward evolution by the Vlasov–Maxwell system, one can define the scattering map

$$\mathscr{S} : (f_{-\infty}, E_{-\infty}, B_{-\infty}) \mapsto (f_{+\infty}, E_{+\infty}, B_{+\infty}),$$

relating the past asymptotic dynamics of the solutions to (1.1)–(1.3) to the future one.

3. Linear scattering does not hold for the distribution function

For a solution (f, E, B) to the Vlasov–Maxwell system arising from generic small initial data, the distribution f does not satisfy a linear scattering dynamic. If we impose f to be nonnegative, it only occurs if and only if $f(0, \cdot, \cdot) = 0$, as it was proved by [8] in the context of the Vlasov–Poisson system.

The obstruction for such a statement to hold is related to the long range effect of the Lorentz force $E + \widehat{v} \times B$, which can already be observed for the solutions to the linearised system as stated in the next proposition. For simplicity, we restrict our analysis to compactly supported distribution functions.

Proposition 8. *Let* $(f^{\mathrm{lin}}, E^{\mathrm{lin}}, B^{\mathrm{lin}})$ *be a solution to the linearised Vlasov–Maxwell system* (1.5)–(1.7) *arising from sufficiently regular initial data and such that*

$$f_0^{\mathrm{lin}} := f^{\mathrm{lin}}(0, \cdot, \cdot) \geq 0$$

does not vanish identically. Assume further that there exists $R > 0$ *such that* f_0^{lin} *is supported in*

$$\big\{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \,\big|\, |v| \leq R, \ |x| \leq R\big\}.$$

Then, the function h *defined as*

$$\partial_t h + \widehat{v} \cdot \nabla_x h + \big(E^{\mathrm{lin}} + \widehat{v} \times B^{\mathrm{lin}}\big) \cdot \nabla_v f^{\mathrm{lin}} = 0, \qquad\qquad h(0, \cdot, \cdot) = f^{\mathrm{lin}}(0, \cdot, \cdot),$$

does not satisfy linear scattering. There exists $\delta > 0$ *and* $(x_0, v_0) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$ *such that*

$$\forall\, |x - x_0| + |v - v_0| \leq \delta, \qquad\qquad |h(t, x_0 + t\widehat{v_0}, v_0)| \geq C \log(t),$$

where $C > 0$ *depends on* R *and the initial data.*

From now on and until the end of this section, the constant hidden by $\lesssim$ may depend on R.

3.1. The decay rate is at the threshold of time integrability

We explain here why we should not expect linear scattering for h. Let $|x| \leq R$ as well as $|v| \leq R$. We have

$$\partial_t \Big[h(t, x + t\widehat{v}, v) \Big] = -\big(E^{\mathrm{lin}}(t, x + t\widehat{v}) + \widehat{v} \times B^{\mathrm{lin}}(t, x + t\widehat{v}) \big) \cdot \nabla_v f^{\mathrm{lin}}(t, x + t\widehat{v}, v).$$

As we shall see, the right hand side can be bounded by t^{-1}. The reasons are the following,

- We will prove that $\langle t - |x| \rangle \gtrsim t$ on the support of f^{lin}.
- If the right hand side does not vanish, we then have $|(E^{\mathrm{lin}}, B^{\mathrm{lin}})|(t, x + t\widehat{v}) \lesssim \langle t \rangle^{-2}$ in view of the decay estimates (1.8) verified by E^{lin} and B^{lin}.
- We have $\| \nabla_v f^{\mathrm{lin}}(t, \cdot, \cdot) \|_{L^\infty_{x,v}} \lesssim t$. Indeed, as $f^{\mathrm{lin}}(t, x, v) = f_0^{\mathrm{lin}}(x - t\widehat{v}, v)$,
$$\partial_{v_i} f^{\mathrm{lin}}(t, x, v) = \partial_{v_i} f_0^{\mathrm{lin}}(x - t\widehat{v}, v) - \sum_{1 \leq j \leq 3} t \frac{\delta_{i,j} - \widehat{v}_i \widehat{v}_j}{\langle v \rangle} \partial_{x_j} f_0^{\mathrm{lin}}(x - t\widehat{v}, v). \tag{3.1}$$

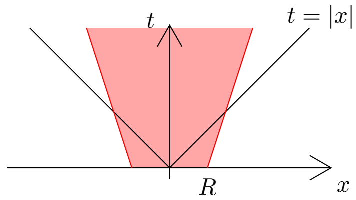
This estimate prevents us to prove that $h(t, x + t\widehat{v}, v)$ converges as $t \to +\infty$. We could then wonder if we were too naive or if we could improve this estimate. As stated by Proposition 8, unless $f_0^{\mathrm{lin}} = 0$, this estimate cannot be improved for all $(x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$.

3.2. Asymptotic behavior of massive relativistic Vlasov fields

Since f^{lin} is conserved along the linear characteristics, that is $f^{\mathrm{lin}}(t, x + t\widehat{v}, v)$ is constant,

$$\operatorname{supp} f^{\mathrm{lin}}(t, \cdot, \cdot) \subset \mathbf{D}_t^R := \left\{ (x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v \;\middle|\; |v| \leq R, \; |x| \leq R + t\frac{R}{\sqrt{1 + R^2}} \right\} \tag{3.2}$$

for all $t \geq 0$. Thus, for $t \geq 4(1 + R^2)$, the spatial support of $f(t, \cdot, \cdot)$ is located inside the light cone $\{t = |x|\}$. Moreover, we have $\langle t - |x| \rangle \geq C_R \langle t \rangle$ on the support of f^{lin}, where $C_R > 0$ is a constant depending only on R.



The spatial support of $f^{\mathrm{lin}}(t, \cdot, \cdot)$.

Remark 9. It reflects that massive Vlasov fields enjoy strong decay properties near and in the exterior of the light cone. This property is important since most of the energy of the electromagnetic field concentrates near $\{t = |x|\}$, as it is suggested by the estimate $|(E, B)|(t, x) \lesssim \langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-1}$, which holds in the nonlinear setting. Hence, on the support of f^{lin}, we one can transform the $t - |x|$ decay into time decay.

If $f^{\mathrm{lin}}(0, \cdot, \cdot)$ is not compactly supported, such properties can be recovered by propagating higher moments of the solution. More precisely, $\langle v \rangle$ and $\langle x - t\widehat{v} \rangle$ are conserved by the linear flow and

$$\langle t - |x| \rangle^{-1} \lesssim \langle t + |x| \rangle^{-1} \langle x - t\widehat{v} \rangle \langle v \rangle^2. \tag{3.3}$$

The nonlinear analysis is however much more complicated in this case (see [2, 4, 35]).

In order to investigate the large time behavior of the velocity averages of f^{lin}, we will make use of the change of variables $y = x - t\widehat{v}$.

Lemma 10. *We define, on the domain $\{u \in \mathbb{R}^3 \mid |u| < 1\}$, the operator* $\breve{}$ *as*

$$u \mapsto \breve{u} = \frac{u}{\sqrt{1 - |u|^2}}, \qquad \text{so that} \qquad \forall \; |u| < 1, \quad v \in \mathbb{R}^3_v, \qquad \widehat{\breve{u}} = u, \quad \breve{\widehat{v}} = v.$$

The Jacobian determinant of the transformation $v \mapsto \widehat{v}$ is equal to $\langle v \rangle^{-5}$.

Remark 11. If $|x| < t$, we have $\frac{\breve{x}}{t} = \frac{x}{\sqrt{t^2-|x|^2}}$.

Let us now determine the leading order term of the asymptotic expansion of the charge and the current density.

Proposition 12. *For all $t \geq 4(1+R^2)$ and $|x| < t$, we have*

$$\left| t^3 \int_{\mathbb{R}^3_v} f^{\mathrm{lin}}(t,x,v)\mathrm{d}v - \int_{\mathbb{R}^3_y} \left[\langle v\rangle^5 f_0^{\mathrm{lin}}\right]\left(y, \frac{x}{\sqrt{t^2-|x|^2}}\right)\mathrm{d}y \right| \lesssim \frac{1}{t}\left(\|f_0^{\mathrm{lin}}\|_{L^\infty} + \|\nabla_v f_0^{\mathrm{lin}}\|_{L^\infty}\right),$$

$$\left| t^3 \int_{\mathbb{R}^3_v} \widehat{v} f^{\mathrm{lin}}(t,x,v)\mathrm{d}v - \frac{x}{t}\int_{\mathbb{R}^3_y} \left[\langle v\rangle^5 f_0^{\mathrm{lin}}\right]\left(y, \frac{x}{\sqrt{t^2-|x|^2}}\right)\mathrm{d}y \right| \lesssim \frac{1}{t}\left(\left\|f_0^{\mathrm{lin}}\right\|_{L^\infty} + \left\|\nabla_v f_0^{\mathrm{lin}}\right\|_{L^\infty}\right),$$

where $\langle v\rangle^5 h$ denotes the function $(x,w) \mapsto \langle w\rangle^5 h(x,w)$.

Proof. Performing the change of variables $y = x - t\widehat{v}$, one gets from Lemma 10,

$$t^3 \int_{\mathbb{R}^3_v} f^{\mathrm{lin}}(t,x,v)\mathrm{d}v = t^3 \int_{\mathbb{R}^3_v} f_0^{\mathrm{lin}}(x - t\widehat{v}, v)\mathrm{d}v = \int_{|y-x|<t} \left[\langle v\rangle^5 f_0^{\mathrm{lin}}\right]\left(y, \frac{x-y}{\sqrt{|t^2-|x-y|^2}}\right)\mathrm{d}y.$$

We decompose the right hand side as

$$\int_{\mathbb{R}^3_y} \left[\langle v\rangle^5 f_0^{\mathrm{lin}}\right]\left(y, \frac{x}{\sqrt{t^2-|x|^2}}\right)\mathrm{d}y + \mathcal{I}_1 + \mathcal{I}_2,$$

where

$$\mathcal{I}_1 := \int_{|x-y|<t} \left[\langle v\rangle^5 f_0^{\mathrm{lin}}\right]\left(y, \frac{x-y}{\sqrt{|t^2-|x-y|^2}}\right)\mathrm{d}y - \int_{|x-y|<t} \left[\langle v\rangle^5 f_0^{\mathrm{lin}}\right]\left(y, \frac{x}{\sqrt{|t^2-|x|^2}}\right)\mathrm{d}y,$$

$$\mathcal{I}_2 := -\int_{|x-y|\geq t} \left[\langle v\rangle^5 f_0^{\mathrm{lin}}\right]\left(y, \frac{x}{\sqrt{|t^2-|x|^2}}\right)\mathrm{d}y.$$

Since $|\nabla_u \breve{u}| \lesssim \sqrt{1-|u|^2}^{\,-3} = \langle \breve{u}\rangle^3$, the mean value theorem gives us

$$|\mathcal{I}_1| \lesssim \int_{|y-x|<t} \frac{|y|}{t} \sup_{v\in\mathbb{R}^3} \langle v\rangle^7 |f_0^{\mathrm{lin}}|(y,v) + \langle v\rangle^8 \left|\nabla_v f_0^{\mathrm{lin}}\right|(y,v)\mathrm{d}y.$$

In order to bound $\mathcal{I}_2$ recall that $|x| < t$ and remark that, for $v = \breve{x/t}$ and any $y \in \mathbb{R}$ such that $|y-x| \geq t$,

$$1 = \langle v\rangle^2 \left(1 - \frac{|x|^2}{t^2}\right) \leq \langle v\rangle^2 \frac{|y|(t+|x|)}{t^2} \leq 2\frac{|y|\langle v\rangle^2}{t}. \tag{3.4}$$

We then finally deduce that

$$|\mathcal{I}_2| \leq \frac{2}{t}\int_{|y-x|\geq t} |y| \left[\langle v\rangle^7 f_0^{\mathrm{lin}}\right]\left(y, \frac{x}{\sqrt{|t^2-|x|^2}}\right)\mathrm{d}y.$$

It remains to use the compact support assumption on f_0^{lin} in order to derive the first estimate. For the one concerning the current density, simply apply the first estimate to $\widehat{v} f^{\mathrm{lin}}$ and remark that $\widehat{v} = \frac{x}{t}$ if $v = \frac{x}{\sqrt{t^2-|x|^2}}$. □

Remark 13. Without a compact support assumption on f_0^{lin}, these estimates still hold provided that the norms of f_0^{lin} in the right hand sides are weighted by powers of $\langle x\rangle$ and $\langle v\rangle$. The behavior of $\int_v f^{\mathrm{lin}}\mathrm{d}v$ for $|x| \geq t$ is governed by the decay of f_0^{lin}.

3.3. The asymptotic Maxwell equations

Now that we have identified the leading order contribution of the source terms in the Maxwell equations (1.6)–(1.7), we introduce the asymptotic Maxwell equations. Let, for $|x| < t$,

$$\rho^{\mathrm{asymp}}(t,x) := \frac{1}{t^3}\int_{\mathbb{R}^3_x} f_0^{\mathrm{lin}}\left(y, \frac{x}{\sqrt{t^2-|x|^2}}\right)\mathrm{d}y,$$

$$j^{\mathrm{asymp}}(t,x) := \frac{x}{t^4}\int_{\mathbb{R}^3_x} f_0^{\mathrm{lin}}\left(y, \frac{x}{\sqrt{t^2-|x|^2}}\right)\mathrm{d}y$$

be respectively the asymptotic charge density and the asymptotic current density. We extend them smoothly by 0 for $|x| \geq t$. Consider further $(E^{\mathrm{asymp}}, B^{\mathrm{asymp}})$ the unique solution to

$$\nabla_x \cdot E^{\mathrm{asymp}} = \rho^{\mathrm{asymp}}, \qquad \partial_t E^{\mathrm{asymp}} = \nabla_x \times B^{\mathrm{asymp}} - j^{\mathrm{asymp}}, \tag{3.5}$$

$$\nabla_x \cdot B^{\mathrm{asymp}} = 0, \qquad \partial_t B^{\mathrm{asymp}} = -\nabla_x \times E^{\mathrm{asymp}}, \tag{3.6}$$

such that

$$\nabla_x \times E^{\mathrm{asymp}}(t=1, \cdot\,) = 0, \qquad \nabla_x \times B^{\mathrm{asymp}}(t=1, \cdot\,) = 0.$$

Remark 14. The initial time is here $t = 1$ instead of $t = 0$ since ρ^{asymp} converges to a multiple of the Dirac delta function as $t \to 0$.

The initial data imply that the divergence-free part of $E^{\mathrm{asymp}}(1, \cdot\,)$ and $B^{\mathrm{asymp}}(1, \cdot\,)$, according to Helmholtz decomposition, both vanish. Since the curl-free part of B^{asymp} vanishes as $\nabla_x \cdot B^{\mathrm{asymp}} = 0$, we get $B^{\mathrm{asymp}}(1, \cdot\,) = 0$. Moreover,

$$E^{\mathrm{asymp}}(1, \cdot\,) = \nabla \phi, \qquad \Delta \phi = \rho^{\mathrm{asymp}}.$$

Exploiting these properties as well as (3.5)–(3.6), we are able to suitably estimate all the initial spacetime derivatives.

Remark 15. For completeness, let us provide the asymptotic expansion of the initial data (see [5, Proposition 3.15] for a detailed proof).

- All the spacetime derivatives of $B^{\mathrm{asymp}}(1, \cdot\,)$ vanish.
- For any multi-index κ and $n \geq 1$, $\partial_t^n \partial_x^\kappa E^{\mathrm{asymp}}(1, \cdot\,)$ is supported in $\{|x| \leq 1\}$.
- For any multi-index κ,

  $$\forall\, |x| \geq 1, \qquad \langle x \rangle^{3+|\kappa|} \left| \partial_x^\kappa \left(E^{\mathrm{asymp}}(1,x) - \frac{Qx}{4\pi |x|^3} \right) \right| < +\infty,$$

  where Q is the total charge of the plasma,

  $$Q := \int_{\mathbb{R}^3_x} \rho^{\mathrm{asymp}}(1,x) \mathrm{d}x = \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_v} f_0^{\mathrm{lin}}(x,v) \mathrm{d}v \mathrm{d}x.$$

  It means that if $Q \neq 0$, E^{asymp} (as well as E^{lin}) has a tail corresponding to the electric field generated by a point charge at $r = 0$. It turns out that the effect of this tail is only relevant in the exterior of the light cone $|x| \geq t$, where $f^{\mathrm{lin}}(t, \cdot\,, \cdot\,)$ and $h(t, \cdot\,, \cdot\,)$ both vanish for $t \geq 4(1 + R^2)$.

Then, in view of Proposition 12, the source term of the Maxwell equations satisfied by $(E^{\mathrm{lin}} - E^{\mathrm{asymp}}, B^{\mathrm{lin}} - B^{\mathrm{asymp}})$ is strongly decaying. Thus, under suitable initial decay assumptions on $(E^{\mathrm{lin}}, B^{\mathrm{lin}})(t = 0, \cdot\,)$ and by performing estimates for solutions to Maxwell equations, we are able to reduce the analysis of the large time behavior of $(E^{\mathrm{lin}}, B^{\mathrm{lin}})$ to the one of the asymptotic electromagnetic field.

Proposition 16. *For all $(t,x) \in [1, +\infty[\times \mathbb{R}^3$, we have*

$$\left| E^{\mathrm{lin}} - E^{\mathrm{asymp}} \right|(t,x) + \left| B^{\mathrm{lin}} - B^{\mathrm{asymp}} \right|(t,x) \lesssim \langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-\frac{3}{2}}.$$

Remark 17. This decay rate is stronger than the one satisfied by $(E^{\mathrm{lin}}, B^{\mathrm{lin}})$, which is $\langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-1}$. It corresponds to an initial decay of $r^{-\frac{5}{2}}$ for $B^{\mathrm{lin}}(0, \cdot\,)$ and for the divergence free part of $E^{\mathrm{lin}}(0, \cdot\,)$. Assuming more decay on the data, it could be improved up to $\langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-3}$

We then deduce the following estimate.

Corollary 18. *If $t \geq 1$ and $(x + t\widehat{v}, v) \in \operatorname{supp} f^{\mathrm{lin}}(t, \cdot\,, \cdot\,) \subset \mathbf{D}_t^R$,*

$$\left| E^{\mathrm{lin}} - E^{\mathrm{asymp}} \right|(t, x + t\widehat{v}) + \left| B^{\mathrm{lin}} - B^{\mathrm{asymp}} \right|(t, x + t\widehat{v}) \lesssim t^{-\frac{5}{2}}.$$

The next step consists in determining the asymptotic behavior of $(E^{\mathrm{asymp}}, B^{\mathrm{asymp}})$ along the trajectories of massive isolated particles $t \mapsto (t, x + t\widehat{v})$. It turns out that we can isolate the term which prevents it to decay faster than t^{-2}.

Proposition 19. *There exists* $(\mathbb{E},\mathbb{B}) \in L^\infty(\mathbb{R}^3_v,\mathbb{R}^6)$ *such that, for all* $t \geq 5(1+R^2)$*,* $|x| \leq R$ *and* $|v| \leq R$*, we have*

$$\left|t^2 E^{\mathrm{asymp}}(t,x+t\widehat{v}) - \mathbb{E}(v)\right| + \left|t^2 B^{\mathrm{asymp}}(t,x+t\widehat{v}) - \mathbb{B}(v)\right| \lesssim t^{-1}.$$

Moreover, $\mathbb{E}$ *and* $\mathbb{B}$ *are functionals of* $\int_{\mathbb{R}^3_x} f_0^{\mathrm{lin}}(x,\cdot)\mathrm{d}x$*. For instance,*

$$\mathbb{E}(v) = \int_{\substack{|y|\leq 1 \\ |y+\widehat{v}|<1-|y|}} \mathbf{w}\left(\frac{y}{|y|},\widecheck{\frac{y+\widehat{v}}{1-|y|}}\right)\int_{\mathbb{R}^3_z}\left[\langle v\rangle^5 f_0^{\mathrm{lin}}\right]\left(z,\widecheck{\frac{y+\widehat{v}}{1-|y|}}\right)\mathrm{d}z\frac{\mathrm{d}y}{(1-|y|)^3|y|^2},$$

where

$$\mathbf{w}(\omega,p) = \frac{\omega+\widehat{p}}{\langle v\rangle^2(1+\omega\cdot\widehat{p})^2}, \qquad\qquad (\omega,p)\in\mathbb{S}^2\times\mathbb{R}^3.$$

Remark 20. If $t \geq 4(1+R^2)$ and $|x|$, $|v| \leq R$, then $(x+t\widehat{v},v)\in\mathbf{D}^R_t$.

Proof. We sketch the proof of this result, which is a consequence of [5, Proposition 5.19]. We focus on the electric field since the magnetic field can be handled similarly. We fix a component $1 \leq k \leq 3$ and we write $E_k^{\mathrm{asymp}} = E_k^{\mathrm{hom}} + E_k^{\mathrm{inh}}$, where

$$\square E_k^{\mathrm{hom}} = 0, \qquad E_k^{\mathrm{hom}}(1,\cdot) = E_k^{\mathrm{asymp}}(1,\cdot), \quad \partial_t E_k^{\mathrm{hom}}(1,\cdot) = \partial_t E_k^{\mathrm{asymp}}(1,\cdot)$$

and

$$\square E_k^{\mathrm{inh}} = -\partial_{x_k}\rho^{\mathrm{asymp}} - \partial_t j_k^{\mathrm{asymp}}, \qquad\qquad E_k^{\mathrm{inh}}(1,\cdot) = 0, \quad \partial_t E_k^{\mathrm{inh}}(1,\cdot) = 0.$$

The initial decay of E^{asymp} allows us to deduce that

$$\forall\, t \geq |x|, \qquad \left|E_k^{\mathrm{hom}}\right|(t,x) \lesssim \langle t+|x|\rangle^{-1}\langle t-|x|\rangle^{-2}.$$

Thus, $|E_k^{\mathrm{hom}}|(t,x+t\widehat{v}) \lesssim t^{-3}$ for $t \geq 4(1+R^2)$ and $|x|,|v| \leq R$.

In order to handle the inhomogeneous part, we use the representation formula for the wave equation. Fix $t \geq 4(1+R^2)$ and $|x| \leq R + t\frac{R}{\sqrt{1+R^2}}$. We have

$$E_k^{\mathrm{inh}}(t,x) = -\int_{|y-x|\leq t}\left[\partial_{x_k}\rho^{\mathrm{asymp}} + \partial_t j_k^{\mathrm{asymp}}\right](t-|y-x|,y)\frac{\mathrm{d}y}{|y-x|},$$

since the integrand vanishes for $t-1 \leq |y-x| \leq t$. Next, in order to gain regularity, we exploit the Glassey–Strauss decomposition of the electromagnetic field, introduced in [17, Theorem 4].

Let us briefly explain the key ideas behind this decomposition, which applies to solutions to the Maxwell equations where the source term is generated by a solution to a Vlasov equation. The operators ∂_t and ∂_{x_k}, applied to functions integrated on the backward light cone $y \mapsto (t-|y-x|,y)$, can be written as a combination of ∂_{y_i} and the free Vlasov operator $\partial_t + \widehat{v}\cdot\nabla_x$, so that integration by parts can be performed. What makes that possible is that they span the 4-dimensional tangent space of any point of the cone. Indeed, ∂_{y_i} are tangential to it whereas the free Vlasov operator is transverse to the cone.

It turns out that ρ^{asymp} and j^{asymp} can be viewed as the charge and the current density of

$$f^{\mathrm{sing}}(t,x,v) := \int_{\mathbb{R}^3_z} f_0^{\mathrm{lin}}(z,v)\mathrm{d}z\,\delta(x-t\widehat{v}),$$

where δ is the Dirac delta distribution in $\mathbb{R}^3$. Moreover, f^{sing} is a singular solution to the free relativistic transport equation (1.5) in $\mathscr{D}'(\mathbb{R}_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v)$. Then, an approximation argument and the Glassey–Strauss decomposition provide that for such t and x,

$$E_k^{\mathrm{inh}}(t,x) = \frac{1}{t^2}\mathbb{E}_k\left(\widecheck{\frac{x}{\sqrt{t^2-|x|^2}}}\right), \qquad\qquad \frac{x}{\sqrt{t^2-|x|^2}} = \frac{\widecheck{x}}{t}.$$

Finally, the estimate for $E_k^{\mathrm{inh}}(t,x+t\widehat{v})$ ensues from the mean value theorem[2]. ☐

[2]Note that for this, one needs to prove that $\mathbb{E}$ is of class C^1.

3.4. The function h does not scatter linearly

Let

$$\mathbb{L}(v) := \mathbb{E}(v) + \widehat{v} \times \mathbb{B}(v)$$

be the asymptotic Lorentz force, generating the effective Lorentz force $t^{-2}\mathbb{L}(v)$. We start by reducing the proof of Proposition 8 to the following statement,

$$\exists\, v \in \mathbb{R}^3_v, \qquad \mathbb{L}(v) - \widehat{v} \cdot \mathbb{L}(v)\, \widehat{v} \neq 0, \qquad f_0^{\mathrm{lin}}(\cdot, v) \neq 0. \tag{3.7}$$

For this, we start by identifying the leading order term in the asymptotic expansion of $\partial_t h$ along the trajectories of massive particles.

Lemma 21. *For all $t \geq 5(1+R^2)$ and $(x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$, we have*

$$\left| \partial_t h(t, x + t\widehat{v}, v) + \frac{1}{t\,\langle v \rangle} \mathbb{L}(v) \cdot \nabla_x f_0^{\mathrm{lin}}(x,v) - \frac{\widehat{v} \cdot \mathbb{L}(v)}{t\,\langle v \rangle}\, \widehat{v} \cdot \nabla_x f^{\mathrm{lin}}(x,v) \right| \leq C t^{-\frac{3}{2}},$$

where the constant $C > 0$ depends on R and the data.

Proof. Recall from (3.1) the expression of $\nabla_v f^{\mathrm{lin}}$. Since we have $|(E^{\mathrm{lin}}, B^{\mathrm{lin}})|(t,y) \lesssim t^{-2}$ if $f^{\mathrm{lin}}(t,y,\cdot\,) \neq 0$, there holds

$$\begin{aligned}
\partial_t h(t, x + t\widehat{v}, v) &= -\big[E^{\mathrm{lin}} + \widehat{v} \times B^{\mathrm{lin}}\big](t,x) \cdot \nabla_v f^{\mathrm{lin}}(t, x + t\widehat{v}, v) \\
&= \frac{t}{\langle v \rangle} \big[E^{\mathrm{lin}} + \widehat{v} \times B^{\mathrm{lin}}\big](t, x + t\widehat{v}) \cdot \big[\widehat{v}\, \widehat{v} \cdot \nabla_x f_0^{\mathrm{lin}} - \nabla_x f_0^{\mathrm{lin}}\big](x,v) + O\big(t^{-2}\big).
\end{aligned}$$

Then, by Corollary 18,

$$\partial_t h(t, x + t\widehat{v}, v) = \frac{t}{\langle v \rangle} \big[E^{\mathrm{asymp}} + \widehat{v} \times B^{\mathrm{asymp}}\big](t, x + t\widehat{v}) \cdot \big[\widehat{v}\, \widehat{v} \cdot \nabla_x f_0^{\mathrm{lin}} - \nabla_x f_0^{\mathrm{lin}}\big](x,v) + O\big(t^{-\frac{3}{2}}\big),$$

so that the result follows from Proposition 19. □

It implies in particular that for all $t \geq 5(1+R^2)$ and $(x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$,

$$h(t, x + t\widehat{v}, v) = -\frac{\log(t)}{\langle v \rangle} \Big[\mathbb{L}(v) - \widehat{v} \cdot \mathbb{L}(v)\, \widehat{v}\Big] \cdot \nabla_x f_0^{\mathrm{lin}}(x,v) + O\big(t^{-\frac{1}{2}}\big). \tag{3.8}$$

Note now that since f_0^{lin} is compactly supported, $x \mapsto \nabla_x f_0^{\mathrm{lin}}(x,v)$ has a constant direction if and only if $f_0^{\mathrm{lin}}(\cdot, v)$ vanishes identically. Thus, if (3.7) holds, we can find $(x_0, v_0) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$ such that the first term on the right hand side of (3.8) does not vanish. It implies, by continuity of $\mathbb{L}$ and $\nabla_x f_0^{\mathrm{lin}}$, Proposition 8.

It then remains us to prove (3.7). For this, we introduce the notation

$$\mathcal{U}_1\big[f_0^{\mathrm{lin}}\big] := \sup \left\{ v_1 \in \mathbb{R} \;\middle|\; \forall\, (v_2, v_2) \in \mathbb{R}^2,\ \int_{\mathbb{R}^3_x} f_0^{\mathrm{lin}}(x, v_1, v_2, v_3) \mathrm{d}x \neq 0 \right\}.$$

We start by dealing with a particular case.

Lemma 22. *If $\mathcal{U}_1\big[f_0^{\mathrm{lin}}\big] = 0$, then* (3.7) *holds.*

Proof. The main part of the proof consists in proving that $\mathbb{E}_1(0) \neq 0$, so that

$$\Big[\mathbb{L}(v) - \widehat{v} \cdot \mathbb{L}(v)\, \widehat{v}\Big]\Big|_{v=0} = \mathbb{E}(0) \neq 0.$$

Then, in view of the assumptions on the support of f_0^{lin}, we would obtain (3.7) by continuity. According to Proposition 19, we have

$$\mathbb{E}_1(0) = \int_{\substack{|y| \leq 1 \\ |y| < 1 - |y|}} (1 - |y|) \frac{y_1}{|y|} \int_{\mathbb{R}^3_z} \Big[\langle v \rangle^5 f_0^{\mathrm{lin}}\Big]\Big(z, \frac{\check{y}}{1 - |y|}\Big) \mathrm{d}z \frac{\mathrm{d}y}{(1 - |y|)^3 |y|^2}.$$

By assumption on $f_0^{\mathrm{lin}} \geq 0$, the integrand vanishes for $y_1 \geq 0$ and is negative for $y_1 < 0$. Since it does not vanish identically on the domain of integration, it implies $\mathbb{E}_1(0) \neq 0$. □

In order to deal with the general case, we use the Lorentz invariance of the linearised Vlasov–Maxwell system. For any Lorentz boost

$$A_\varphi := \begin{pmatrix} \mathrm{ch}(\varphi) & \mathrm{sh}(\varphi) & 0 & 0 \\ \mathrm{sh}(\varphi) & \mathrm{ch}(\varphi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in SO_0(1,3), \qquad \varphi \in \mathbb{R},$$

which corresponds to a change of inertial frame in special relativity, one can construct from $(f^{\mathrm{lin}}, E^{\mathrm{lin}}, B^{\mathrm{lin}})$ an other solution $(f^\varphi, E^\varphi, B^\varphi)$ to (1.5)–(1.7), where

$$f^\varphi(t,x,v) = f^{\mathrm{lin}}\big(\mathrm{ch}(\varphi)t + \mathrm{sh}(\varphi)x_1, \mathrm{ch}(\varphi)x_1 + \mathrm{sh}(\varphi)t, x_2, x_3, \mathrm{ch}(\varphi)v_1 + \mathrm{sh}(\varphi)\langle v\rangle, v_2, v_3\big).$$

The new electromagnetic field (E^φ, B^φ) is slightly more complicated to express and we do not require its exact expression. The Lorentz force transforms under A_φ as

$$\big[E^\varphi + \widehat{v} \times B^\varphi\big](t,x) = \big[E^{\mathrm{lin}} + \widehat{V}_\varphi \times B^{\mathrm{lin}}\big]\big(\mathrm{ch}(\varphi)t + \mathrm{sh}(\varphi)x_1, \mathrm{ch}(\varphi)x_1 + \mathrm{sh}(\varphi)t, x_2, x_3\big),$$

where $V_\varphi := \big(\mathrm{ch}(\varphi)v_1 + \mathrm{sh}(\varphi)\langle v\rangle, v_2, v_3\big)$.

Remark 23. One can introduce the Faraday tensor F^{lin}, which is a 2-form and such that its cartesian components $F_{\mu\nu}$ are either equal to 0, $\pm E_k$ or $\pm B_k$. Then,

$$F^\varphi(t,x) = \big[A_\varphi^{-1}\big]^T F^{\mathrm{lin}}\big(A_\varphi(t,x)\big) A_\varphi^{-1}.$$

Note then that

$$\int_{\mathbb{R}^3_x} f^\varphi(0,x,v)\mathrm{d}x = \int_{\mathbb{R}^3_x} f_0^{\mathrm{lin}}\big(\mathrm{ch}(\varphi)x_1 - \mathrm{sh}(\varphi)x_1\widehat{V}_{\varphi,1}, x_2 - \mathrm{sh}(\varphi)x_1\widehat{V}_{\varphi,2}, x_3 - \mathrm{sh}(\varphi)x_1\widehat{V}_{\varphi,3}, \\ \mathrm{ch}(\varphi)v_1 + \mathrm{sh}(\varphi)\langle v\rangle, v_2, v_3\big)\mathrm{d}x$$

and that $\mathrm{ch}(\varphi) - \mathrm{sh}(\varphi)\widehat{V}_{\varphi,1} > 0$. Thus, there exists (a unique) $\varphi \in \mathbb{R}$ such that

$$\mathcal{U}_1\big[f^\varphi(0,\cdot,\cdot)\big] = 0.$$

Then, by applying Lemma 22, we get that (3.7) holds for $(f^\varphi, E^\varphi, B^\varphi)$. Thus, we obtain Proposition 8 for[3]

$$h^\varphi(t,x,v) := h\big(\mathrm{ch}(\varphi)t + \mathrm{sh}(\varphi)x_1, \mathrm{ch}(\varphi)x_1 + \mathrm{sh}(\varphi)t, x_2, x_3, \mathrm{ch}(\varphi)v_1 + \mathrm{sh}(\varphi)\langle v\rangle, v_2, v_3\big)$$

and then for h. Indeed, the set of the timelike straight lines $\{(t, x + t\widehat{v}, v) \,|\, (t,x,v) \in \mathbb{R} \times \mathbb{R}^3_x \times \mathbb{R}^3_v\}$ is invariant by the elements of the Lorentz group $SO_0(1,3)$. Alternatively, we have

$$\mathbb{L}^\varphi(v) - \widehat{v} \cdot \mathbb{L}^\varphi(v)\,\widehat{v} = \mathbb{L}(V_\varphi) - \widehat{V}_\varphi \cdot \mathbb{L}(V_\varphi)\,\widehat{V}_\varphi.$$

4. Modified scattering for the distribution function

Instead, we could expect a modified scattering statement for the distribution function. That is, given a global solution (f, E, B) to the Vlasov–Maxwell system, we could still expect

$$\lim_{t \to +\infty} f(t, x + t\widehat{v} + \mathscr{C}_{t,x,v}, v) = f_\infty(x,v),$$

where the correction $\mathscr{C}_{t,x,v}$ to the linear characteristics is a lower order term, $\mathscr{C}_{t,x,v} = o(|x + t\widehat{v}|)$. By the previous discussion, we should in fact expect it to grow as $\log(t)$.

Remark 24. If such a convergence holds, then the correction coefficient is not unique.

In our simpler setting, the function h enjoys a modified scattering dynamic. In order to identify the correction term through a heuristic discussion, recall the asymptotic Lorentz force

$$\mathbb{L}(v) := \mathbb{E}(v) + \widehat{v} \times \mathbb{B}(v).$$

The characteristics of the Vlasov equation verified by h satisfy, for $t \gg 1$,

$$\dot{X} = \widehat{V}, \qquad \dot{V} \approx t^{-2}\,\mathbb{L}(V),$$

[3]Note that h^φ verifies $\partial_t h^\varphi + \widehat{v} \cdot \nabla_x h^\varphi + \big(E^\varphi + \widehat{v} \times B^\varphi\big) \cdot \nabla_v f^\varphi = 0$.

so that we expect $V(t) \to v$ as $t \to +\infty$. It yields to the approximations

$$V(t) \approx v - \frac{1}{t}\mathbb{L}(v), \qquad \dot{X}(t) \approx \widehat{v} + \partial_t \mathscr{C}_{t,v} + O\big(t^{-2}\big), \qquad \mathscr{C}^k_{t,v} := \frac{\log(t)}{\langle v \rangle}\left(\widehat{v} \cdot \mathbb{L}(v)\widehat{v}^k - \mathbb{L}^k(v)\right),$$

and then $X(t) \approx x + t\widehat{v} + \mathscr{C}_{t,v}$. Consistently with these heuristic computations, one can prove that

$$\partial_t \Big[h(t, x + t\widehat{v} + \mathscr{C}_{t,v}, v) \Big] = O\big(t^{-2}\big),$$

which implies

$$\exists\, h_\infty : \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}, \qquad \lim_{t \to +\infty} h\big(t, x + t\widehat{v} + \mathscr{C}_{t,v}, v\big) = h_\infty(x, v).$$

5. Scattering results for the Vlasov–Maxwell system

5.1. Presentation of the results

We proved in [4, Theorem 2.11] that asymptotic completeness holds for the solutions to the Vlasov–Maxwell system arising from sufficiently regular initial data and a small distribution function. In [5, Theorem 1.8], we obtained that existence and uniqueness of the scattering states hold true as well for this class of solutions. We state these results here and, in order to simplify the presentation, we stay vague concerning the assumptions on the data[4].

Theorem 25. *Let (f, E, B) be the unique maximal solution to the Vlasov–Maxwell system arising from a sufficiently regular initial data set. Then, if $f(0, \cdot, \cdot)$ is small enough,*

- *(f, E, B) is a global-in-time solution.*
- *There exists an asymptotic Lorentz force $v \mapsto \mathbb{L}(v)$ such that*
$$\forall\, (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v, \qquad \Big| t^2 \big[E + \widehat{v} \times B\big](t, x + t\widehat{v}) - \mathbb{L}(v) \Big| \lesssim \frac{\langle x \rangle^{\frac{3}{2}} \langle v \rangle^3}{\sqrt{1+t}}.$$
- *The distribution function f verifies modified scattering along logarithmical corrections of the linear characteristics, defined teleologically. There exists $f_\infty : \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}$, such that*
$$\forall\, (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v, \qquad \big| f(t, x + t\widehat{v} + \mathscr{C}_{t,v}, v) - f_\infty(x, v) \big| \lesssim \frac{\log^3(2+t)}{\sqrt{1+t}}.$$
Moreover, the asymptotic Lorentz force $\mathbb{L}$ is a functional of $\int_{\mathbb{R}^3_x} f_\infty(x, \cdot)\mathrm{d}x$.
- *The electromagnetic field (E, B) verifies linear scattering. There exists $(E_\infty, B_\infty) : \mathbb{R}_u \times \mathbb{S}^2_\omega \to \mathbb{R}^6$ such that, for all $(u, \omega) \in \mathbb{R}_u \times \mathbb{S}^2_\omega$,*
$$\big| rE(r+u, r\omega) - E_\infty(u, \omega) \big| + \big| rB(r+u, r\omega) - B_\infty(u, \omega) \big| \lesssim \langle r + |u| \rangle^{-1}.$$
Moreover, (E_∞, B_∞) verifies the constraint equations (2.2) as well as additional ones, relating $\omega \mapsto \int_{\mathbb{R}_u} E_\infty(u, \omega)\mathrm{d}u$ to functionals of $v \mapsto \int_{\mathbb{R}^3_x} f_\infty(x, v)\mathrm{d}x$.
- *The total energy is conserved up to $t = +\infty$, that is (2.6) holds.*

Conversally, given a sufficiently regular scattering state $(f_\infty, E_\infty, B_\infty)$ satisfying the same constraint equations, there exists a unique global classical solution (f, E, B) to the Vlasov–Maxwell system verifying the previous convergence estimates.

Remark 26. The additional constraint equations satisfied by $\int_{\mathbb{R}_u} E_\infty(u, \cdot)\mathrm{d}u$ can be found in [5, Equations (22)–(23)]. It turns out that for a sufficiently regular solution $(E^{\mathrm{hom}}, B^{\mathrm{hom}})$ to the homogeneous Maxwell equations, its radiation field $(E^{\mathrm{hom}}_\infty, B^{\mathrm{hom}}_\infty)$ verifies

$$\int_{\mathbb{R}_u} E^{\mathrm{hom}}_\infty(u, \cdot)\mathrm{d}u = \int_{\mathbb{R}_u} B^{\mathrm{hom}}_\infty(u, \cdot)\mathrm{d}u = 0.$$

[4] We state here decay rates corresponding to the choice $\delta \geq 1/2$ in [4, Theorem 2.11].

Thus, in some sense, (E,B) can be asymptotically approximated by a homogeneous solution merely at first order. It is for this reason that, for the purpose of constructing a satisfactory scattering theory for the Vlasov–Maxwell system, one must consider the free dynamic of the linearised system (1.5)–(1.7) as a reference instead of the one of the free transport equation and the homogeneous Maxwell equations.

Remark 27. The convergence estimates are slightly better than the ones stated in [4, Theorem 2.11]. It is our analysis in [5] which provides these improvements.

We emphasize that we proved convergence in stronger topologies as well. In particular, provided that the data are sufficiently regular, we derived scattering results for the derivatives of the solutions. It allowed us to prove that f_∞ and (E_∞, B_∞) are differentiable and decay at polynomial rates.

Finally, let us mention the work of Ben Artzi–Pankavich [27]. Given a small data solution to the Vlasov–Maxwell system (f,E,B) constructed by Glassey–Strauss in [18], they proved modified scattering for the distribution function f.

5.2. Related results for the Vlasov–Poisson system

The study of the modified scattering dynamics of the small data solutions to the Vlasov–Poisson system started with [9]. More recently, [20] clearly identified the asymptotic electrostatic force field responsible for this phenomenon (see also [26]). It is related, through the Poisson equation, to the spatial average of the limit distribution function f_∞. The understanding of these asymptotic dynamics allowed Flynn–Ouyang–Pausader–Widmayer to construct a scattering map for the small data solutions to the Vlasov–Poisson system [11].

The asymptotic stability of other steady states has been recently adressed. The perturbations of a point charge exhibits a modified scattering dynamics [28] and the ones of the Poisson equilibrium scatter to linear solutions [19]. For the study of the solutions to the linearised Vlasov–Poisson system near more general spatially homogenous equilibria, we refer to [21, 25]. Finally, let us mention that modified scattering holds as well for the small data solutions to the Vlasov–Poisson system with the external potential $-\frac{|x|^2}{2}$ (see [6]). For these class of plasmas, certain particles are trapped, leading to the exponential growth of most of the microscopic derivatives of f.

5.3. Key ideas of the proof

We conclude these notes by presenting the main steps of the proof of the modified scattering statement for f in Theorem 25. We then consider a solution (f,E,B) to the Vlasov–Maxwell system arising from sufficiently regular initial data and a small distribution function. Consistently with the linear analysis, up to certain logarithmical loss, the following estimates holds.

Proposition 28. *For all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3_x$,

$$\int_{\mathbb{R}^3_v} f(t,x,v)\mathrm{d}v \lesssim \langle t \rangle^{-3}, \quad \big|(E,B)\big|(t,x) + \langle t-|x|\rangle \big|\nabla_{t,x}(E,B)\big|(t,x) \lesssim \langle t+|x|\rangle^{-1}\langle t-|x|\rangle^{-1}.$$

The derivatives of f *satisfy for all* $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v$, *any* $1 \leq i \leq 3$ *and any* $0 \leq q \leq 6$,

$$\langle x - t\widehat{v}\rangle^q \langle v \rangle^9 \big|f\big|(t,x,v) + \langle x - t\widehat{v}\rangle^q \langle v \rangle^9 \big|\nabla_x f\big|(t,x,v) \lesssim \log^q(2+t), \tag{5.1}$$

$$\langle x - t\widehat{v}\rangle^q \langle v \rangle^7 \bigg| \sum_{1\leq j\leq 3} t\frac{\delta_{i,j}-\widehat{v}_i\widehat{v}_j}{\langle v \rangle}\partial_{x_j} f(t,x,v) + \partial_{v_i} f(t,x,v)\bigg| \lesssim \log^{q+1}(2+t). \tag{5.2}$$

Recall then the inequality (3.3), so that

$$\big|(E,B)\big|(t,x) \lesssim \langle t+|x|\rangle^{-2}\langle x - t\widehat{v}\rangle \langle v \rangle^2, \tag{5.3}$$

$$\big|\nabla_{t,x}(E,B)\big|(t,x) \lesssim \langle t+|x|\rangle^{-3}\langle x - t\widehat{v}\rangle^2 \langle v \rangle^4. \tag{5.4}$$

We now present the strategy of the proof of $f(t, x+t\widehat{v}+\mathscr{C}_{t,v}, v) \to f_\infty(x,v)$ as $t \to \infty$.

Step 1. Although we cannot expect f to have a linear behavior for large times, we can still expect a quantity containing less informations to verify such a property. It turns out that the spatial average of f, which is conserved in the linear case, converges as $t \to +\infty$. Indeed, using the Vlasov equation (1.1) and performing integration by parts in x, we have

$$\partial_t \int_{\mathbb{R}^3_x} f(t,x,v)\mathrm{d}x + \int_{\mathbb{R}^3_x} \Big[E + \widehat{v} \times B\Big](t,x) \cdot \nabla_v f(t,x,v)\mathrm{d}x = 0.$$

Then, using (5.2), (5.3) and performing integration by parts, we obtain for all $t \geq 2$,

$$\langle v\rangle^5 \left|\partial_t \int_{\mathbb{R}^3_x} f(t,x,v)\mathrm{d}x\right| \lesssim t \int_{\mathbb{R}^3_x} \big|\nabla_{t,x}(E,B)\big|(t,x)\big|\langle v\rangle^5 f(t,x,v)\big|\mathrm{d}x + \frac{\log^6(t)}{t^2}\int_{\mathbb{R}^3_x} \frac{\mathrm{d}x}{\langle x - t\widehat{v}\rangle^4}.$$

Thus, by (5.1) and (5.4), we get

$$\forall\ (t,v) \in [2,+\infty[\times \mathbb{R}^3_v, \qquad\qquad \langle v\rangle^5 \left|\partial_t \int_{\mathbb{R}^3_x} f(t,x,v)\mathrm{d}x\right| \lesssim \frac{\log^6(t)}{t^2},$$

so that there exists $Q_\infty \in L^\infty(\mathbb{R}^3_v)$ such that

$$\forall\ (t,v) \in [2,+\infty[\times \mathbb{R}^3_v, \qquad\qquad \langle v\rangle^5 \left|\int_{\mathbb{R}^3_x} f(t,x,v)\mathrm{d}x - Q_\infty(v)\right| \lesssim \frac{\log^6(t)}{t}.$$

Step 2. Then, we can isolate the leading order contribution of the charge and current density. More precisely, one can prove, for all $|x| < t$,

$$\left|t^3 \int_{\mathbb{R}^3_v} f(t,x,v)\mathrm{d}v - \Big[\langle v\rangle^5 Q_\infty\Big]\left(\frac{x}{\sqrt{t^2-|x|^2}}\right)\mathrm{d}y\right| \lesssim \frac{\log^{50}(t)}{t}, \tag{5.5}$$

$$\left|t^3 \int_{\mathbb{R}^3_v} \widehat{v} f(t,x,v)\mathrm{d}v - \frac{x}{t}\Big[\langle v\rangle^5 Q_\infty\Big]\left(\frac{x}{\sqrt{t^2-|x|^2}}\right)\mathrm{d}y\right| \lesssim \frac{\log^{50}(t)}{t}. \tag{5.6}$$

Recall that Vlasov fields strongly decay near and in the exterior of the light cone. Thus, we define the asymptotic charge and current densities as

$$\rho^{\mathrm{asymp}}(t,x) := \frac{1}{t^3}\Big[\langle v\rangle^5 Q_\infty\Big]\left(\frac{x}{\sqrt{t^2-|x|^2}}\right)\mathbb{1}_{|x|<t},$$

$$j^{\mathrm{asymp}}(t,x) := \frac{x}{t^4}\Big[\langle v\rangle^5 Q_\infty\Big]\left(\frac{x}{\sqrt{t^2-|x|^2}}\right)\mathbb{1}_{|x|<t},$$

which turn out to be continous, even at the points $t = |x|$.

Step 3. It allows us to consider the asymptotic Maxwell equations (3.5)–(3.6). Since (5.5)–(5.6) holds, we can in fact prove

$$\big|E - E^{\mathrm{asymp}}\big|(t,x) + \big|B - B^{\mathrm{asymp}}\big|(t,x) \lesssim \langle t+|x|\rangle^{-1}\langle t-|x|\rangle^{-\frac{3}{2}}.$$

Thus, from the analysis of $(E^{\mathrm{asymp}}, B^{\mathrm{asymp}})$ and the previous estimate, we obtain

$$\forall\ (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v, \qquad \big|t^2\big(E,B\big)(t,x+t\widehat{v}) - \big(\mathbb{E},\mathbb{B}\big)(v)\big| \lesssim \frac{\langle x\rangle^{\frac{3}{2}}\langle v\rangle^3}{\sqrt{1+t}},$$

implying the existence of the asymptotic Lorentz force $v \mapsto \mathbb{L}(v)$, from which the correction coefficient $\mathscr{C}_{t,v}$ are defined.

Step 4. From these informations, we are finally able to prove that f converges along logarithmical correction of the linear characteristics. ☐

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