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Abstract

The focusing cubic NLS is a canonical model for the propagation of laser beams. In dimensions 2 and 3, it is known that a large class of initial data leads to finite time blow-up. Now, physical experiments suggest that this blow-up does not always occur. This might be explained by the fact that some physical phenomena neglected by the standard NLS model become relevant at large intensities of the beam. We derive from Maxwell's equations some known variants of NLS and propose some new ones, providing rigorous error estimates for all the models considered. These notes result from the work [9], in collaboration with D. Lannes and J. Szeftel.

1. Introduction

The cubic, focusing, nonlinear Schrödinger equation in space dimension d, given by

$$i\partial_\tau v + \Delta v + |v|^2 v = 0, \quad \tau > 0, \quad x \in \mathbb{R}^d, \tag{1.1}$$

is a canonical model for the propagation of laser beams. It is locally well-posed in $H^1 = H^1(\mathbb{R}^d)$ for $d = 1, 2, 3$ (cf. [10]): when $v_0 \in H^1$, there exists $0 < T \leq +\infty$ and a unique solution $v \in \mathcal{C}([0,T), H^1)$ to (1.1) with $v(0) = v_0$. Then, either $T = +\infty$, and the solution is global, or $T < +\infty$, and the solution blows up in finite time, i.e. $\lim_{\tau \uparrow T} \|\nabla v(t)\|_{L^2} = +\infty$.

The NLS equation (1.1) also admits the following (formal) conservation laws:

$$\begin{aligned} L^2-\text{norm:} \quad & \|v(\tau)\|_{L^2}^2 = \|v_0\|_{L^2}^2; \\ \text{Energy:} \quad & E(v(\tau)) = \frac{1}{2}\int |\nabla v(\tau,x)|^2 \mathrm{d}x - \frac{1}{4}\int |v(\tau,x)|^4 \mathrm{d}x = E(v_0); \\ \text{Momentum:} \quad & \mathrm{Im}\left(\int \nabla v(\tau,x)\overline{v(\tau,x)}\mathrm{d}x\right) = \mathrm{Im}\left(\int \nabla v_0(x)\overline{v_0(x)}\mathrm{d}x\right). \end{aligned}$$

Moreover, a large group of symmetries leaves the equation invariant: if v solves (1.1), then for all $(\lambda_0, \tau_0, x_0, \beta_0, \gamma_0) \in \mathbb{R}_*^+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$, so does

$$(\tau, x) \mapsto u(\tau, x) = \lambda_0 v(\lambda_0^2 \tau + \tau_0, \lambda_0 x + x_0 - \beta_0 t) e^{i\frac{\beta_0}{2}\cdot(x - \frac{\beta_0}{2}\tau)} e^{i\gamma_0}. \tag{1.2}$$

The scaling symmetry $u(\tau, x) = \lambda_0 v(\lambda_0^2 \tau, \lambda_0 x)$ leaves the homogeneous Sobolev space $\dot{H}^{s_c}(\mathbb{R}^d)$ invariant, where $s_c = \frac{d}{2} - 1$.

Referring to conservation of the L^2 norm by the flow, (1.1) is said to be L^2-subcritical if $s_c < 0$, L^2-critical if $s_c = 0$ and L^2-supercritical if $s_c > 0$. Thus, (1.1) is L^2-subcritical if $d = 1$, L^2-critical if $d = 2$, and L^2-supercritical if $d \geq 3$. In the subcritical case, global existence (in $\mathcal{C}([0,\infty), L^2)$) holds for arbitrarily large data in L^2. It turns out that in this case, global existence (in $\mathcal{C}([0,\infty), H^1)$) also holds for arbitrarily large data in H^1, due to the conservation of mass and energy. In the critical and supercritical cases however, there exist stable finite time blow-up dynamics. This has been known since the 60ies using global obstructive arguments based on the virial identity (see e.g. [18]).

There is however a discrepancy between the blow-up results predicted by (1.1) and physical observations. Indeed, while the blow-up signifies a break-down of the solution v, physical observations show in many cases that lasers begin to focus according to the scenarios associated to (1.1) but depart from this behavior slightly before the focusing time. The reason advanced by physicists is that some physical phenomena that have been neglected to derive (1.1) become relevant at high intensities, and therefore near focusing. This phenomenon is called *filamentation*: defocusing physical phenomena are triggered at high intensities and halt the collapse of the beam. This interplay between diffraction, self-focusing, and defocusing mechanisms allow for the beam to propagate along several times the focusing distance and the resulting structure is called *filament*.

Many variants of (1.1) have been derived in optics to take into account these additional physical phenomena and reproduce the filamentation mechanism. In many cases, it is a mathematical open problem to prove whether these additional terms prevent focusing or not, and a fortiori to understand the modification of the dynamics induced by them.

Rather than adding as usual ad hoc modifications to (1.1) in order to take new physical effects into account, we have chosen to rigorously derive such modifications from Maxwell's equations. We indicate the advantage of each model, as well as some of the most physically relevant open mathematical problems that these modified equations raise and that are natural milestones towards the understanding of filamentation.

These notes are structured as follows: in Section 2, we give an abstract setting adapted to the high frequency framework for Maxwell's equations. In Section 3, we derive successively various asymptotic models. Finally, in Section 4, we explain how to get a more complete model, taking into account ionization of the medium in which light propagates.

2. Maxwell's equations and abstract formulation

Maxwell's equations. The Maxwell equations in a non magnetizable medium are a set of two equations coupling the evolution of the *magnetic field* $\mathtt{B}$ to the *electric induction* $\mathtt{D}$,

$$\begin{cases} \partial_t \mathtt{B} + \mathrm{curl}\mathtt{E} = 0, \\ \partial_t \mathtt{D} - \frac{1}{\mu_0}\mathrm{curl}\mathtt{B} = 0, \end{cases} \tag{2.1}$$

where $\mathtt{D}$ is given in terms of the *electric field* $\mathtt{E}$ and the *polarization* $\mathtt{P}$ — modeling the way the dipole moment per unit volume depends on the strength of the electric field — by the relation

$$\mathtt{D} = \epsilon_0 \mathtt{E} + \mathtt{P}, \tag{2.2}$$

and where the positive constants ϵ_0 and μ_0 denote the *electric permittivity* and *magnetic permeability* in vacuum. The evolution equations (2.1) go along with two constitutive laws,

$$\nabla \cdot \mathtt{D} = 0, \qquad \nabla \cdot \mathtt{B} = 0. \tag{2.3}$$

Remark 2.1.

(i) In the description above, we assume absence of electric charge and current density in the medium. These will be added to the model in order to take into account ionization phenomena in Section 4.

(ii) In the sequel, we shall consider the Cauchy problem associated with (2.1)–(2.3). Since the constitutive laws (2.3) are propagated by the evolution equations if they are initially satisfied, we shall omitt them.

Polarization. There exist various ways to describe the polarization $\mathtt{P}$. We use here a simple and natural model called "nonlinear anharmonic oscillator", according to which the polarization is found by solving the second order ODE

$$\partial_t^2 \mathtt{P} + \Omega_1 \partial_t \mathtt{P} + \Omega_0^2 \mathtt{P} - \nabla V_{NL}(\mathtt{P}) = \epsilon_0 b \mathtt{E}, \tag{2.4}$$

where $b \in \mathbb{R}$ is a coupling constant and $\Omega_0, \Omega_1 > 0$ are frequencies, and where V_{NL} accounts for nonlinear effects. When such effects are neglected, the description (2.4) goes back to Lorentz [16] and expresses the fact that electrons are bound to the nucleus by a spring force. Nonlinearities

have been added to this description by Bloembergen [5] and Owyyoung [17] and the mathematical investigation of their influence was initiated by Donnat, Joly, Métivier and Rauch [8, 11] (see also [14]).

Remark 2.2. In order for the electromagnetic field to interact with matter, the frequency of the field (i.e. of the laser beam) has to be tuned so that it is of the same order as the electronic eigenfrequency Ω_0 in (2.4).

Typical examples for V_{NL} are

(i) cubic nonlinearity:

$$V_{NL}(\mathtt{P}) = \frac{a_3}{4}|\mathtt{P}|^4, \quad \text{and therefore} \quad \nabla V_{NL}(\mathtt{P}) = a_3|\mathtt{P}|^2\mathtt{P};$$

(ii) cubic/quintic nonlinearity:

$$V_{NL}(\mathtt{P}) = \frac{a_3}{4}|\mathtt{P}|^4 - \frac{a_5}{6}|\mathtt{P}|^6, \quad \text{and therefore} \quad \nabla V_{NL}(\mathtt{P}) = a_3|\mathtt{P}|^2\mathtt{P} - a_5|\mathtt{P}|^4\mathtt{P};$$

(iii) saturated nonlinearity: there exists a function $\mathcal{V}_{sat} : \mathbb{R}^+ \to \mathbb{R}$, with $\mathcal{V}'_{sat}$ and $s \mapsto \mathcal{V}''_{sat}(s)s$ bounded on $\mathbb{R}^+$ and such that

$$V_{NL}(\mathtt{P}) = \frac{1}{2}\mathcal{V}_{sat}(|\mathtt{P}|^2), \quad \text{and therefore} \quad \nabla V_{NL}(\mathtt{P}) = \mathcal{V}'_{sat}(|\mathtt{P}|^2)\mathtt{P};$$

for instance, one can take

$$\mathcal{V}_{sat}(r) = \frac{a_3}{2}\frac{r^2}{1 + \frac{2a_5}{3a_3}r},$$

in which case $\nabla V_{NL}(\mathtt{P}) = a_3|\mathtt{P}|^2\mathtt{P} - a_5|\mathtt{P}|^4\mathtt{P} + h.o.t$, and is thus the same at the origin as in (ii) above, up to higher order terms (seventh order terms here).

Remark 2.3. It is easy to show that, replacing the nonlinearity $|v|^2v$ in NLS (1.1) by $\mathcal{V}(v)$, with $\mathcal{V}$ saturated as in (iii) above, one gets a global solution $v \in \mathcal{C}([0,\infty), L^2)$ for each initial data $v_0 \in L^2$. In the case of a cubic focusing / quintic defocusing nonlinearity (replacing in (1.1) $|v|^2v$ by $a_3|v|^2v - a_5|v|^4v$, with $a_3, a_5 > 0$), one also gets global solutions (in H^1; see [9] for the case $d = 2$, and [19] for the case $d = 3$). It is in fact conjectured (but not proved) that these solutions depend almost peridodically on time (see [9], Section 4.1.1, and references therein).

Scaling and abstract formulation. Laser experiments described here correspond to high-frequency regimes. This means that a small parameter $\varepsilon > 0$ naturally shows up: the ratio of the duration of an optical cycle (i.e. the inverse of the laser frequency) over the duration of the pulse (typically measured by the time the optical power is at least half-maximum). Introducing this small parameter, and after nondimensionalization (for details, see the Appendix of [9]), Maxwell's equations can be put under the following form for all the nonlinearities considered above,

$$\begin{cases} \partial_t\mathtt{B} + \mathrm{curl}\mathtt{E} = 0, \\ \partial_t\mathtt{E} - \mathrm{curl}\mathtt{B} + \frac{1}{\varepsilon}\sqrt{\gamma}\mathtt{Q}^\sharp = 0, \\ \partial_t\mathtt{Q}^\sharp + \varepsilon^{1+p}\omega_1\mathtt{Q}^\sharp - \frac{1}{\varepsilon}(\sqrt{\gamma}\mathtt{E} - \omega_0\mathtt{P}^\sharp) = \varepsilon\frac{\gamma}{\omega_0^3}\big(1 + f(\varepsilon^q|\mathtt{P}^\sharp|^2)\big)|\mathtt{P}^\sharp|^2\mathtt{P}^\sharp, \\ \partial_t\mathtt{P}^\sharp - \frac{1}{\varepsilon}\omega_0\mathtt{Q}^\sharp = 0, \end{cases} \tag{2.5}$$

where γ, ω_0, ω_1, p and q are positive constants, while f is a smooth function vanishing at the origin. These equations have the form

$$\partial_t\mathbf{U} + A(\partial)\mathbf{U} + \frac{1}{\varepsilon}E\mathbf{U} + \varepsilon^{1+p}A_0\mathbf{U} = \varepsilon F(\varepsilon, \mathbf{U}), \tag{2.6}$$

where $\mathbf{U}$ is a $\mathbb{R}^n$ $(n \geq 1)$ valued function depending on the time variable t and the space variable $x \in \mathbb{R}^d$ $(d \geq 1)$,

$$\mathbf{U} : \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^n.$$

The operator $A(\partial)$ is defined as

$$A(\partial) = \sum_{j=1}^{d} A_j \partial_j,$$

where ∂_j is the differentiation operator with respect to the j-th space coordinate. The matrix A_0 has size $n \times n$, and p is a positive number. The following assumption is made on the matrices A_j and E, and on the nonlinearity F.

Assumption 2.4.

(i) *The matrices A_j ($j = 1, \dots, d$) are constant coefficient $n \times n$, real valued,* symmetric *matrices.*

(ii) *The matrix E is a constant coefficient $n \times n$, real valued,* skew symmetric *matrix.*

(iii) *There exists a smooth mapping $f : \mathbb{R}^+ \to \mathbb{R}$ vanishing at the origin, a real number $q > 0$, a quadratic form $Q : \mathbb{C}^n \mapsto \mathbb{R}^+$ and a trilinear symmetric mapping $T : (\mathbb{C}^n)^3 \to \mathbb{C}^n$ (with $T(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \subset \mathbb{R}^n$) such that*

$$\forall\, U \in \mathbb{C}^n, \qquad F(\varepsilon, U) = \big(1 + f\big(\varepsilon^q Q(U)\big)\big)\, T(U, \bar{U}, U).$$

Remark 2.5. There exist of course situations where the leading order of the nonlinearity is not cubic (it can be quadratic for non centro-symmetric crystals for instance) or not of this form; since we are interested here in deriving variants of the standard cubic nonlinear Schrödinger equation, we restrict ourselves to this framework for the sake of simplicity.

The Cauchy problem and the profile equation. We shall consider initial conditions corresponding to laser pulses, i.e. fast oscillating *wave packets* slowly modulated by an envelope. The evolution equation (2.6) being of semilinear nature, it is natural to work with some Banach algebra in view of a resolution by Picard iterations. Thus, we assume

$$\mathbf{U}_{|_{t=0}} = u^0(x) e^{i\frac{\mathbf{k}\cdot x}{\varepsilon}} + \text{c.c.}, \quad u^0 \in B = H^{t_0}(\mathbb{R}^d)^n, \tag{2.7}$$

where $\mathbf{k} \in \mathbb{R}^d$ is the (spatial) *wave-number* of the oscillations.

The space $H^{t_0}(\mathbb{R}^d)^n$ is stable by translations in Fourier space, so that with $u^0 \in B$ in (2.7) we get $\mathbf{U}_{|_{t=0}} \in B$. Furthermore, when $t_0 > d/2$, $B = H^{t_0}(\mathbb{R}^d)^n$ is a Banach algebra in the sense that

$$\forall\, f, g \in B, \qquad f \cdot g \in B \quad \text{and} \quad |f \cdot g|_B \lesssim |f|_B |g|_B,$$

and by Moser's inequality, the mapping F from Assumption 2.4 acts on B and is locally Lipschitz: uniformly with respect to $\varepsilon \in (0, 1]$,

$$\forall\, f \in B, \qquad F(f) \in B \quad \text{and} \quad |F(\varepsilon, f)|_B \le C(|f|_B)|f|_B,$$
$$\forall\, f, g \in B, \qquad |d_f F(\varepsilon, \cdot) g|_B \le C(|f|_B)|g|_B.$$

For all $k \in \mathbb{N}$, we also define

$$B^{(k)} = \{f \in B, \quad \forall\, \alpha \in \mathbb{N}^d, \quad \forall\, |\alpha| \le k, \quad \partial^\alpha f \in B\} = H^{t_0+k}(\mathbb{R}^d)^n.$$

Remark 2.6. Another possible choice for the Banach algebra B is the so-called Wiener algebra $W(\mathbb{R}^d)^n := \{f \in \mathcal{S}'(\mathbb{R}^d)^n, |f|_B := |\widehat{f}|_{L^1} < \infty\}$, which is better adapted than $H^{t_0}(\mathbb{R}^d)^n$ to handle short pulses, see [6, 15]. In this case, the Lipschitz property holds for analytic nonlinearities F (see [12, Theorem 8.6]).

Due to Assumption 2.4, the Cauchy problem for the (semilinear, symmetric hyperbolic) system (2.6), with initial data (2.7) and $t_0 > d/2$, has a unique local-in-time solution. However, the size of the data in H^s is of the order ε^{-s}, and the existence time of the corresponding solution may shrink to zero as ε goes to zero. We shall use a non-singular representation of the initial data and solution to circumvent this difficulty and make a first step towards an asymptotic description of the solution. Precisely, we seek a solution $\mathbf{U}$ to the initial value problem (2.6)–(2.7) given by means of a *profile* U,

$$\mathbf{U}(t, x) = U\left(t, x, \frac{\mathbf{k}\cdot x - \omega t}{\varepsilon}\right), \tag{2.8}$$

with $U(t,x,\theta)$ periodic with respect to θ and for any $\omega \in \mathbb{R}$. Then $\mathbf{U}$ solves (2.6)-(2.7) provided that U solves the Cauchy problem associated with the *profile equation*

$$\begin{cases} \partial_t U + A(\partial)U + \frac{i}{\varepsilon}\mathcal{L}(\omega D_\theta, \mathbf{k}D_\theta)U + \varepsilon^{1+p}A_0 U = \varepsilon F(\varepsilon, U), \\ U_{|_{t=0}}(x,\theta) = u^0(x)e^{i\theta} + \text{c.c.}. \end{cases} \tag{2.9}$$

Here, we used the notation

$$\mathcal{L}(\omega D_\theta, \mathbf{k}D_\theta) = -\omega D_\theta + A(\mathbf{k})D_\theta + \frac{E}{i}, \tag{2.10}$$

with $D_\theta = -i\partial_\theta$ and $A(\mathbf{k}) = \sum_{j=1}^d A_j k_j$.

Thanks to the Lipschitz property of the nonlinearity and its size ε, a standard iteration procedure and fixed-point argument provides a solution $U \in C([0,T/\varepsilon]; H^k(\mathbb{T};B))$ to (2.6)-(2.7) for all $k \geq 1$, where

$$H^k(\mathbb{T};B) = \left\{ f = \sum_{n\in\mathbb{Z}} f_n e^{in\theta}, \ \ |f|_{H^k(\mathbb{T},B)} < \infty \right\} \tag{2.11}$$

and with $|f|^2_{H^k(\mathbb{T},B)} = \sum_{n\in\mathbb{Z}}(1+n^2)^k |f_n|^2_B$. For $k \geq 1$, $H^k(\mathbb{T},B)$ is a Banach algebra, on which the evolution operator $S(t)$ associated to the linear part of (2.9),

$$S(t) = \exp\left(-tA(\partial) - \frac{i}{\varepsilon} t\mathcal{L}(\omega D_\theta, kD_\theta) \right),$$

is unitary, thanks to point (i) and (ii) of Assumption 2.4. Hence, we get

Theorem 2.7. *Let $t_0 > d/2$. Under Assumption 2.4, there exists $T > 0$ such that for all $0 < \varepsilon \leq 1$ there exists a unique solution $\mathbf{U} \in C([0,T/\varepsilon]; B)$ to* (2.6)-(2.7)*. Moreover, for any given $\omega \in \mathbb{R}$, one can write $\mathbf{U}$ under the form*

$$\mathbf{U}(t,x) = U\left(t, x, \frac{\mathbf{k}\cdot x - \omega t}{\varepsilon}\right),$$

where U solves the profile equation (2.9)*.*

Note that the above argument provides a bound

$$\sup_{0\leq t\leq T/\varepsilon} |U(t)|_B \leq C(T, |u^0|_B),$$

which induces an L^∞ bound, since $t_0 > d/2$. But Theorem 2.7 does not give information about the propagation of the initial oscillations. In the sequel, we investigate this point.

3. Asymptotic models

Here, we derive various approximations to the solutions provided by Theorem 2.7, filterring oscillations, then getting simpler (scalar, then local) equations, and also refining the description to capture extreme behaviors.

3.1. The slowly varying envelope approximation

The slowly varying envelope approximation (SVEA) consists in writing the profile U under the form

$$U(t,x,\theta) \sim u_{env}(t,x)e^{i\theta} + \text{c.c.} \tag{3.1}$$

with an *envelope* u_{env} possibly depending on ε, but not through fast oscillations, which will be ensured by the boundedness of ∇u_{env}.

Plugging this approximation into the profile equation (2.9) and keeping only the first harmonic in the Fourier expansion yields easily (writing $u = u_{env}$),

$$\partial_t u + A(\partial)u + \frac{i}{\varepsilon}\mathcal{L}(\omega, \mathbf{k})u + \varepsilon^{1+p}A_0 u = \varepsilon F^{env}(\varepsilon, u),$$

where

$$F^{env}(\varepsilon,u) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta} F(\varepsilon, ue^{i\theta} + \text{c.c.})\, \mathrm{d}\theta. \tag{3.2}$$

Example 3.1. With $F(u) = |u|^2 u$, one gets $F^{env}(u) = (u \cdot u)\overline{u} + 2|u|^2 u$.

Denoting $D = -i\nabla$, we observe that

$$\begin{aligned} A(\partial) + \frac{i}{\varepsilon}\mathcal{L}(\omega, \mathbf{k}) &= A(\partial) + \frac{i}{\varepsilon}(-\omega \,\mathrm{Id} + A(\mathbf{k})) \\ &= \frac{i}{\varepsilon}(-\omega \,\mathrm{Id} + A(\mathbf{k} + \varepsilon D)) \\ &:= \frac{i}{\varepsilon}\mathcal{L}(\omega, \mathbf{k} + \varepsilon D), \end{aligned}$$

where the last notation is of course consistent with (2.10).

As a consequence of these computations, we see that in order for (3.1) to hold, it is necessary that $u = u_{env}$ satisfies the Cauchy problem associated with the *envelope equation*

$$\begin{cases} \partial_t u + \frac{i}{\varepsilon}\mathcal{L}(\omega, \mathbf{k} + \varepsilon D)u + \varepsilon^{1+p} A_0 u = \varepsilon F^{env}(\varepsilon, u), \\ u_{|_{t=0}} = u^0. \end{cases} \tag{3.3}$$

Now, following the same lines as in the proof of Theorem 2.7, we are able to construct solution u_{env} to this problem which is bounded in $B = H^{t_0}(\mathbb{R}^d)^n$. Taking $t_0 > d/2 + 1$, we get the boundedness of ∇u_{env}. However, the approximation (3.1) may fail because of the following mechanisms:

- The singular part of the linear term in (3.3) creates fast oscillations with frequencies $\omega - \omega_j(\mathbf{k})$, where the $\omega_j(\mathbf{k})$ stand for the eigenvalues of $\mathcal{L}(0, \mathbf{k})$.
- The nonlinearity creates other oscillations that may resonate with the linear propagator.

There is one way to avoid the catastrophic effects of these two scenarios. Choosing $\omega = \omega_j(\mathbf{k})$ for some j and assuming that, up to $O(\varepsilon)$ terms, the initial envelope u^0 is contained in the corresponding eigenspace prevents the creation of oscillations by the linear propagator. This is the *polarization condition*. The nonlinearity will however create harmonics of the main oscillation $\mathbf{k}\cdot x - \omega_j(\mathbf{k})t$ and it is necessary to make a non resonance assumption. What is called *characteristic variety* in the assumption below is the set $\mathcal{C}_{\mathcal{L}} \subset \mathbb{R}^{d+1}$ defined as

$$\mathcal{C}_{\mathcal{L}} = \{(\omega', \mathbf{k}') \in \mathbb{R}^{1+d}, \;\; \det \mathcal{L}(\omega', \mathbf{k}') = 0\}.$$

Let us also recall that we assumed that the nonlinearity is under the form

$$F(\varepsilon, U) = \big(1 + f\big(\varepsilon^q Q(U)\big)\big)\, T(U, \bar{U}, U),$$

with $f(0) = 0$, Q a quadratic form and T a trilinear symmetric mapping. If U is a monochromatic oscillation, the nonlinearity $\varepsilon F(\varepsilon, U)$ creates third harmonic with size $O(\varepsilon)$, a fifth harmonic (if $f'(0) \neq 0$) with size $O(\varepsilon^{1+q})$, etc. The non-resonance condition stated below holds for the $(2k+3)$-th harmonics, for all $k \geq 0$ such that $kq < 1$ (the contribution of higher harmonics is small enough to be controlled even if it is resonant).

Assumption 3.2. *The characteristic variety $\mathcal{C}_{\mathcal{L}}$ and the frequency/wave number couple $(\omega, \mathbf{k})$ satisfy:*

(i) *There exist m functions $\omega_j \in C^\infty(\mathbb{R}^d \backslash \{0\})$ $(j = 1, \dots, m)$ such that*

$$\mathcal{C}_{\mathcal{L}} \backslash \{0\} = \bigcup_{j=1}^{m} \big\{(\omega_j(\mathbf{k}'), \mathbf{k}'), \;\; \mathbf{k}' \in \mathbb{R}^d \backslash \{0\}\big\};$$

up to a renumbering, we assume that $(\omega, \mathbf{k}) = (\omega_1(\mathbf{k}), \mathbf{k})$.

(ii) *There exists a constant $c_0 > 0$ such that*

$$\inf_{\mathbf{k}' \in \mathbb{R}^d} |\omega - \omega_j(\mathbf{k}')| \geq c_0, \qquad j = 2, \dots m.$$

(iii) *(Non resonance assumption) One has $\pm(2k+3)(\omega, \mathbf{k}) \notin \mathcal{C}_{\mathcal{L}}$, for all $k \geq 0$ such that $kq < 1$, with q from Assumption 2.4.*

Notation 3.3. We denote by $\pi_j(\mathbf{k})$ $(j = 1, \dots, m)$ the eigenprojectors of the eigenvalues $\omega_j(\mathbf{k})$ of $A(\mathbf{k}) + \frac{1}{i}E$; in particular, we have

$$\mathcal{L}(0, \mathbf{k}) = A(\mathbf{k}) + \frac{1}{i}E = \sum_{j=1}^{m} \omega_j(\mathbf{k})\pi_j(\mathbf{k}).$$

Example 3.4. For Maxwell's equations, it is shown in Appendix B of [9] that Assumption 3.2 is satisfied with $m = 7$ in dimension $d = 3$, for $\omega \neq 0$. Explicit expressions for the eigenprojectors $\pi_j(\mathbf{k})$ are also provided in this appendix.

Theorem 3.5. *Let $t_0 > d/2$, and $u^0, r \in B^{(1)}$. Let Assumptions 2.4 and 3.2 be satisfied and assume that $u^0 = \pi_1(\mathbf{k})u^0 + \varepsilon r$. Then*

(1) *There exist $T > 0$ and, for all $\varepsilon \in (0, 1]$, a unique solution $u \in C([0, T/\varepsilon]; B^{(1)})$ to (3.3) with initial condition u^0, and*

$$\sup_{0 \leq t \leq T/\varepsilon} |u(t)|_{B^{(1)}} \leq C(T, |u^0|_{B^{(1)}}).$$

(2) *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution $\mathbf{U}$ to (2.6) provided by Theorem 2.7 exists on $[0, T/\varepsilon]$ and*

$$|\mathbf{U} - \mathbf{U}_{SVEA}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leq \varepsilon C(T, |u^0|_B)(1 + |\nabla u^0|_B),$$

where $\mathbf{U}_{SVEA}(t, x) = u(t, x)e^{i\frac{\mathbf{k}\cdot x - \omega t}{\varepsilon}} + c.c.$.

Main steps of the proof.

Step 1. Existence and bounds of the solution u to (3.3) are established by a fixed point argument as in Theorem 2.7.

Step 2. We decompose u as

$$u = u_1 + u_{II}, \quad \text{with} \quad u_{II} = \sum_{j=2}^{m} u_j,$$

and where $u_j = \pi_j(\mathbf{k} + \varepsilon D)u$ (see Notation 3.3).

Step 3. Thanks to the assumption that $\omega = \omega_1(\mathbf{k})$ one gets from the equation obtained by applying $\pi_1(\mathbf{k} + \varepsilon D)$ to (3.3) that $|\partial_t u_1(t)|_B$ is uniformly bounded on $[0, T/\varepsilon]$ (with a bound depending on $|\nabla u^0|_B$; for details, see [6, Lemma 2]).

Step 4. Using a non-stationary phase argument (on the semigroup formulation) relying on point (ii) of Assumption 3.2 and the bound on $\partial_t u_1$ established in Step 3, and taking advantage of the Lipschitz property of the nonlinearity, we get that $\frac{1}{\varepsilon}|u_{II}(t)|_B$ remains uniformly bounded on $[0, T/\varepsilon]$.

Step 5. Using the non-resonance condition (iii) of Assumption 3.2, one can show that the third and higher harmonics created by the nonlinearity remain of order $O(\varepsilon)$. More precisely, the solution $U \in H^1(\mathbb{T}; B)$ to (2.9) provided by Theorem 2.7 can be written as

$$U(t, x, \theta) = U_{\text{app}}(t, x, \theta) + \varepsilon V(t, x, \theta),$$

where $U_{\text{app}}(t, x, \theta) = u(t, x)e^{i\theta} + \text{c.c.}$, and V remains bounded (with respect to ε) in $C([0, T/\varepsilon]; H^1(\mathbb{T}; B)^n)$.

Step 6. Since $U(t) - U_{\text{app}}(t) = \varepsilon V(t)$, it follows from the above that

$$\sup_{t \in [0,T/\varepsilon]} |U(t) - U_{\text{app}}(t)|_{H^1(\mathbb{T};B)} \leq \varepsilon C(T, |u^0|_B)(1 + |\nabla u^0|_B + |r|_B),$$

and the theorem follows therefore from the observation that

$$|\mathbf{U} - \mathbf{U}_{SVEA}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leq \sup_{t \in [0,T/\varepsilon]} |U(t) - U_{\text{app}}(t)|_{H^1(\mathbb{T};B)}. \qquad \square$$

3.2. The Full Dispersion model

As a by-product of the proof of Theorem 3.5, we infer an approximation with the same precision, *via* the solution of an equation with a *scalar* transport part: setting

$$u_{FD} = \pi_1(\mathbf{k} + \varepsilon D)u_{env},$$

which is nothing but the u_1 term from the previous subsection, we have the same result as Theorem 3.5, replacing $\mathbf{U}_{SVEA}$ by $\mathbf{U}_{FD}$, where

$$\mathbf{U}_{FD}(t,x) = u_{FD}(t,x)e^{i\frac{\mathbf{k}\cdot x-\omega t}{\varepsilon}} + \text{c.c.},$$

and $u = u_{FD}$ solves the Cauchy problem associated with the *full dispersion equation*,

$$\begin{cases} \partial_t u + \frac{i}{\varepsilon}(\omega_1(\mathbf{k}+\varepsilon D)-\omega)u + \varepsilon^{1+p}\pi_1(\mathbf{k}+\varepsilon D)A_0 u = \varepsilon\pi_1(\mathbf{k}+\varepsilon D)F^{env}(\varepsilon,u), \\ u\,|_{t=0}(x) = u^0(x). \end{cases} \tag{3.4}$$

This "full dispersion" model is related to the so-called "unidirectional pulse propagation equation" used in nonlinear optics [4, 13].

3.3. The nonlinear Schrödinger (NLS) equation

Another way of simplifying the equation defining the envelope of the approximate solution consists in approximating the pseudodifferential operators $\omega_1(\mathbf{k}+\varepsilon D)$ and $\pi_1(\mathbf{k}+\varepsilon D)$ involved in (3.4) by *differential* operators: using Taylor expansions,

$$\frac{i}{\varepsilon}(\omega_1(\mathbf{k}+\varepsilon D)-\omega) = \mathbf{c}_g\cdot\nabla - \frac{i}{2}\nabla\cdot H_{\mathbf{k}}\nabla + O(\varepsilon^2),$$
$$\pi_1(\mathbf{k}+\varepsilon D) = \pi_1(\mathbf{k}) + O(\varepsilon),$$

where $\mathbf{c}_g = \nabla\omega_1(\mathbf{k})$ and $H_{\mathbf{k}}$ stands for the Hessian of ω_1 at $\mathbf{k}$. Neglecting the $O(\varepsilon^2)$ terms in (3.4) we define the NLS approximation $u = u_{NLS}$ as the solution to

$$\begin{cases} \partial_t u + \mathbf{c}_g\cdot\nabla u - \varepsilon\frac{i}{2}\nabla\cdot H_{\mathbf{k}}\nabla u + \varepsilon^{1+p}\pi_1(\mathbf{k})A_0 u = \varepsilon\pi_1(\mathbf{k})F^{env}(\varepsilon,u). \\ u\,|_{t=0}(x) = u^0(x) \end{cases} \tag{3.5}$$

We then get easily (see [6, 15]) the following justification of the NLS approximation.

Corollary 3.6 (Schrödinger approximation)**.** *Under the assumptions of Theorem 3.5, one has for all $u^0 \in B^{(3)}$ such that $u_0 = \pi_1(\mathbf{k})u_0$,*

(1) *There exists $T > 0$ and, for all $\varepsilon \in (0,1]$, a unique solution $u \in C([0,T/\varepsilon];B^{(3)})$ to (3.5) with initial condition u^0.*

(2) *There exists $\varepsilon_0 > 0$ and $\mathfrak{c}_{NLS} > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution $\mathbf{U}$ to (2.6) provided by Theorem 2.7 exists on $[0,T/\varepsilon]$ and*

$$|\mathbf{U} - \mathbf{U}_{NLS}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \le \varepsilon C(T,|u^0|_B)(1 + |\nabla u^0|_B + \mathfrak{c}_{NLS}|u^0|_{B^{(3)}}),$$

where $\mathbf{U}_{NLS}(t,x) = u(t,x)e^{i\frac{\mathbf{k}\cdot x-\omega t}{\varepsilon}} + c.c.$

Remark 3.7. We assumed here that the polarization of the initial condition is exact (i.e. $r = 0$ in Theorem 3.5) for the sake of simplicity; indeed, the solution remains polarized along $\pi_1(\mathbf{k})$ for all times and computations on real physical models are much easier.

Example 3.8. In the frequent case where $\omega_1(\,\cdot\,)$ has a radial symmetry, and writing with a slight abuse of notation $\omega_1(\mathbf{k}) = \omega_1(k)$, with $k = |\mathbf{k}|$, (3.5) boils down to

$$\partial_t u + \omega_1'(k)\partial_z u - \varepsilon\frac{i}{2}\frac{\omega_1'(k)}{k}\Delta_\perp u - i\frac{\varepsilon}{2}\omega_1''(k)\partial_z^2 u + \varepsilon^{1+p}\pi_1(\mathbf{k})A_0 u = \varepsilon\pi_1(\mathbf{k})F^{env}(\varepsilon,u),$$

with $(0z)$ the direction of the wave number $\mathbf{k}$ and $\Delta_\perp = \partial_x^2 + \partial_y^2$ the Laplace operator in transverse variables.

3.4. The NLS equation with improved dispersion relation

A more accurate model. The term $\mathfrak{c}_{NLS}|u^0|_{B^{(3)}}$ in the error estimate of Corollary 3.6 is due to the bad frequency behavior of the Schrödinger equation when the envelope of the oscillations ceases to be well localized in frequency. This is for instance the case for short pulses, chirped pulses ([6, 7, 15]), and near a focusing point. To describe such extreme situations, the standard NLS approximation does a poor job, and this is why various variants have been derived in physics (e.g. [3]).

The dispersion relation $\omega_1(\,\cdot\,)$ of (2.6) is approximated by the second order polynomial

$$\omega_{NLS}(\mathbf{k}') = \omega_1(\mathbf{k}) + \mathbf{c}_g \cdot (\mathbf{k}' - \mathbf{k}) + \frac{1}{2}(\mathbf{k}' - \mathbf{k}) \cdot H_{\mathbf{k}}(\mathbf{k}' - \mathbf{k}). \tag{3.6}$$

For Maxwell's equations and in dimension $d = 1$, Figure 3.1 shows that this dispersion relation is very poor when $\mathbf{k}'$ is not close to $\mathbf{k}$.

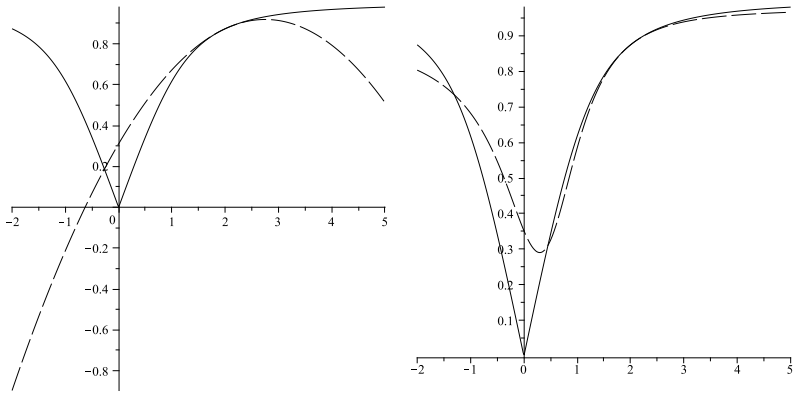

Figure 3.1: One component of the characteristic variety of Maxwell's equation (solid) and the dispersion relation corresponding to the Schrödinger approximation (left) and improved Schrödinger (right).

The idea introduced in [6] is to replace the *linear* part of the Schrödinger approximation by a linear operator that differs from the linear part of the Schrödinger approximation by $O(\varepsilon^2)$ terms only, but whose dispersion relation is far better.

More precisely, we consider an approximation under the form

$$\mathbf{U}_{imp}(t,x) = u_{imp}(t,x)e^{i\frac{\mathbf{k}\cdot x - \omega t}{\varepsilon}} + \text{c.c.}, \tag{3.7}$$

where $u = u_{imp}$ solves the Cauchy problem associated with the *nonlinear Schrödinger equation with improved dispersion relation*,

$$\begin{cases} \big(1 - i\varepsilon\mathbf{b}\cdot\nabla - \varepsilon^2\nabla\cdot B\nabla\big)\partial_t u \\ \qquad + \mathbf{c}_g\cdot\nabla u - \varepsilon\frac{i}{2}\nabla\cdot\big(H_{\mathbf{k}} + 2\mathbf{c}_g\otimes\mathbf{b}\big)\nabla u + \varepsilon^2 C_3(\nabla)u \\ \qquad + \varepsilon^{1+p}\pi_1(\mathbf{k})A_0 u = \varepsilon\pi_1(\mathbf{k})F^{env}(\varepsilon,u), \\ u\,|_{t=0}(x) = u^0(x), \end{cases} \tag{3.8}$$

where $\mathbf{b}\in\mathbb{C}^d$, $B\in\mathcal{M}_{d\times d}(\mathbb{R})$ and $C_3(\nabla)$ is a third order homogeneous differential operator. We assume moreover that

$$B \text{ is symmetric positive}, \quad \mathbf{b}\in \mathit{Range}(B), \quad \text{and} \quad 4 - \mathbf{b}\cdot(B^{-1}\mathbf{b}) > 0 \tag{3.9}$$

(note that even though $B^{-1}\mathbf{b}$ is not unique when B is not definite, the scalar $\mathbf{b}\cdot(B^{-1}\mathbf{b})$ is uniquely defined). These assumptions ensure that the operator $(1 - i\varepsilon\mathbf{b}\cdot\nabla - \varepsilon^2\nabla\cdot B\nabla)$ is invertible. This new model allows one to replace (3.6) by

$$\omega_{imp}(\mathbf{k}') = \omega_1(\mathbf{k}) + \frac{\mathbf{c}_g\cdot(\mathbf{k}'-\mathbf{k}) + \frac{1}{2}(\mathbf{k}-\mathbf{k}')\cdot(H_{\mathbf{k}} + 2\mathbf{c}_g\otimes\mathbf{b})(\mathbf{k}'-\mathbf{k}) - C_3(\mathbf{k}'-\mathbf{k})}{1 + \mathbf{b}\cdot(\mathbf{k}'-\mathbf{k}) + (\mathbf{k}'-\mathbf{k})\cdot B(\mathbf{k}'-\mathbf{k})}.$$

A good choice of $\mathbf{b}$, B and C_3 allows a much better approximation of $\omega_1(\,\cdot\,)$, as shown in Figure 3.1 for Maxwell's equation in dimension $d = 1$, where Padé approximant can be used.

Exactly as for Corollary 3.6 we get the following result, where the only difference in the error estimate with respect to Corollary 3.6 is the constant $\mathfrak{c}_{imp}$ (with is much smaller than $\mathfrak{c}_{NLS}$ for good choices of $\mathbf{b}$, B and C). We refer to [6] for the proof and numerical validations of this model for the approximation of short pulses and chirped pulses.

Corollary 3.9 (Schrödinger approximation with improved dispersion)**.** *Under the assumptions of Theorem 3.5, one has, for all $u^0 \in B^{(3)}$ such that $\pi_1(\mathbf{k})u_0 = u_0$,*

(1) *There exists $T > 0$ and, for all $\varepsilon \in (0, 1]$, a unique solution $u \in C([0, T/\varepsilon]; B^{(3)})$ to (3.8) with initial condition u^0.*

(2) *There exists $\varepsilon_0 > 0$ and $\mathfrak{c}_{imp} > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution $\mathbf{U}$ to (2.6) provided by Theorem 2.7 exists on $[0, T/\varepsilon]$ and*

$$|\mathbf{U} - \mathbf{U}_{imp}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leq \varepsilon C(T, |u^0|_B)(1 + |\nabla u^0|_B + \mathfrak{c}_{imp}|u^0|_{B^{(3)}}),$$

where $\mathbf{U}_{imp}(t, x) = u(t, x)e^{i\frac{\mathbf{k}\cdot x-\omega t}{\varepsilon}} + c.c.$.

Example 3.10. In the framework of Example 3.8, i.e. if $\omega_1(\mathbf{k}) = \omega_1(k)$ with $k = |\mathbf{k}|$ and $\mathbf{k} = k\mathbf{e}_z$, (3.8) can be written

$$(1 - i\varepsilon\mathbf{b}\cdot\nabla - \varepsilon^2\nabla\cdot B\nabla)\partial_t u + \omega_1'(k)\partial_z u - \varepsilon\frac{i}{2}\big(\frac{\omega_1'(k)}{k}\Delta_\perp + \omega_1''(k)\partial_z^2\big)u$$
$$- i\varepsilon\omega_1'(k)\mathbf{b}\cdot\nabla\partial_z u + \varepsilon^2 C_3(\nabla)u + \varepsilon^{1+p}\pi_1(\mathbf{k})A_0 = \varepsilon\pi_1(\mathbf{k})F^{env}(\varepsilon, u).$$

A promizing model in extreme situations. This more precise model concerning the dispersion relation may also be useful to describe the exact solution to Maxwell's equations in extreme cases, when the standard NLS model predicts blow-up of the approximate solution $\mathbf{U}_{NLS}$.

The nonlinear Schrödinger equation with improved dispersion relation has the typical form

$$iP_\varepsilon\partial_t u + \Delta u + |u|^{2\sigma}u = 0, \tag{3.10}$$

with $P_\varepsilon = 1 - \sum_{j=1}^k \partial_{x_j}^2$, taking into account the *off-axis variation of the group velocity* (or *space-time focusing*, see [4]). The case $k = d$ corresponds to *full off-axis variation*, whereas $k < d$ is *partial off-axis variation*. In [9], we proved that in the cubic case $\sigma = 1$ with full off-axis variation, in dimension $d = 2$ or 3, the Cauchy problem associated with (3.10) is globally well-posed in $H^1(\mathbb{R}^d)$; thus, taking off-axis variation into account may prevent from blow-up.

This result has been extended by Antonelli, Arbunich and Sparber in [1], using the unknown $v = P_\varepsilon^{1/2}u$, for which (3.10) reads

$$i\partial_t v + P_\varepsilon^{-1}\Delta v + P_\varepsilon^{-1/2}(|P_\varepsilon^{-1/2}v|^{2\sigma}P_\varepsilon^{-1/2}v) = 0.$$

Balancing regularization by P_ε^{-1} in the first k variables and dispersive properties of the propagator $S(t) = e^{itP_\varepsilon^{-1}\Delta}$ in the other directions, they obtain global existence of the solution in the case of full off-axis variation, in any dimension d, for $\sigma \leq 2/(d-2)_+$; in the case of partial off-axis variation, they show the following:

Theorem 3.11 ([1, Theorem 1.1])**.** *Let $d > k \geq 0$ and*

- *either $k \leq 2$ and $0 \leq \sigma < \frac{2}{d-k}$,*
- *or $k > 2$ and $0 \leq \sigma \leq \frac{2}{d-k}$.*

Then for any $u_0 \in L^2(\mathbb{R}^{d-k}_{x''}; H^1(\mathbb{R}^k_{x'}))$, there exists a unique global in-time solution $u \in C(\mathbb{R}_t; L^2(\mathbb{R}^{d-k}_{x''}; H^1(\mathbb{R}^k_{x'})))$ to (3.10) with $u(0) = u_0$. Furthermore, this solution depends continuously on the initial data and satisfies $\|P_\varepsilon^{1/2}u(t,\cdot)\|_{L^2} = \|P_\varepsilon^{1/2}u_0\|_{L^2}$ for all $t \in \mathbb{R}$.

Thus, they get global existence of the solution u to (3.10) even in cases $\sigma > 2/d$, which are supercritical for the standard NLS, as soon as partial off-axis variation ($k > 0$) occurs.

3.5. The NLS equation with frequency dependent polarization

Another possible modification of the NLS model (3.5) consists in improving the rough approximation $\pi_1(\mathbf{k}+\varepsilon D) \sim \pi_1(\mathbf{k}) + O(\varepsilon)$. Indeed, when the description of the envelope of the laser pulse requires a broad band of frequencies as in the situations mentioned in Section 3.4, the variations of the polarization term $\pi_1(\mathbf{k}+\varepsilon D)$ in front of the nonlinearity in (3.4) become important and should be taken into account. We therefore make the following approximation,

$$\pi_1(\mathbf{k}+\varepsilon D) \sim (1 - i\varepsilon \mathbf{b}\cdot\nabla - \varepsilon^2 \nabla\cdot B\nabla)^{-1}\big[\pi_1(\mathbf{k}) + \varepsilon\pi_1'(\mathbf{k})\cdot D - i\varepsilon(\mathbf{b}\cdot\nabla)\pi_1(\mathbf{k})\big],$$

where $\mathbf{b}$ and B are the same as in the NLS approximation with improved dispersion (3.8). In particular, if $\mathbf{b}=0$ and $B=0$ (standard NLS), then the above approximation coincides with the first order Taylor expansion

$$\pi_1(\mathbf{k}+\varepsilon D) = \pi_1(\mathbf{k}) + \varepsilon\pi_1'(\mathbf{k})\cdot D.$$

The general formula has the same accuracy as this Taylor expansion as $\varepsilon \to 0$. The corresponding approximation is given by

$$\mathbf{U}_{pol}(t,x) = u_{pol}(t,x)e^{i\frac{\mathbf{k}\cdot x - \omega t}{\varepsilon}} + \text{c.c.}, \tag{3.11}$$

where $u = u_{pol}$ solves the Cauchy problem associated with the *nonlinear Schrödinger equation with frequency dependent polarization*,

$$\begin{cases} \big(1 - i\varepsilon\mathbf{b}\cdot\nabla - \varepsilon^2\nabla\cdot B\nabla\big)\partial_t u \\ \quad + \mathbf{c}_g\cdot\nabla u - \varepsilon\frac{i}{2}\nabla\cdot\big(H_{\mathbf{k}} + 2\mathbf{c}_g\otimes\mathbf{b}\big)\nabla u + \varepsilon^2 C_3(\nabla)u + \varepsilon^{1+p}\pi_1(\mathbf{k})A_0 u \\ \quad = \varepsilon\big[\pi_1(\mathbf{k}) + \varepsilon\pi_1(\mathbf{k})\pi_1'(\mathbf{k})\cdot D - i\varepsilon(\mathbf{b}\cdot\nabla)\pi_1(\mathbf{k})\big]F^{env}(\varepsilon, \pi_1(\mathbf{k})u) \\ u\,|_{t=0}(x) = u^0(x), \end{cases} \tag{3.12}$$

where $\mathbf{b}$, B and $C_3(\nabla)$ are the same as in (3.4).

Contrary to all the previous models, the nonlinearity in (3.12) is a *derivative nonlinearity*. It turns out, as already mentioned, that the operator $\big(1 - i\varepsilon\mathbf{b}\cdot\nabla - \varepsilon^2\nabla\cdot B\nabla\big)$ in front of the time derivative plays a smoothing role allowing the control of one or several first order derivatives. If the first order derivatives involved in the nonlinearity are all controlled by this smoothing operator, then the nonlinearity remains semilinear in nature. If not, the nonlinearity is rather quasilinear; this is probably responsible for the *self-steepening* effect which may lead to "optical schocks" (see [4]).

In fact, the component $-i\varepsilon(\mathbf{b}\cdot\nabla)\pi_1(\mathbf{k})F^{env}(\varepsilon,\pi_1(\mathbf{k})u)$ of the nonlinearity is always semilinear in this sense. This is not the case for the component $\varepsilon\pi_1(\mathbf{k})\pi_1'(\mathbf{k})\cdot DF^{env}(\varepsilon,\pi_1(\mathbf{k})u)$ that may be of quasilinear nature, in which case a symmetry assumption (satisfied by Maxwell's equations (2.5)) is needed on the nonlinearity to ensure local well-posedness.

Assumption 3.12. *For all $v \in W^{1,\infty}(\mathbb{R}^d)^n$ and $u \in L^2(\mathbb{R}^d)^n$ such that $\pi_1(\mathbf{k})u = u$, one has*

$$\forall\ 1 \le j \le d, \qquad \Re\Big(\pi_1(\mathbf{k})\pi_1'(\mathbf{k})\cdot\mathbf{e}_j d_v F^{env} D_j u, u\Big) \le Cst|v|_{W^{1,\infty}}|u|_*^2,$$

where $\mathbf{e}_j$ is the unit vector in the j-th direction of $\mathbb{R}^d$, $d_v F^{env}$ is the derivative at v of the mapping $u \mapsto F^{env}(\varepsilon,u)$, and

$$|u|_*^2 = \big(u, (1 - i\varepsilon\mathbf{b}\cdot\nabla - \varepsilon^2\nabla\cdot B\nabla)u\big).$$

The approximation furnished by (3.12) is justified by the following corollary. The difference in the estimate with respect to Corollary 3.9 is just a better nonlinear constant, denoted C_{pol} to insist on this point. Also, the local well-posedness for a time scale of order $O(1/\varepsilon)$ does not follow the same lines as in the proof of Theorem 2.7 and must be established. We can show that such a local well-posedness result holds if $u_0 \in H^{s+1}(\mathbb{R}^d)^n$ with $s > t_0 + 1$, controlling a natural energy associated to (3.12), namely $E^s(u)$, given for all $s \ge 0$ by

$$E^s(u) = \frac{1}{2}\big((1 - i\varepsilon\mathbf{b}\cdot\nabla - \varepsilon^2\nabla\cdot B\nabla)\Lambda^s u, \Lambda^s u\big) = \frac{1}{2}|\Lambda^s u|_*^2,$$

where $\Lambda = (1+|D|^2)^{1/2}$. Under the assumptions (3.9) on B and $\mathbf{b}$, $E^s(u)^{1/2}$ defines a norm that controls uniformly the H^s-norm. Assumption 3.12 ensures that it also controls first order derivatives in the direction $\mathbf{b}\cdot\nabla$. This leads (see [9] for details) to

Corollary 3.13 (Schrödinger approximation with frequency dependent polarization)**.** *Let the assumptions of Theorem 3.5 be satisfied, as well as Assumption 3.12. Then for all $u_0 \in B^{(3)}$ such that $\pi_1(\mathbf{k})u_0 = u_0$, one has*

(1) *There exists $T > 0$ and, for all $\varepsilon \in (0,1]$, a unique solution $u \in C([0,T/\varepsilon]; B^{(3)})$ to (3.12) with initial condition u^0.*

(2) *There exists $\varepsilon_0 > 0$ and $\mathfrak{c}_{imp} > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution $\mathbf{U}$ to (2.6) provided by Theorem 2.7 exists on $[0,T/\varepsilon]$ and*

$$|\mathbf{U} - \mathbf{U}_{pol}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leq \varepsilon C_{pol}(T, |u^0|_B)(1 + |\nabla u^0|_B + \mathfrak{c}_{imp}|u^0|_{B^{(3)}}),$$

where $\mathbf{U}_{pol}(t,x) = u(t,x)e^{i\frac{\mathbf{k}\cdot x - \omega t}{\varepsilon}} + c.c.$.

Example 3.14. In the framework of Examples 3.8 and 3.10, we can check that $v(t,x,z) = u(\varepsilon t, x, z - \omega_1'(k)t)$ solves

$$(1 - i\varepsilon\mathbf{b}\cdot\nabla - \varepsilon^2\nabla\cdot B\nabla)\partial_t v - \frac{i}{2}\left(\frac{\omega_1'(k)}{k}\Delta_\perp + \omega_1''(k)\partial_z^2\right)v + \varepsilon^p\pi_1(\mathbf{k})A_0 v$$
$$= \left[\pi_1(\mathbf{k}) + \varepsilon\pi_1'(\mathbf{k})\cdot D - i\varepsilon(\mathbf{b}\cdot\nabla)\pi_1(\mathbf{k})\right]F^{env}(\varepsilon, \pi_1(\mathbf{k})v).$$

It is an open question to get a precise description of the abovementioned "optical shock" phenomenon. In [2], Arbunich, Klein and Sparber have addressed this question, in particular from the numerical point of view. The typical form of (3.12) is

$$iP_\varepsilon\partial_t u + \Delta u + (1 + i\delta\cdot\nabla)|u|^{2\sigma}u = 0, \tag{3.13}$$

with the same notations as in (3.10), and for some $\delta \in \mathbb{R}^d$. In [2], the authors show that in dimension $d = 2$, in the cubic case $\sigma = 1$, for partial off-axis variation of the group velocity ($k = 1$), when the derivative of the nonlinearity is parallel to the regularization (corresponding to Assumption 3.12), solutions to the Cauchy problem associated with (3.13) are global in time (Theorem 6.3). They present numerical evidences, in the case of derivative of the nonlinearity in the orthogonal direction, of blow-up in the absence of off-axis variation, whereas the L^∞-norm of the computed solution stabilizes when partial off-axis variations are present.

4. Ionization processes

The main mechanism at stake in laser filamentation is certainly the local ionization of the medium: once a powerful self-focusing laser beam reaches high enough intensities, it ionizes the medium around itself. It leaves behind a narrow channel of plasma, hereby causing local defocusing that prevents blowup. We propose here a description of this light-matter coupling, and exhibit the kind of asymptotic models deduced from the methods presented in the previous section.

Taking current density into account, Maxwell's equations (2.1) have to be replaced with

$$\begin{cases} \partial_t \mathtt{B} + \mathrm{curl}\mathtt{E} = 0, \\ \partial_t \mathtt{D} - \frac{1}{\mu_0}\mathrm{curl}\mathtt{B} = -\mathtt{J}, \end{cases} \tag{4.1}$$

where the current density $\mathtt{J}$ has the form

$$\mathtt{J} = \mathtt{J}_\mathrm{e} + \mathtt{J}_\mathrm{i}, \tag{4.2}$$

and $\mathtt{J}_\mathrm{e}$ and $\mathtt{J}_\mathrm{i}$ are respectively the free electron and ionization current densities.

Free electron current density. Partial ionization of the material medium by the laser generates free electrons, with charge $q_\mathrm{e}(= -1.6\times10^{-19}C)$. This induces a free electron current density $\mathtt{J}_\mathrm{e} = q_\mathrm{e}\rho_\mathrm{e}v_\mathrm{e}$, where ρ_e is the electron density, and v_e is the electron velocity. We propose a relation between the current density $\mathtt{J}_\mathrm{e}$ and the electric field $\mathtt{E}$ that fits with the usual slowly varying envelope approximation, but which may used in Maxwell's equations (4.1), namely

$$\mathtt{J}_\mathrm{e} = \frac{q_\mathrm{e}^2}{\omega_\mathrm{l} m_\mathrm{e}}\mathcal{H}\left(\frac{\mathbf{k}_\mathrm{l}}{k_\mathrm{l}^2}\cdot D\right)(\rho_\mathrm{e}\mathtt{E}),$$

where $m_{\rm e}$ is the electron mass, $\mathbf{k}_{\rm l}$ the laser wavevector, $k_{\rm l} = |\mathbf{k}_{\rm l}|$, and $\mathcal{H}$ is the regularization of the Hilbert transform given by the Fourier multiplier

$$\mathcal{H}(D_z) = \frac{\sqrt{2}\, iD_z}{(1+D_z^2)^{1/2}}. \tag{4.3}$$

Then, the evolution of the electron density $\rho_{\rm e}$ is given by a source term S representing external plasma sources. Taking into account photo-ionization and collisional ionization, but neglecting electron recombination (see for instance [3] for richer models), we have

$$S = W(I)(\rho_{\rm nt} - \rho_{\rm e}) + \frac{\sigma}{U_{\rm i}}\rho_{\rm e} I,$$

where the intensity is $I = |E|^2$ and $\rho_{\rm nt}$ is the constant density of neutral species. In the regime considered here, $\rho_{\rm e}$ is negligible compared to $\rho_{\rm nt}$ and the photo-ionization rate $W(I)$ takes the form

$$W(I) = \sigma_K I^{2K},$$

for some constant coefficient $\sigma_K > 0$ and with $K > 1$ the number of photons needed to liberate one electron. The collisional ionization cross-section σ depends on the laser frequency, and $U_{\rm i}$ is the ionization potential. Summing up, we get the following expression for the free electron current $\mathtt{J}_{\rm e}$ and $\rho = \rho_{\rm e}$,

$$\begin{cases} \mathtt{J}_{\rm e} = \frac{q_{\rm e}^2}{\omega_{\rm l} m_{\rm e}} \mathcal{H}\left(\frac{\mathbf{k}_{\rm l}}{k_{\rm l}^2}\cdot D\right)(\rho \mathtt{E}), \\ \partial_t \rho = \sigma_K \rho_{\rm nt} |\mathtt{E}|^{2K} + \frac{\sigma}{U_{\rm i}} \rho |\mathtt{E}|^2. \end{cases} \tag{4.4}$$

Ionization current density. It is also necessary to take into account losses due to photo-ionization. We introduce a ionization current density $\mathtt{J}_{\rm i}$ such that $\mathtt{J}_{\rm i}\cdot E$ represents the energy lost by the laser to extract electrons (per time and volume unit). More precisely, $\mathtt{J}_{\rm i}\cdot E$ is equal to the energy necessary to extract one electron (given by the ionization potential $U_{\rm i}$) multiplied by the number of electrons per time and volume unit (given by $\partial_t \rho$). Using the second equation of (4.4), this gives

$$\mathtt{J}_{\rm i}\cdot \mathtt{E} = U_{\rm i}\sigma_K \rho_{\rm nt} |\mathtt{E}|^{2K} + \sigma \rho |\mathtt{E}|^2.$$

Therefore, we take

$$\mathtt{J}_{\rm i} = \big(U_{\rm i}\sigma_K \rho_{\rm nt} |\mathtt{E}|^{2K-2} + \sigma\rho\big)\mathtt{E}. \tag{4.5}$$

After nondimensionalization, the set of equations (4.1)-(4.2)-(2.4)-(4.4)-(4.5) becomes

$$\begin{cases} \partial_t \mathtt{B} + \mathrm{curl}\mathtt{E} = 0, \\ \partial_t \mathtt{E} - \mathrm{curl}\mathtt{B} + \frac{1}{\varepsilon}\sqrt{\gamma}\mathtt{Q}^\sharp = -\varepsilon \mathcal{H}\big(\varepsilon \frac{\mathbf{k}}{k^2}\cdot D_x\big)(\rho\mathtt{E}) - \varepsilon c_0\big(c_1|\mathtt{E}|^{2K-2} + c_2\rho\big)\mathtt{E}, \\ \partial_t \mathtt{Q}^\sharp + \varepsilon^{1+p}\omega_1 \mathtt{Q}^\sharp - \frac{1}{\varepsilon}(\sqrt{\gamma}\mathtt{E} - \omega_0 \mathtt{P}^\sharp) = \varepsilon \frac{\gamma}{\omega_0^3}\big(1 + f(\varepsilon^q |\mathtt{P}^\sharp|^2)\big)|\mathtt{P}^\sharp|^2 \mathtt{P}^\sharp, \\ \partial_t \mathtt{P}^\sharp - \frac{1}{\varepsilon}\omega_0 \mathtt{Q}^\sharp = 0, \\ \partial_t \rho = \varepsilon c_1 |\mathtt{E}|^{2K} + \varepsilon c_2 \rho |\mathtt{E}|^2, \end{cases} \tag{4.6}$$

with the same notations as in (2.5).

It is then possible to derive a slowly varying envelope approximation and all the previous NLS approximations as in the non-ionized case. We simply indicate the general form of the equations extending (3.12) when taking ionization into account, with the same notations as in (3.10):

$$\begin{cases} iP_\varepsilon \partial_t u + \Delta u + \varepsilon(1 + i\delta\cdot\nabla)[(|u|^2 - \rho)u + ic(\alpha_1 |u|^{2K-2}u + \alpha_2 \rho u)] = 0, \\ \partial_t \rho = \varepsilon\alpha_1 |u|^{2K} + \varepsilon\alpha_2 \rho |u|^2. \end{cases} \tag{4.7}$$

For such a system, it is easy to prove existence of the approximate solution (say in H^s, for $s > d/2$) up to times $O(1/\varepsilon)$, but it would be desirable to identify coefficients yielding global solutions. This is still an open question.

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