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Asymptotics for vectorial Allen–Cahn type problems

Fabrice Bethuel

Abstract

These notes present some recent results concerning the convergence of solutions to the elliptic vectorial Allen–Cahn equation in dimension two as the parameter ε tends to zero, and its connections to minimal surface theory in the weak sense of stationary varifolds. We first describe the results obtained so far in the scalar theory, which can be considered as quite satisfactory, and provide some ideas about the proofs and their main steps. We then present some adaptations necessary to handle the vectorial case in dimension two.

1. The problem

1.1. The system of elliptic equations

We investigate the asymptotic behavior, as $\varepsilon \to 0$ of families of solutions $(u_\varepsilon)_{\varepsilon>0}$ to gradient-type equations

$$-\Delta u_\varepsilon = -\varepsilon^{-2}\nabla V_u(u_\varepsilon), \text{ on a domain } \Omega \subset \mathbb{R}^N, \ N \geq 1, \qquad (\mathbf{VAC}_\varepsilon)$$

where $0 < \varepsilon < 1$ represents a (small) parameter, u_ε denotes a map from the subdomain Ω of $\mathbb{R}^N$ into an euclidean space $\mathbb{R}^k$, $k \in \mathbb{N}^*$ and where V, the *potential*, is a map from $\mathbb{R}^k$ into $\mathbb{R}^+$, hence scalar. In particular, equation $(\mathbf{VAC}_\varepsilon)$ represents a *system* of k coupled scalar elliptic equations. Solutions to $(\mathbf{VAC}_\varepsilon)$ are critical points of the energy

$$\mathrm{E}_\varepsilon(u) = \int_\Omega e_\varepsilon(u) = \int_\Omega \varepsilon\frac{|\nabla u|^2}{2} + \frac{V(u)}{\varepsilon}, \quad \text{for } u : \Omega \mapsto \mathbb{R}^k. \qquad (1.1)$$

If V is convex, then E_ε is strictly convex and non negative. Throughout, we will impose a bound of the type

$$\mathrm{E}_\varepsilon(u) \leq M_0, \qquad (1.2)$$

for some constant $M_0 > 0$. This bound will allow us to recover suitable *compactness properties*, under specific assumptions on V.

Remark 1. For a strictly convex potential which is bounded below, solutions to $(\mathbf{VAC}_\varepsilon)$ are locally bounded. If one imposes a mild compactness assumption on the sequence $(u_\varepsilon)_{\varepsilon>0}$, then one may prove that

$$u_\varepsilon \to \sigma, \text{ as } \varepsilon \to 0, \qquad (1.3)$$

where σ represents the unique minimizer. The speed of the convergence depends on the properties of the potential near σ, in particular the second derivative.

Remark 2. The parabolic case

$$\frac{\partial u}{\partial t} - \Delta u_\varepsilon = -\varepsilon^{-2}\nabla V_u(u_\varepsilon). \qquad (\mathbf{PVAC}_\varepsilon)$$

is also of great interest, but will not be discussed in details here.

1.2. Assumptions on the potential V

The behavior of solutions to $(\mathbf{VAC}_\varepsilon)$ depend crucially on properties of the potential $V : \mathbb{R}^k \to \mathbb{R}$. The main assumptions on the potential V in these notes are the following :

$$\inf V = 0 \text{ and the set of minimizers } \Sigma \equiv \{y \in \mathbb{R}^k, V(y) = 0\} \tag{H$_1$}$$

is finite and contains *at least two distinct* elements, i.e.

$$\Sigma = \{\sigma_1, \dots, \sigma_q\}\,,\ q \geq 2,\ \ \sigma_i \in \mathbb{R}^k,\ \forall\, i = 1, \dots, q.$$

Besides (H_1), we impose a behavior at infinity, namely

$$V(x) \underset{|x|\to+\infty}{\longrightarrow} +\infty. \tag{H$_\infty$}$$

Assumption (H_1) hence excludes in particular the case of a strict convex potential, for which the infimum would be unique, as discussed in Remark 1. A potential with such properties is represented in Figure 1.1.

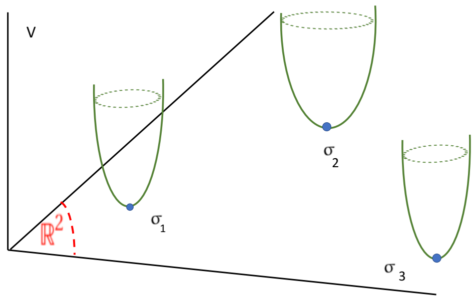


Figure 1.1: Shape of the graph of V with three zeroes.

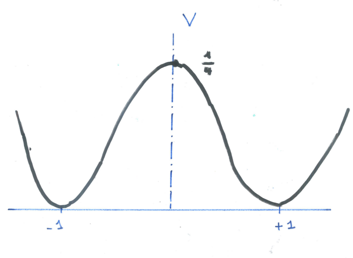
For potentials satisfying assumptions (H_1) and (H_∞), we may construct solutions which take values, for small ε, close to different minimizers in Σ. Such a phenomenon is sometimes called *phase segregation*, since it divides the domain in regions where the solution takes values near one of the minimizers, creating *interfaces* between the minimizers. The geometric interpretation of this interfaces is one of the central questions of the theory.

1.3. A classical scalar example: The Allen–Cahn potential

In the *scalar case* $k = 1$, a classical and typical example is given the *Allen–Cahn* potential:

$$V(s) = \frac{(1 - s^2)^2}{4}, \quad s \in \mathbb{R}. \tag{AC}$$

The minimizers here are hence $\sigma_1 = +1$ and $\sigma_2 = -1$, $\Sigma = \{+1, -1\}$



2. One-dimensional problems $N = 1$

We first turn to the one-dimension case. It provides us with first examples yielding segregation. One dimensional solutions turn out to be one of the building blocks for solutions in higher dimensions. The study for $N = 1$ also highlights important differences between the scalar and the vectorial case.

2.1. Heteroclinic connections

For $\Omega = \mathbb{R}$, we may assume, by scaling, that $\varepsilon = 1$, and ($\mathbf{VAC}_\varepsilon$) reduces to the *second order* ODE

$$-\frac{\mathrm{d}^2 w}{\mathrm{d}s^2} = \nabla_w V(w) \text{ on } \mathbb{R}, \ \ w(x) = u(\varepsilon x). \tag{\textbf{ODE}}$$

In view of the form (1.1) of the energy, finite energy solutions necessarily connect at $\pm\infty$ two minimizers σ^- and σ^+: They are called *profiles* or *heteroclinic connections*, if $\sigma^- \neq \sigma^+$, and *homoclinic connection* otherwise. Multiplying (**ODE**) by $\frac{\mathrm{d}w}{\mathrm{d}x}$, we obtain the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}x}\xi = 0, \text{ where } \xi(x) = V(w) - \frac{|w'|^2}{2}, \text{ for } x \in \mathbb{R}, \tag{2.1}$$

so that the function ξ is constant. For profiles, the function ξ has to vanish, since the energy given in (1.1) is assumed finite, so that one is led to the identity

$$|w'| = \sqrt{2V(w)}, \text{ on } \mathbb{R}. \tag{2.2}$$

2.2. The scalar case

In the *scalar* case, we may remove the absolute value in (2.2), which yields a *first order* equation

$$w' = \pm\sqrt{2V(w)}, \text{ on } \mathbb{R}.$$

We may integrate this equation using standard separation of variables. One then shows that profiles connect only *nearby minimizers* σ^- and σ^+, and that the solution is unique up to translations and symmetries. For the Allen–Cahn potential, it is given by

$$w(s) = \tanh\left(\frac{s}{\sqrt{2}}\right).$$

2.3. The vectorial case

In contrast, in the *vectorial* case, the integration of the equation (**ODE**) is far from being straighforward. As a matter of fact, the problem of finding profiles is a very active field of research: Several approaches have been proposed and results have been obtained recently see e.g. [2, 13, 14, 18], among many others). However, there is no general result, both for existence and uniqueness. The potential whose graph is given in Figure 2.1 possesses a symmetry according to one of the axis, and has therefore two minimizing connections between the two minimizers (and most perhaps a third one using moutain pass).

2.4. The limit $\varepsilon \to 0$

We recover u_ε from w scaling back, that is setting $u_\varepsilon(x) = w(x/\varepsilon)$, $x \in \mathbb{R}$, so that u_ε solves equation ($\mathbf{VAC}_\varepsilon$). We have the scaling law

$$\mathrm{E}_\varepsilon(w_\varepsilon) = \mathrm{E}(w), \text{ for any } \varepsilon > 0.$$

This motivates condition (1.2).

As seen on Figure 2.2 for the Allen–Cahn equation, the solution u_ε has now a steep slope at the origin, of order C/ε. On the left of the origin, u_ε takes values close to -1, whereas on the right of the origin, it takes values close to $+1$. The transition from one minimizer of V to another occurs on a region of caracteristic size ε near the origin.

In the limit $\varepsilon \to 1$, the sequence $(u_\varepsilon)_{\varepsilon>0}$ converges to the function $u_\star(x) = \mathrm{sgn}(x)$.

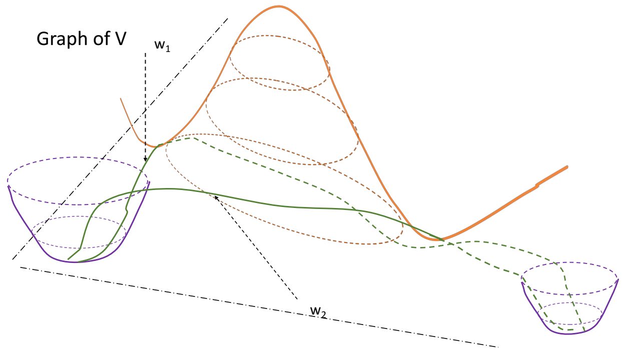


Figure 2.1: Two symmetric minimizing heteroclinic connections joining the same minimizers.

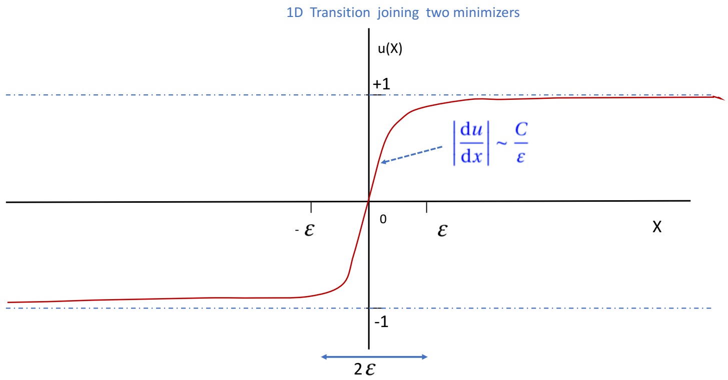


Figure 2.2: Graph of u_ε and phase segregation

3. Higher dimensional problems

In dimension $N > 1$, one expect similar phase segregation results, the interfaces being now codimension 1 hypersurfaces with area minimizing properties. These type of results have been establishes in the scalar case $k = 1$.

3.1. An overview of the results for the scalar case

In the scalar case, the limit $\varepsilon \to 0$ of solutions to $(\mathbf{VAC}_\varepsilon)$ or $(\mathbf{PVAC}_\varepsilon)$ is well understood. The result can roughly be stated as follows: The domain Ω decomposes into regions where the solution takes values either close to $+1$ or close to -1(for the Allen–Cahn potential). These two regions are separated by interfaces of width $\simeq \varepsilon$. In the limit $\varepsilon \to 0$, these interfaces converge to codimension 1 *generalized minimal surfaces*. Figure 3.1 shows a typical example in dimension two, where the interface is a (slightly fatened) line. In the parabolic case $(\mathbf{PVAC}_\varepsilon)$, the interfaces are moved by mean curvature. In a simplest cases, near a core point x_0 on the interface, one has (see e.g. [9])

$$u_\varepsilon(x_1, x_2, \ldots, x_N) \underset{x_1 \to x_{0,N}}{\simeq} \tanh\left(\frac{x_N - x_{0,N}}{\sqrt{2}\varepsilon}\right),$$

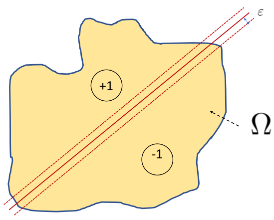


Figure 3.1: Interface near a regular point x_0 in the scalar case, with an Allen–Cahn type potential.

where the unit vector $\mathbf{e}_1, \mathbf{e}_3, \ldots, \mathbf{e}_{N-1}$ are chosen to be tangent to the interface, and where the unit vector $\mathbf{e}_N$ is orthonormal to the interface. In the general case, there are also solutions which correspond to gluing several such one-dimensional solutions, but we will not discuss this here. Ultimately, the results in [9, 10] provide a rather simple picture of the solutions. They involve a minimal surface, the solution may be represented as one-dimensional profiles glued on the surface in the transversal direction, so that one is tempted to write the correspondance

$$\text{solutions to } (\mathbf{VAC}_\varepsilon) \sim \text{minimal surface} + \text{glued profiles}. \tag{3.1}$$

The general structure of solution is hence fairly well understood. As a matter of fact, the correspondance goes to some extent in either way, since, conversely, given a minimal surface, one may construct solutions to the scalar Allen–Cahn equation having the previous behavior (see e.g. [15]).

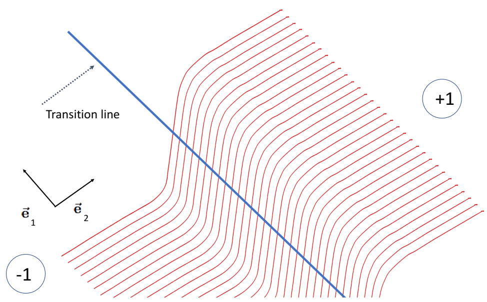


Figure 3.2: Graph of u_ε at the interface near a regular point x_0.

3.2. The vectorial case for $N \geq 2$

One may wonder if similar results hold in the vectorial case. More precisely, given a family $(u_\varepsilon)_{\varepsilon>0}$ of solutions to $(\mathbf{VAC}_\varepsilon)$, with $\mathrm{E}_\varepsilon(u_\varepsilon) \leq M_0$, do we have:

($\mathcal{Q}_1$) *Is there concentration near a codimension 1 hypersurface (or rectifiable set) $\mathfrak{S}_\star$?*

($\mathcal{Q}_2$) *It is the set $\mathfrak{S}_\star$ minimal in some suitable weak sense?*

($\mathcal{Q}_3$) *Do we have, near a point x_0 on this interface $\mathfrak{S}_\star$*

$$u_\varepsilon(x_1, x_2, \ldots, x_N) \underset{x_1 \to x_{0,1}}{\simeq} w\left(\frac{x_1 - x_{0,1}}{\varepsilon}\right), \tag{Profile}$$

where $w : \mathbb{R} \to \mathbb{R}^k$ is a one-dimensional profil? Uniqueness of w?

It turns out that the three questions are *linked*.

3.3. $(\mathcal{Q}_3)$: new profiles or pseudo-profiles

3.3.1. The Alama–Bronsard–Gui result

In [1], the authors construct a solution to the vectorial ($\mathbf{VAC}_\varepsilon$) equation on $\mathbb{R}^2$ does not rely on a single profile, as for instance in (3.1). The assumption on the potential V involves a situation similar to the one described in Figure 2.1:

$$\text{There exists two minimizers } \sigma^+ \text{ and } \sigma^- \text{ of } V\text{, and two } \textit{distinct minimizing solutions } w_1 \text{ and } w_2 \text{ of the } \textit{differential equation} \quad -w' = \nabla_w V(w) \text{ on } \mathbb{R},\ \ w(x) \underset{x\to\pm\infty}{\longrightarrow} \sigma^{\pm}. \tag{H$_4$}$$

Alama, Bronsard and Gui provided a solution connecting the two profiles:

Theorem 1 ([1])**.** *Under the above assumptions* (H$_4$)*, there exists a (locally minimizing) solution* $u : \mathbb{R}^2 \to \mathbb{R}^k$ *of* ($\mathbf{VAC}_\varepsilon$) *such that*

$$\begin{cases} u(x_1, x_2) \to w_1(x_2) & \text{as } x_1 \to -\infty \\ u(x_1, x_2) \to w_2(x_2) & \text{as } x_1 \to +\infty \end{cases}$$

For $\varepsilon > 0$, consider the scaled map u_ε defined by $u_\varepsilon(\mathbf{x}) = u\left(\frac{\mathbf{x}}{\varepsilon}\right)$, $\mathbf{x} \in \mathbb{R}^2$ has then the following properties:

- u_ε is locally minimizing for E_ε
- concentrates on the line $\mathfrak{S}_\star = \{(x_1, 0), x_1 \in \mathbb{R}\}$, and

$$\begin{cases} u_\varepsilon(x_1, x_2) \simeq w_1(\frac{x_2}{\varepsilon}) & \text{for } x_1 < 0. \\ u_\varepsilon(x_1, x_2) \simeq w_2(\frac{x_2}{\varepsilon}) & \text{as } x_1 > 0. \end{cases}$$

The asymptotics (Profile) are verified on the half-lines, but not on the whole line.

3.3.2. Periodic Pseudo-profiles

With R. Oliver-Bonafoux, we construct solutions to ($\mathbf{VAC}_\varepsilon$) on the cylinder

$$\Lambda_{\mathrm{L}} = [-\mathrm{L}, \mathrm{L}] \times \mathbb{R}, \text{ where } L > 0,$$

with periodic boundary conditions in the x_1 direction, namely such that

$$u(-\mathrm{L}, x_2) = u(\mathrm{L}, x_2) \text{ and } \frac{\partial u}{\partial x_1}(-\mathrm{L}, x_2) = \frac{\partial u}{\partial x_1}(\mathrm{L}, x_2), \text{ for any } x_2 \in \mathbb{R}. \tag{Periodic}$$

Theorem 2 ([6])**.** *Under assumption* (H$_4$)*, there exists a constant* $L_0 > 0$*, such that, if* $L > L_0$*, then there exists a (smooth) solution to* $-\Delta u = \nabla_u V(u)$ *on* Λ_L *such that* (Periodic) *holds and such that*

$$\frac{\partial u}{\partial x_1} \neq 0. \tag{Transverse}$$

In view of condition (Transverse), the obtained solution is not derived from a 1D profile. The proof is variational, and relies on a mountain-pass argument. Consider the set $W = \{u : \Lambda_L \to \mathbb{R}^k$, $\mathrm{E}(u) < +\infty, u(-\mathrm{L}, x_2) = u(\mathrm{L}, x_2)\}$ and the number

$$c_L = \inf_{p\in\mathcal{P}} \left(\sup_{s\in[0,1]} \mathrm{E}(p(s)) \right),$$

where $\mathcal{P}$ denotes the set of all paths joining the maps u_1 and u_2 defined by

$$u_i(x_1, x_2) = w_i(x_2), \quad \forall\, x_1 \in [-\mathrm{L}, \mathrm{L}],\ x_2 \in \mathbb{R}.$$

w_1 and w_2 being the two minimizing heteroclinic connections. It turns out that, thanks to the mountain-pass Lemma, c_L is a critical value, if L is large enough. □

The scaled map on $\mathbb{R}^2$ defined for $\mathbf{x} = (x_1, x_2)$ by

$$u_\varepsilon(\mathbf{x}) = u\left(\frac{\mathbf{x} - (L(2N-1)\varepsilon\mathbf{e}_1}{\varepsilon}\right), \text{ if } x_1 \in [2LN\varepsilon, 2L(N+1)\varepsilon] \tag{3.2}$$

solves ($\mathbf{VAC}_\varepsilon$) on $\mathbb{R}^2$. Notice that, for the transversal derivative

$$\varepsilon \left| \frac{\partial u_\varepsilon}{\partial x_1} \right|^2 \rightharpoonup \upmu_{\star,1,1} \neq 0,$$

where $\upmu_{\star,1,1} = c\mathcal{H}^1(D)$, $D = \{(x_1, 0), x_1 \in \mathbb{R}\}$, for some constant $c > 0$.

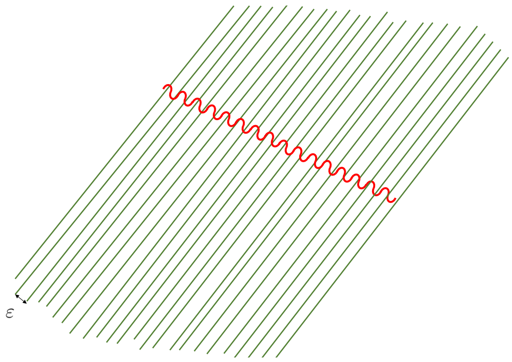


Figure 3.3: Graph of u_ε given in (3.2). The map does not involve a 1D-profile, but rather a periodic structure of size $2L\varepsilon$.

3.4. $\mathcal{Q}_1$: Concentration on lower dimensional sets

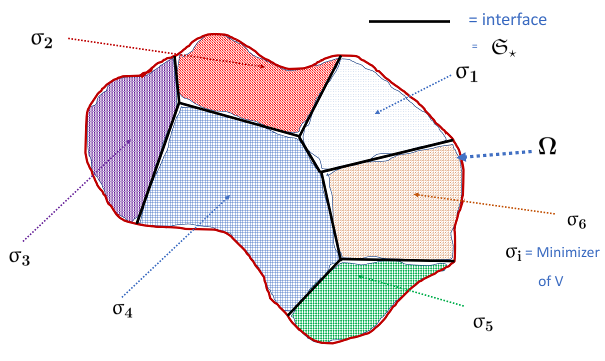
The case of minimizing solutions concentration on minimal surfaces has been established in [4, 8, 12]. Concentration for arbitrary stationary solutions is still *a widely open subject*. Only results available for $N = 2$, i.e. $\Omega \subset \mathbb{R}^2$. Recall that we consider a family $(u_\varepsilon)_{0<\varepsilon\leq 1}$ of solutions of the equation ($\mathbf{VAC}_\varepsilon$) satisfying the natural energy bound $\mathrm{E}_\varepsilon(v_\varepsilon) \leq \mathrm{M}_0$, where $\mathrm{M}_0 > 0$ is given.

Theorem 3 ([5])**.** *There exists a subset $\mathfrak{S}_\star$ of Ω, and a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ tending to 0 such that the following holds:*

(i) *$\mathfrak{S}_\star$ is closed,* rectifiable *of dimension 1, with locally a finite number of connected components and such that $\mathcal{H}^1(\mathfrak{S}_\star) \leq \mathrm{C_H}\,\mathrm{M}_0$.*

(ii) *Let $\mathfrak{U}_\star = \Omega \setminus \mathfrak{S}_\star$, and $(\mathfrak{U}^i_\star)_{i\in I}$ be the connected components $\mathfrak{U}_\star$. For $i \in I$, there exists $\upsigma_i \in \Sigma$ such that*

$$u_\varepsilon \to \upsigma_i \text{ uniformly on every compact subset of } \mathfrak{U}^i_\star.$$



The interface have a very simple structure, since they are merely an union of segments, hence locally minimizing.

Theorem 4 ([5]). *There exists a set* $\mathfrak{E}_\star \subset \mathfrak{S}_\star$ *such that* $\mathcal{H}^1(\mathfrak{E}_\star) = 0$ *and such that, for* $x_0 \in \mathfrak{S}_\star \backslash \mathfrak{E}_\star$, the set $\mathfrak{S}_\star$ is, locally near x_0, a segment. *More precisely, there exists a unit vector* $\vec{e}_{x_0}$ *and a radius* $r_0 > 0$, *depending on* x_0, *such that*

$$\mathfrak{S}_\star \cap \mathbb{D}^2(x_0, r_0) = (x_0 - r_0 \vec{e}_{x_0}, x_0 + r_0 \vec{e}_{x_0}) . \tag{3.3}$$

The set $\mathfrak{E}_\star$ is the set of singular points, which is nonempty in general. The proof of Theorem 4 is related is related to the presence of measures concentrating on $\mathfrak{S}_\star$ and their stationarity properties. Consider the positive measure ζ_ε defined on Ω by

$$\zeta_\varepsilon \equiv \frac{V(u_\varepsilon)}{\varepsilon} \mathrm{d}x, \text{ so that } \zeta_\varepsilon(\Omega) \leq M_0. \tag{3.4}$$

so that the family of measures $(\zeta_\varepsilon)_{\varepsilon > 0}$ is bounded. Passing possibly to a further subsequence, we have

$$\zeta_{\varepsilon_n} \equiv \frac{V(u_{\varepsilon_n})}{\varepsilon_n} \mathrm{d}x \rightharpoonup \zeta_\star, \text{ in the sense of measures on } \Omega, \text{ as } n \to +\infty, \tag{3.5}$$

It turns out that the measure $\zeta_\star$ concentrates on $\mathfrak{S}_\star$, and that it is absolutely continuous with respect to the $\mathcal{H}^1$-measure on $\mathfrak{S}_\star$. This property implies that the measure $\zeta_\star$ is determined by the set $\mathfrak{S}_\star$ and the density $\Theta_\star$, and we have

$$\zeta_\star = \Theta_\star (\mathcal{H}^1 \llcorner \mathfrak{S}_\star) = \Theta_\star \mathrm{d}\lambda, \text{ where } \mathrm{d}\lambda = \mathcal{H}^1 \llcorner \mathfrak{S}_\star. \tag{3.6}$$

Theorem 5 ([5]). *The rectifiable one-varifold* $\mathbf{V}(\mathfrak{S}_\star, \Theta_\star)$ *corresponding to the measure* $\zeta_\star$ *is stationary.*

The statement is equivalent to the following : Given any smooth vector field $\vec{X} \in C_c^\infty(\Omega, \mathbb{R}^2)$ on Ω with compact support, the following identity holds

$$\int_\Omega \mathrm{div}_{T_x \mathfrak{S}_\star} \vec{X} \mathrm{d}\zeta_\star = 0. \tag{3.7}$$

For $x \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, the number $\mathrm{div}_{T_x \mathfrak{S}_\star} \vec{X}(x)$ is well-defined by

$$\mathrm{div}_{T_x \mathfrak{S}_\star} \vec{X}(x) = \left(\vec{e}_x \cdot \vec{\nabla} \vec{X}(x) \right) \cdot \vec{e}_x, \text{ for } x \in \mathfrak{S}_\star. \tag{3.8}$$

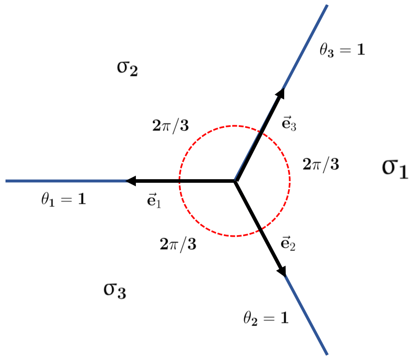
Identity (3.7) expresses a local stationarity property of the integral of $\zeta_\star$ with respect to local deformation, and for smooth sets $\mathfrak{S}_\star$, is equivalent, in dimension one, to the fact that the curvature vanishes. Allard and Almgren showed in [3] that such one-dimensional varifolds have a network structure and are the sum of segments with constant densities, so that Theorem 4 follows from Theorem 5. A typical example of a stationary one-varifold with a singularity at 0 is given by the union of d half-lines, intersecting at the origin, with constant densities. Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_d$ be d-distinct unit vectors in $\mathbb{R}^2$. Set

$$\mathcal{S}_\star = \bigcup_{i=1}^{d} \mathbb{H}_i, \text{ where for } i = 1, \ldots, d, \text{ we set } \mathbb{H}_i = \{t \vec{e}_i, t \geq 0\}, \tag{3.9}$$

and let $\theta_1, \ldots, \theta_d$ be d positive numbers. If θ_i represents the density on $\mathbb{H}_i$ (which is hence constant there), then $\mathbf{V}(\mathcal{S}_\star, \Theta)$ is a stationary one-dimensional rectifiable varifold if and only

$$\sum_{i=1}^{d} \theta_i \vec{e}_i = 0. \tag{3.10}$$

Such singularities may actually occur as limits of $(\mathbf{VAC}_\varepsilon)$ solutions as constructed in [7, 17].



We next present the ideas of the proofs in the scalar case, and then turn to the vectorial case in dimension two.

4. About the proofs in the scalar case

4.1. The discrepancy function

The discrepancy function ξ_ε defined in (2.1) may be generalized in higher dimensions as

$$\xi_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon}V(u_\varepsilon) - \varepsilon\frac{|\nabla u|^2}{2}. \tag{4.1}$$

Recall that for $N = 1$ we have, for finite energy solutions $\xi_\varepsilon = 0$. We cannot expect a similar result in higher dimension. However, following earlier results by Sperb, Payne, Stackgold, and Serrin, L. Modica proved the following groundbreaking result:

Theorem 6 ([11])**.** *Let $V \in C^2(\mathbb{R})$ be non-negative, $f = V'$, and $v \in C^3(\mathbb{R}^N)$ be a bounded solution to $\Delta u = f(u)(= V'(u))$. Then*

$$\xi(x) \equiv V(u(x)) - |\nabla u(x)|^2 \geq 0, \forall\, x \in \mathbb{R}^N.$$

The idea of the proof relies on the inequality

$$-|\nabla u|^2\Delta\xi \geq \frac{1}{2}|\nabla\xi|^2 - 2f(u)\nabla u.\nabla\xi \tag{4.2}$$

and the use of a suitable maximum principle. For general domains and general solutions, one might still obtain a slightly weaker version: $\xi \gtrsim 0$ (in some suitable sense).

4.2. Monotonicity formulas

Let u_ε be a solution to $(\mathbf{VAC}_\varepsilon)$. For $r > 0$, $x_0 \in \Omega$ such that $\mathbb{B}(x_0, r) \subset \Omega$, we set $\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{B}(x_0, r)) = \int_{\mathbb{B}(x,r)} e(u_\varepsilon(x))\mathrm{d}x$ and $\mathbb{V}_\varepsilon(u, \mathbb{B}(x, r)) = \int_{\mathbb{B}(x,r)} \varepsilon^{-1}V(u_\varepsilon(x))\mathrm{d}x$. We have the identity

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{N-2}}\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{B}(x_0, r))\right) = 2\frac{\mathbb{V}_\varepsilon(u, \mathbb{B}(r))}{r^{N-1}} + \frac{\varepsilon}{r^{N-2}}\int_{\partial\mathbb{B}(r)}\left|\frac{\partial u}{\partial r}\right|^2 \geq 0, \tag{4.3}$$

which holds for *any potential, even in the vectorial case.* It implies in particular that the local energy density is smaller then Cr^{N-2}, i.e.

$$\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{B}(x_0, r)\right) \leq Cr^{n-2}, \text{ for } r \leq r_0, C = \mathrm{E}_\varepsilon(u, \mathbb{B}(x_0 r_0)), \text{ for any } r_0 \text{ s. t } \mathbb{B}(x_0, r_0) \subset \Omega.$$

As we will see, together with the clearing-out property which we will introduce later, it allows concentration on sets of dimension *not smaller then* $N - 2$ (see Figure 4.1).

For concentration on $N - 1$ *dimensional sets*, the stronger monotonicity formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{N-1}}\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{B}^N(x_0, r)\right)\right) \gtrsim 0, \text{ for any } x_0 \in \Omega, \tag{4.4}$$

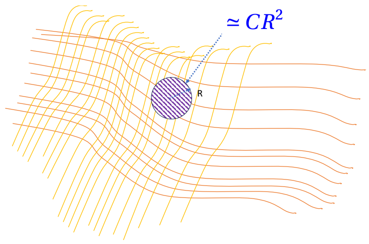


Figure 4.1: $N-1$ concentration on a surface in $\mathbb{R}^3$.

is more appropriate. Starting from (4.3), we derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{N-1}}\mathrm{E}_\varepsilon(u,\mathbb{B}(r))\right) &= \frac{1}{r^{N-1}}\frac{\mathrm{d}}{\mathrm{d}r}(\mathrm{E}_\varepsilon(u,\mathbb{B}(r))) - \frac{N-1}{r^N}(\mathrm{E}_\varepsilon(u,\mathbb{B}(r)) \\ &= \frac{1}{r^N}(2\mathbb{V}_\varepsilon(u,B(r)) - \mathrm{E}_\varepsilon(u,\mathbb{B}(r)) + \frac{1}{r^{N-1}}\int_{\partial\mathbb{B}(r)}|\frac{\partial u}{\partial r}|^2 \\ &= \frac{1}{r^N}\int_{\mathbb{B}(r)}\xi_\varepsilon(u) + \frac{1}{r^{N-1}}\int_{\partial\mathbb{B}(r)}|\frac{\partial u}{\partial r}|^2 \gtrsim 0\end{aligned} \tag{4.5}$$

Formula (4.4) has been established in [9, 10] in the *scalar case*.

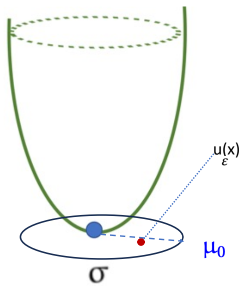


Figure 4.2: The map u_ε takes values near σ on $\mathbb{B}\left(x_0, \frac{3r}{4}\right)$. Hence, we are in a situation similar to a convex potential, yielding improved estimates.

4.3. The Clearing-out property in the scalar case

As a direct consequence of the monotonicity formula, we have the following:

Lemma 1. *Given any* $\mu_0 > 0$, $\exists\, \eta_0 > 0$ *such that, if for* $r \geq \varepsilon$, *we have*

$$\frac{1}{r^{N-1}}\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{B}^N(x_0,r)\right) \leq \eta_0 \implies \exists\, \sigma \in \Sigma,\ \ s.t\ |u_\varepsilon(x) - \sigma| \leq \mu_0,\ \ x \in \mathbb{B}\left(x_0, \frac{3r}{4}\right).$$

Proof. Assume by contradiction that there exists $x_1 \in \mathbb{B}(x_0, \frac{3r}{4})$ such that

$$|u(x_1) - \sigma| > \mu_0,\ \forall\, \sigma \in \Sigma. \tag{4.6}$$

Set $\mathrm{E}_\varepsilon(r) = \mathrm{E}_\varepsilon(u, \mathbb{B}(x_1, r))$. In view of monotonicity, we have, for any $\varepsilon \leq r_1 \leq \frac{r}{4}$

$$r_1^{-(N-1)}\mathrm{E}_\varepsilon(r_1) \leq \left(\frac{r}{4}\right)^{-(N-1)}\mathrm{E}_\varepsilon\left(\frac{r}{4}\right) \leq r^{-(N-1)}4^{N-1}\mathrm{E}_\varepsilon(r) \leq 4^{N-1}\eta_0.$$

Take $r_1 = \varepsilon$. This yields

$$\varepsilon^{-N}\int_{B(x_1,\varepsilon)} V(u(x))\,\mathrm{d}x \leq 4^{N-1}\eta_0. \tag{4.7}$$

It follows from (4.6) that $V(u(x_1)) \geq \beta_0 > 0$, where β_0 is some constant. Invoking the bound $|\nabla u| \leq K/\varepsilon$, we obtain $|V(u(x)) - V(u(x_1))| \leq C\gamma K$, for $x \in \mathbb{B}(x_1, \gamma\varepsilon)$. If $0 < \gamma < 1$ is sufficiently small, then $V(u(x)) \geq \frac{\beta_0}{2}$, for $x \in \mathbb{B}(x_1, \gamma\varepsilon)$. Integrating, we are led to

$$\int_{B(x_1,\gamma\varepsilon)} V(u(x))\mathrm{d}x \leq C\frac{\beta_0}{2}\gamma^N\varepsilon^N,$$

Combining with(4.7), we are led to $C\frac{\beta_0}{2}\gamma^N \leq 4^{N-1}\eta_0$, a contradiction if η_0 is chosen sufficiently small. □

By elliptic estimates, we are finally led to a clearing out result for the energy:

Lemma 2 ([10])**.** *$\exists\, \eta_0 > 0$ such that, if for $r \geq \varepsilon$, then*

$$\frac{1}{r^{N-1}}\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{B}^N(x_0, r)\right) \leq \eta_0 \implies \mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{B}(x_0, \frac{r}{2})) \leq C\exp\left(-\frac{\sqrt{\lambda_1}r}{32\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

4.4. Limiting measures

For $\varepsilon > 0$, we define the associated energy measure $\nu_\varepsilon = e_\varepsilon(u_\varepsilon)\mathrm{d}x$ so that $\nu_\varepsilon(\overline{\Omega}) \leq M_0$. By compactness, we have, up to a *subsequence* $\varepsilon_n \to 0$, in the sense of measures

$$\begin{cases} \nu_{\varepsilon_n} \rightharpoonup \nu_\star \text{ on } \Omega \text{ as } n \to +\infty, \\ \varepsilon_n u_{\varepsilon_n x_i} \cdot u_{\varepsilon_n x_j} \rightharpoonup \mu_{\star,i,j} \text{ as } n \to +\infty, \\ \zeta_{\varepsilon_n} \rightharpoonup \zeta_\star, \text{ as } n \to +\infty, \text{ where } \zeta_\varepsilon = \varepsilon^{-1}V(u_\varepsilon)\mathrm{d}x. \end{cases} \tag{4.8}$$

We will see next how our previous results translate to the limiting measure $\nu_\star$. Passing to the limit in the *integrated form* of the monotonicity formula, we obtain

Proposition 1 (monotonicity for the measure)**.** *Assume $k = 1$ and let $0 < r_1 < r_2$. We have the $(n-1)$ monotonicty for $\nu_\star$*

$$r_1^{-N+1}\nu_\star\big(\overline{\mathbb{B}(r_1)}\big) \leq r_2^{-N+1}\nu_\star\big(\overline{B(r_2)}\big). \tag{4.9}$$

This property implies that the measure *concentrates on sets* of dimension at least $(N-1)$. Similarily, passing to the limit $\varepsilon \to 0$ in Lemma 2, we deduce that the measure $\nu_\star$ verifies the following $(N-1)$ *clearing-out property*

Proposition 2. *Let $x_0 \in \Omega$ and $r > 0$ be given such that $\mathbb{B}(x_0, r) \subset \Omega$. There exists a constant $\eta_0 > 0$ such that, if we have*

$$\frac{\nu_\star\big(\overline{\mathbb{B}(x_0,r)}\big)}{r^{N-1}} < \eta_0, \text{ then it holds } \nu_\star\left(\overline{\mathbb{B}\Big(x_0, \frac{r}{2}\Big)}\right) = 0. \tag{4.10}$$

This property implies concentration on sets of dimension $\leq N-1$. We are hence led to consider the $(N-1)$ -dimensional lower density of the measure $\nu_\star$ defined, for $x_0 \in \Omega$, by

$$\oplus_\star(x_0) = \liminf_{r\to 0}\frac{\nu_\star\big(\overline{\mathbb{B}(x_0,r)}\big)}{r^{N-1}} \text{ and } \oplus^\star(x_0) = \limsup_{r\to 0}\frac{\nu_\star\big(\overline{\mathbb{B}(x_0,r)}\big)}{r^{N-1}}. \tag{4.11}$$

It follows from the monotonicity formula (4.9) that

$$0 \leq \oplus_\star(x_0) = \oplus^\star(x_0) \leq r_0^{-N+1}\nu_\star\big(\overline{\mathbb{B}(x_0, r_0)}\big), \text{ where } r_0 = \mathrm{dist}(x_0, \partial\Omega). \tag{4.12}$$

We define the set $\mathfrak{S}_\star$ as the *concentration set* for the measure $\nu_\star$. More precisely, we set

$$\mathfrak{S}_\star = \{x \in \Omega, \oplus_\star(x_0) \geq \eta_0\}, \tag{4.13}$$

where $\eta_0 > 0$ is the constant of the clearing-out. From its definition, the set $\mathfrak{S}_\star$ is closed, and by a standard covering argument, we obtain thanks to Proposition 2 that, for some constant $C_\mathrm{H} > 0$,

$$\mathcal{H}^{N-1}(\mathfrak{S}_\star) \leq C_\mathrm{H}M_0.$$

Relations (4.12) together with the clearing-out lemma combined with a deep result by D. Preiss [16], allows to show that $\mathfrak{S}_\star$ is rectifiable. Recall that rectifiability implies that $\mathfrak{S}_\star$ there exists an approximate tangent hypersurface $T_x\mathfrak{S}_\star$ for $\mathcal{H}^{N-1}$ almost every $x \in \mathfrak{S}_\star$. The measure $\nu_\star$ is *absolutely continuous with respect to* $\mathrm{d}\lambda$, the $\mathcal{H}^{N-1}$ measure on $\mathfrak{S}_\star$, so that one may write $\nu_\star = \oplus_\star\,\mathrm{d}\lambda$, where

$\mathbb{e}_\star$ is locally bounded in view of (4.12), hence an integrable function on $\mathfrak{S}_\star$. Going to the limit in the discrepancy, one may also prove a limiting discrepancy relation, quite analogous to the 1D case

$$2\zeta_\star \geq \sum_{i=0}^{k} \mu_{\star,i,i}. \tag{4.14}$$

4.5. $\mathfrak{S}_\star$ is a weak-minimal surface

Recall that a submanifold M of dimension m is said to be minimal if its a critical point of *the area functional* $\mathcal{A}(m) = \int_M \mathrm{d}\mathcal{H}^m$. The related Euler Lagrange equation writes $\vec{H}_M(x) = 0$, for any $x \in M$, where $\vec{H}_M(x)$ denotes the mean-curvature vector. This can be recast in an integral form as

$$\int_M \mathrm{div}_{T_x M} \vec{X} \mathrm{d}\mathcal{H}^m = 0, \quad \forall\, \vec{X} \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N). \tag{4.15}$$

an identity which makes sense for rectifiable sets. A related relation for $\mathfrak{S}_\star$ is derived using the stress-energy tensor $\alpha_{ij,\varepsilon} = e_\varepsilon(u_\varepsilon)\delta_{i,j} - \frac{\partial u_\varepsilon}{\partial x_i} \cdot \frac{\partial u_\varepsilon}{\partial x_j}$ which satisfies, thanks to $(\mathbf{VAC}_\varepsilon)$

$$\int_\Omega \alpha_{ij,\varepsilon} \frac{\partial Xj}{\partial x_i} = 0, \quad \forall\, \vec{X} \in C_c^\infty(\Omega, \mathbb{R}^n),$$

and which expresses the fact that the energy of the solution u_ε is stationary with respect to variations of the domain. Setting $A_\varepsilon = \left(\mathrm{Id} - \frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{e_\varepsilon(u_\varepsilon)}\right)$, and passing to the limit $\varepsilon_n \to 0$, we define a limiting matrix-valued measure $A_\star$ satisfying the relation

$$\int_\Omega A_{i,j,\star} \frac{\partial X^j}{\partial x_i} \mathrm{d}\nu_\star = 0, \quad \forall\, \vec{X} \in C_c^\infty(\Omega, \mathbb{R}^N) \tag{4.16}$$

Notice that $A_\star$ symmetric, $A_\star \leq \mathrm{Id}$ and $\mathrm{Tr}(A_\star) = N - 1$. Using these facts, one may show that

$$\int_{\mathfrak{S}_\star} \mathrm{div}_{T_x \mathfrak{S}_\star} \vec{X} \mathbb{e}_\star \mathrm{d}\mathcal{H}^{N-1} = 0, \tag{4.17}$$

which is, up to the presence of the density $\mathbb{e}_\star$, similar to (4.15). A couple $(\mathfrak{S}_\star, \mathbb{e}_\star)$ verifiying relation (4.17) is called a stationary varifold.

4.6. Summary of methods in the scalar Allen–Cahn case

Thanks to the sign of the discrepancy, the chain of arguments in the scalar case goes as follows: sign of discrepancy $\implies$ monotonicity $\implies$ clearing-out, whereas

$$\begin{cases} \text{clearing out + monotonicity} \implies \text{concentration on } N-1 \text{ dimensional sets} \\ \text{monotonicity} \implies \text{(Preiss) rectifiability of concentration set} \end{cases}$$

$$\text{and} \quad \text{sign of discrepancy + stress-energy tensor}$$
$$\Downarrow$$
$$\text{stationary sets or motion by mean-curvature}$$

Conclusion: Sign of discrepancy is crucial!

Turning to the vectorial Allen–Cahn equation, the main difficulty is that the *discrepancy function has no sign*, in general, as the existence of periodic *pseudo-profiles* shows. Hence, the chain of argument used in the scalar case is broken, in particular there is no monotonicity formula for solutions to $(\mathbf{VAC}_\varepsilon)$. New ideas are therefore required!

5. On the proofs in the vectorial case

5.1. Clearing-out

A central part of the argument and the starting point is to establish the "clearing-out" statement remains true.

Theorem 7. *The statement of Proposition 2 remains true for $N = 2$ and $k \geq 2$. Consequently, the set $\mathfrak{S}_\star$ defined in* (4.13) *is closed and such that $\mathcal{H}^1(\mathfrak{S}_\star) \leq \mathrm{C_H}\,\mathrm{M}_0$.*

As already mentioned, whereas the clearing-out result is a rather direct consequence, in the scalar case, of the monotonicity formula, the proof of Theorem 7 has to circumvent the absence of monotonicity formula at the level of the PDE. It relies merely on elliptic PDE methods, some of which are quite specific to dimension two.

5.2. Rectifiability

Proposition 3. *The set $\mathfrak{S}_\star$ is rectifiable.*

The argument for establishing the rectifiability of $\mathfrak{S}_\star$ in Proposition 3 is specific to 1-dimensional sets, and relies on the following property: a compact connected set of dimension 1 is rectifiable.

5.3. Stationarity and the Hopf Differential

In dimension two, we may recast the stress-energy tensor as the complex-valued measure referred to as the *limiting Hopf differential*

$$\omega_\star = (\mu_{\star,1,1} - \mu_{\star,2,2}) - 2i\mu_{\star,1,2}, \tag{5.1}$$

where the measures $\mu_{\star,i,j}$ have been defined in (4.8). Notice that, since the measures $\mu_{\star,i,j}$ depend on the choice of orthonormal basis, the expression of the Hopf differential also strongly depends on this choice. The measures $\zeta_\star$ and $\omega_\star$ are related in view as follows.

Lemma 3. *We have, in the sense of distributions,*

$$\frac{\partial \omega_\star}{\partial \bar{z}} = 2\frac{\partial \zeta_\star}{\partial z} \quad \textit{in } \mathcal{D}'(\Omega). \tag{5.2}$$

Relation (5.2) is equivalent to the stress-energy tensor identity (4.16), and the two-dimensional analog of the conservation law (2.2) for the ordinary differential equation.

At this stage, we only know that the measures are supported by $\mathfrak{S}_\star$, but we do not know that they are absolutely with respect to $\mathcal{H}^1$ measure. Hence we need to decompose the measures into absolutely continuous parts with respect to $\mathrm{d}\lambda = \mathcal{H}^1 \llcorner \mathfrak{S}_\star$ and singular parts

$$\mu_{\star,i,j} = \mu^s_{\star,i,j} + \mu^{ac}_{\star,i,j} \text{ with } \mu^s_{\star,i,j} \perp \mu^{ac}_{\star,i,j} \text{ and } \zeta_\star = \zeta^s_\star + \zeta^{ac}_\star, \text{ with } \zeta^s_\star \perp \zeta^{ac}_\star, \tag{5.3}$$

where the measures $\mu^{ac}_{\star,i,j}$ and $\zeta^{ac}_\star$ are absolutely continuous with respect to the measure $\mathrm{d}\lambda = \mathcal{H}^1 \llcorner \mathfrak{S}_\star$. Relations (5.3) imply that there exists a set $\mathfrak{B}_\star \subset \mathfrak{S}_\star$ such that $\mathcal{H}^1 \llcorner \mathfrak{S}_\star(\mathfrak{B}_\star) = 0$ and

$$\nu^s_\star(\mathfrak{S}_\star \setminus \mathfrak{B}_\star) = 0,\ \zeta^s_\star(\mathfrak{S}_\star \setminus \mathfrak{B}_\star) = 0, \text{ and } \mu^s_{\star,i,j}(\mathfrak{S}_\star \setminus \mathfrak{B}_\star) = 0, \text{ for } i,j = 1,2. \tag{5.4}$$

Since, by construction, the measures $\zeta^{ac}_\star$, $\nu^{ac}_\star$ and $\mu^{ac}_{\star,i,j}$ are absolutely continuous with respect to $\mathrm{d}\lambda$, there exist functions $\Theta_\star$, $\mathbb{e}_\star$ and $\mathbb{m}_{\star,i,j}$ defined on $\mathfrak{S}_\star$, such that we have

$$\zeta^{ac}_\star = \Theta_\star\,\mathrm{d}\lambda, \nu^{ac}_\star = \mathbb{e}_\star\,\mathrm{d}\lambda, \text{ and } \mu^{ac}_{\star,i,j} = \mathbb{m}_{\star,i,j}\mathrm{d}\lambda, \tag{5.5}$$

Let $\mathfrak{A}_\star$ denote the set of points of $\mathfrak{S}_\star$ having no approximate tangent line. We introduce a *third class of exceptional points*, the set $\mathfrak{C}_\star$, defined as the complementary of the set of Lebesgue points for the densities of the measures $\mu^{ac}_\star, \zeta^{ac}_\star, \mu^{ac}_{\star,i,j}$ with respect to $\mathrm{d}\lambda = \mathcal{H}^1 \llcorner \mathfrak{S}_\star$. The set $\mathfrak{S}_\star \setminus \mathfrak{C}_\star$, then corresponds to the intersection of the set of Lebesgue's points of the functions $\Theta_\star, \mathbb{e}_\star$ and $\mathbb{m}_{\star,i,j}$. We consider the union of all exceptional points

$$\mathfrak{E}_\star = \mathfrak{A}_\star \cup \mathfrak{B}_\star \cup \mathfrak{C}_\star, \tag{5.6}$$

which is precisely the set appearing in Theorem 4. We first describe properties of the absolutely continuous part.

Proposition 4. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$. Assume that the orthonormal frame $(\mathbf{e}_1, \mathbf{e}_2)$ is chosen so that $\mathbf{e}_1 = \vec{e}_{x_0}$. We have the identities, for the functions $\Theta_\star$, $\mathbb{m}_{\star,i,j}$ defined in* (5.5)*,*

$$\begin{cases} 2\Theta_\star(x_0) = \mathbb{m}_{\star,2,2}(x_0) - \mathbb{m}_{\star,1,1}(x_0) \quad \textit{and} \\ \mathbb{m}_{\star,1,2}(x_0) = 0. \end{cases} \tag{5.7}$$

Remark 3. Let $\omega_\star^{ac} = \left(\mu_{\star,1,1}^{ac} - \mu_{\star,2,2}^{ac}\right) - 2i\mu_{\star,1,2}^{ac}$ denote the absolutely continuous part of $\omega_\star$ with respect to $\mathrm{d}\lambda$. As mentioned, the expression of $\omega_\star^{ac}$ depends on the basis. For a given orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$, we have the identity

$$\omega_\star^{ac} = -2\exp(-2i\gamma_\star)\zeta_\star^{ac} = -2(\cos 2\gamma_\star - i\sin 2\gamma_\star)\zeta_\star^{ac}, \tag{5.8}$$

where $\gamma_\star(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denotes, for $x \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, the angle between $\mathbf{e}_1$ and $\vec{e}_{x_0}$.

5.4. Monotonicity for $\zeta_\star$ and its consequences

The next important step in the proof of the theorems is to show that the singular part of all measures introduced so far vanish. We first establish this statement for the measure $\zeta_\star$. Our argument involves a new ingredient, the monotonicity formula for $\zeta_\star$, which actually directly yields the absolute continuity of $\zeta_\star$ with respect to $\mathcal{H}^1 \llcorner \mathfrak{S}_\star$.

Proposition 5. *Let $x_0 \in \Omega$, let $\rho > 0$ be such that $\mathbb{D}^2(x_0, \rho) \subset \Omega$. If $0 < r_0 \le r_1 \le \rho$, then we have the inequality*

$$\frac{\zeta_\star(\mathbb{D}^2(x_0, r_1))}{r_1} \ge \frac{\zeta_\star(\mathbb{D}^2(x_0, r_0))}{r_0}. \tag{5.9}$$

For every $x_0 \in \Omega$ the quantity $\frac{\zeta_\star(\mathbb{D}^2(x_0,r))}{r}$ has a limit when $r \to 0$ and we have the estimate

$$\Theta_\star(x_0) = \lim_{r\to 0} \frac{\zeta_\star(\mathbb{D}^2(x_0, r))}{r} \le \frac{\zeta_\star(\mathbb{D}^2(x_0, \rho))}{\rho} \le \frac{M_0}{d(x_0, \partial\Omega)}. \tag{5.10}$$

The measure $\zeta_\star$ is hence absolutely continuous with respect to the $\mathcal{H}^1$-measure on $\mathfrak{S}_\star$.

The starting point of the proof of Proposition 5 is a monotonicity formula for solutions u_ε to $(\mathbf{VAC}_\varepsilon)$ involving the potential $V(u_\varepsilon)$, for $0 < r_0 < r_1 \le \rho$ such that $\mathbb{D}^2(x_0, \rho) \subset \Omega$

$$\frac{\zeta_\varepsilon(\mathbb{D}^2(x_0, r_1))}{r_1} - \frac{\zeta_\varepsilon(\mathbb{D}^2(x_0, r_0))}{r_0} = \int_{\mathbb{D}^2(x_0,r_1)\setminus\mathbb{D}^2(x_0,r_0)} \frac{1}{4r} \mathrm{d}\mathscr{N}_{x_0,\varepsilon}, \tag{5.11}$$

with $r = |x - x_0|$, and where we have set

$$\mathscr{N}_{x_0,\varepsilon} = \left(\frac{2}{\varepsilon} V(u_\varepsilon) - \varepsilon r^{-2} \left|\frac{\partial u_\varepsilon}{\partial \theta}\right|^2 + \varepsilon \left|\frac{\partial u_\varepsilon}{\partial r}\right|^2\right) \mathrm{d}x.$$

Here (r, θ) denote radial coordinates, so that $x_1 - x_{0,1} = r\cos\theta$ and $x_2 - x_{0,2} = r\sin\theta$. Passing to the limit $\varepsilon \to 0$ in identity (5.11), we obtain:

Lemma 4. *Let $x_0 \in \Omega$, let $\rho > 0$ and assume that $\mathbb{D}^2(x_0, \rho) \subset \Omega$. For almost every radii $0 < r_0 < r_1 \le \rho$, we have the identity*

$$\frac{\zeta_\star(\mathbb{D}^2(x_0, r_1))}{r_1} - \frac{\zeta_\star(\mathbb{D}^2(x_0, r_0))}{r_0} = \int_{\mathbb{D}^2(x_0,r_1)\setminus\mathbb{D}^2(x_0,r_0)} \frac{1}{4r} \mathrm{d}\mathscr{N}_{x_0,\star} \tag{5.12}$$

where $\mathscr{N}_{x_0,\star} = 2\zeta_\star - r^{-2}\mu_{\star,\theta,\theta} + \mu_{\star,r,r}$, with

$$\begin{cases} \mu_{\star,r,r} = \cos^2\theta\mu_{\star,1,1} + \sin^2\theta\mu_{\star,2,2} + 2\sin\theta\cos\theta\mu_{\star,1,2} & \textit{and} \\ r^{-2}\mu_{\star,\theta,\theta} = \sin^2\theta\mu_{\star,1,1} + \cos^2\theta\mu_{\star,2,2} - 2\sin\theta\cos\theta\mu_{\star,1,2}. \end{cases} \tag{5.13}$$

The next step in the proof of Proposition 5 is the fact that, as a consequence of Proposition 4, the absolutely continuous part of $\mathcal{N}_\star$ is non-negative. We then perform a few manipulations which allow to get rid of the singular part in (5.9), and lead to the completion of the proof of Proposition 5.

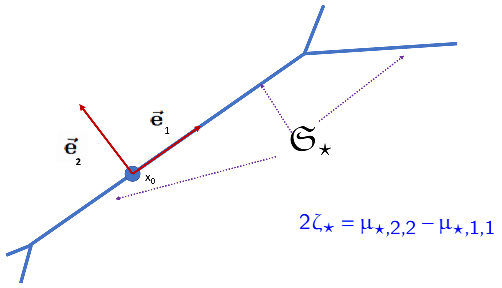
In order to prove that $\nu_\star$ is also absolutely continuous with respect to $\mathrm{d}\lambda$, we will invoke the fact that $\nu_\star$ is "dominated" by the measure $\zeta_\star$. Whereas the reverse statement is straightforward, since we have the inequality $\zeta_\star \le \nu_\star$, the fact that $\nu_\star$ is "dominated" by the measure $\zeta_\star$ is a consequence of several estimates at the ε-level, requiring some PDE analysis.

5.5. A new discrepancy relation

In view of the previous results, we may now write Proposition 4 as:

Proposition 6. *Let x_0 be a regular point of $\in \mathfrak{S}_\star$. Assume that the orthonormal frame $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ is choosen so that $\vec{\mathbf{e}}_1 = \vec{e}_{x_0}$. We have locally near x_0 the identity*

$$2\zeta_\star = \mu_{\star,2,2} - \mu_{\star,1,1} \text{ and } \mu_{\star,1,2}(x_0) = 0.$$



Remark 4. In the scalar case, one obtains $\mu_{\star,1,1} = \mu_{\star,1,2} = 0$ so that $\omega_\star = \mu_{\star,2,2} = 2\zeta_\star$.

References

[1] S. Alama, L. Bronsard & C. Gui, "Stationary layered solutions in $\mathbb{R}^2$ for an Allen–Cahn system with multiple well potential", *Calc. Var. Partial Differ. Equ.* **5** (1997), no. 4, p. 359-390.

[2] N. D. Alikakos, S. I. Betelú & X. Chen, "Explicit stationary solutions in multiple well dynamics and non-uniqueness of interfacial energy densities", *Eur. J. Appl. Math.* **17** (2006), no. 5, p. 525-556.

[3] W. K. Allard & F. J. Almgren, Jr, "The structure of stationary one dimensional varifolds with positive density", *Invent. Math.* **34** (1976), p. 83-97.

[4] S. Baldo, "Minimal interface criterion for phase transitions in mixtures of Cahn- Hilliard fluids", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **7** (1990), no. 2, p. 67-90.

[5] F. Bethuel, "Asymptotics for two-dimensional vectorial Allen–Cahn systems", to appear in *Acta Math.*, 2020, `https://arxiv.org/abs/2003.10189`.

[6] F. Bethuel & R. Oliver-Bonafoux, "Pseudo-profiles for vectorial Allen–Cahn systems", work in progress, 2023.

[7] L. Bronsard, C. Gui & M. Schatzman, "A three-layered minimizer in $\mathbb{R}^2$ for a variational problem with a symmetric three-well potential", *Commun. Pure Appl. Math.* **49** (1996), no. 7, p. 677-715.

[8] I. Fonseca & L. Tartar, "The gradient theory of phase transitions for systems with two potential wells", *Proc. R. Soc. Edinb., Sect. A, Math.* **111** (1989), no. 1-2, p. 89-102.

[9] J. E. Hutchinson & Y. Tonegawa, "Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory.", *Calc. Var. Partial Differ. Equ.* **10** (2000), no. 1, p. 49-84.

[10] T. Ilmanen, "Convergence of the Allen–Cahn equation to Brakke's motion by mean curvature", *J. Differ. Geom.* **38** (1993), no. 2, p. 417-461.

[11] L. Modica, “A gradient bound and a Liouville theorem for nonlinear Poisson equations”, *Commun. Pure Appl. Math.* **38** (1985), p. 679-684.

[12] L. Modica & S. Mortola, “Un esempio di Γ^--convergenza”, *Boll. Unione Mat. Ital., V. Ser., B* **14** (1977), p. 285-299.

[13] A. Monteil & F. Santambrogio, “Metric methods for heteroclinic connections”, *Math. Methods Appl. Sci.* **41** (2018), no. 3, p. 1019-1024.

[14] R. Oliver-Bonafoux, “Non-minimizing connecting orbits for multi-well systems”, *Calc. Var. Partial Differ. Equ.* **61** (2022), no. 2, p. 27, Id/No 69.

[15] F. Pacard & J. Wei, “Stable solutions of the Allen–Cahn equation in dimension 8 and minimal cones”, *J. Funct. Anal.* **264** (2013), no. 5, p. 1131-1167.

[16] D. Preiss, “Geometry of measures in R^n : Distribution, rectifiability, and densities”, *Ann. Math.* **125** (1987), p. 537-643.

[17] P. Sternberg & W. P. Zeimer, “Local minimisers of a three-phase partition problem with triple junctions”, *Proc. R. Soc. Edinb., Sect. A, Math.* **124** (1994), no. 6, p. 1059-1073.

[18] A. Zuniga & P. Sternberg, “On the heteroclinic connection problem for multi-well gradient systems”, *J. Differ. Equations* **261** (2016), no. 7, p. 3987-4007.

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