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Wavelet transform modulus: phase retrieval and scattering

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Abstract

We discuss the problem that consists in reconstructing a function from the modulus of its wavelet transform. In the case where the wavelets are Cauchy wavelets, all analytic functions are uniquely determined by this modulus. Additionally, although it is not uniformly continuous, the reconstruction operator enjoys a form of local stability. We describe these two results, and give an idea of the proof of the first one. To conclude, we present a related result on a more sophisticated operator, based on the wavelet transform modulus: the scattering transform.

1. Introduction

The wavelet transform of a signal is a complex-valued function, computed from the signal by convolving it with a family of filters called "wavelets". Since its introduction in the late eighties, it has become an essential tool for various data analysis tasks. Indeed, the wavelet transform of a signal is often much easier to interpret than the signal itself. This is notably the case for audio signals, with the additional peculiarity, in this case, that, most of the time, only the modulus of the wavelet transform is used; the phase is discarded, because it does not seem to be useful.

The reason for this fact is that, apparently, the modulus alone accurately models human perception: it empirically seems that two audio signals have almost the same wavelet transform modulus if and only if they seem almost identical to a human listener [4, 10]. This is not true when the phase is kept: two signals that "sound" identical can have wavelet transforms with significantly different phases.

To give a more formal content to this observation, we propose to study the following problem:

To what extent is it possible to recover a function $f \in L^2(\mathbb{R})$ from its wavelet transform modulus?

More specifically, we ask the following two questions: is any function f uniquely determined from its wavelet transform modulus? Can we describe all pairs of functions (f, g) such that f and g have "almost" the same wavelet transform modulus?

This problem belongs to a family of inverse problems called phase retrieval problems, that consist in recovering an element in a complex vector space from the modulus of linear measurements. Phase retrieval problems are known to be difficult to study: given a specific linear measurement operator, it is in general not possible to determine whether any element can be uniquely recovered, or which pairs of elements yield measurements with very similar modulus. Apart from cases where the linear operator is random (with a simple distribution), the main situation that is relatively well understood is when the linear operator is the Fourier transform, and the function to be recovered is compactly supported. It is indeed known that, when $d = 1$, no compactly supported function in $L^2(\mathbb{R}^d)$ is uniquely determined from the modulus of its Fourier transform (except trivial ones) [1], and it is possible to describe all pairs of functions with the same Fourier transform modulus. When $d \geq 2$, almost all functions become uniquely determined, but not all [5]. A similar analysis can be done for the fractional Fourier transform [7].

In the case of the wavelet transform, for a specific family of wavelets ("Cauchy wavelets"), a precise analysis happens to be possible, and the result is quite different from the Fourier case. All (analytic) functions in $L^2(\mathbb{R})$ are uniquely determined by their wavelet transform modulus. Additionally, we can describe, at least partially, the pairs of functions whose wavelet transform modulus are very close.

The aim of this article is to present these results, then to discuss an operator related to, but more sophisticated than the wavelet transform: the scattering transform. The scattering transform is a cascade of wavelet transform modulus, organized in layers; since its introduction in [8], it has proven a valuable tool for difficult and diverse data analysis tasks. Ideally, we would like to extend to the scattering transform the analysis that we can do for the wavelet transform modulus. The result that we present in this article is only a first step towards this goal: it describes a relation between the decay of the scattering coefficients of a function along the layers, and the decay of the Fourier transform of the function. In particular, it shows that scattering coefficients of even moderately regular functions decay exponentially fast, thus suggesting that further analysis of the scattering transform should focus on the first layers only.

Section 2 contains the definition of the wavelet transform and Cauchy wavelets. In Section 3, we prove that two analytic functions with the same Cauchy wavelet transform modulus are (almost) equal. In Section 4, we give a description of pairs of functions whose wavelet transform modulus are very similar. The decay of scattering coefficients is discussed in Section 5.

This review article is based on [9] and [13].

2. Definitions

If f is a function in $L^1(\mathbb{R})$, we define its Fourier transform by

$$\forall \omega \in \mathbb{R}, \quad \hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} dt.$$

Let $L^2_+(\mathbb{R})$ be the set of square-integrable *analytic* functions:

$$L^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \text{ such that for a.e. } \omega \leq 0, \hat{f}(\omega) = 0\}.$$

2.1. Wavelets and wavelet transforms

An element ψ of $L^1(\mathbb{R}) \cap L^2_+(\mathbb{R})$ is a *wavelet* if

$$\int_{\mathbb{R}} \psi(t) dt = 0.$$

(The assumption that ψ is analytic is useless for most of the following definitions, but will simplify some.)

From a fixed wavelet ψ (the *mother wavelet*), we can define a whole family of wavelets $(\psi_j)_{j\in\mathbb{Z}}$:

$$\forall t \in \mathbb{R}, \quad \psi_j(t) = 2^{-j} \psi(2^{-j} t). \tag{2.1}$$

The *wavelet transform* associated to this family is the operator W defined as

$$\begin{array}{rccc} W: & L^2(\mathbb{R}) & \to & (L^2(\mathbb{R}))^{\mathbb{Z}} \\ & f & \to & (f \star \psi_j)_{j\in\mathbb{Z}}. \end{array}$$

It is a linear operator. Provided that the wavelets satisfy a so-called *Littlewood-Paley condition*, its restriction to $L^2_+(\mathbb{R})$ is continuous and invertible; its pseudo-inverse has an explicit expression,

$$\begin{array}{rccc} W^\dagger: & (L^2(\mathbb{R}))^{\mathbb{Z}} & \to & L^2_+(\mathbb{R}) \\ & (h_j)_{j\in\mathbb{Z}} & \to & \sum_j h_j \star \tilde{\psi}_j, \end{array}$$

where $(\tilde{\psi}_j)$ is the *dual wavelet family*:

$$\begin{array}{lccll} \forall j \in \mathbb{Z}, \forall \omega \in \mathbb{R},, & \hat{\tilde{\psi}}_j(\omega) & = & \frac{\overline{\hat{\psi}_j(\omega)}}{\sum_{k\in\mathbb{Z}} |\hat{\psi}_k(\omega)|^2} & \text{if } \omega \geq 0 \\ & & = & 0 & \text{if } \omega < 0. \end{array}$$

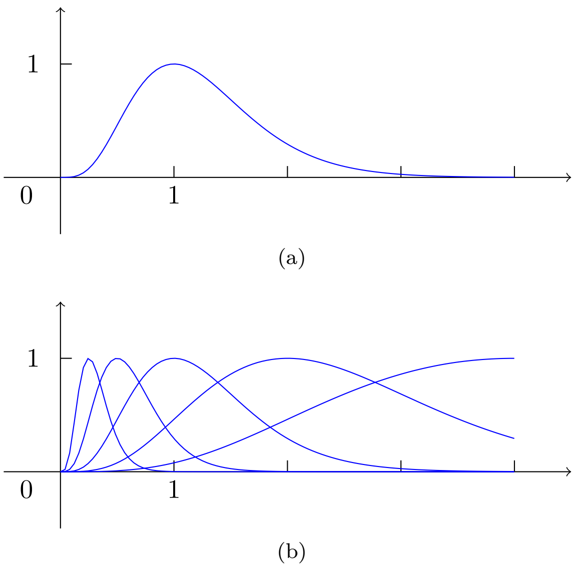

Figure 2.1. (A) Mother wavelet, in the Fourier domain (Cauchy wavelet, defined with $p = 4$). (B) Family of wavelets defined from this mother wavelet.

In this article, in addition to the wavelet transform itself, we will often consider its modulus,

$$\begin{array}{rccc} |W|: & L^2(\mathbb{R}) & \to & (L^2(\mathbb{R}))^{\mathbb{Z}} \\ & f & \to & (|f \star \psi_j|)_{j\in\mathbb{Z}}. \end{array}$$

In applications, the mother wavelet ψ is usually chosen so that its Fourier transform is well-localized, and reaches its value of maximal amplitude around 1 (see Figure 2.1a). The family of wavelets can then be understood as a family of band-pass filters (whose j-th filter has a characteristic frequency equal to 2^{-j}), and the wavelet transform as a decomposition in (overlapping) frequency bands.

2.2. Cauchy wavelets

The *Cauchy wavelet family* is defined by applying Equation (2.1) to the following mother wavelet:

$$\forall \omega \in \mathbb{R}, \qquad \begin{array}{rcll} \hat{\psi}^{\text{Cauchy}}(\omega) & = & \omega^p e^{-\omega} & \text{if } \omega \geq 0 \\ & = & 0 & \text{if } \omega < 0, \end{array}$$

where $p > 0$ can be any real parameter, that we assume fixed for the whole article.

We call *Cauchy wavelet transform* the wavelet transform associated to this family, and denote it by W_{Cauchy}. Its modulus is correspondingly denoted by $|W_{\text{Cauchy}}|$.

3. Uniqueness for Cauchy wavelets

In this section, we consider the following problem:

$$\text{Is any } f \in L^2(\mathbb{R}) \text{ uniquely determined from } |W_{\text{Cauchy}}|f = (|f \star \psi_j^{\text{Cauchy}}|)_{j\in\mathbb{Z}}?$$

The main result of this section is that any analytic function f is uniquely determined by its Cauchy wavelet transform modulus, up to multiplication by a complex unitary number. We state it in Paragraph 3.1, and give an idea of its proof in Paragraph 3.2.

3.1. Uniqueness theorem

We first note that, for any function $f \in L^2(\mathbb{R})$ and any $\phi \in \mathbb{R}$,

$$|W_{\text{Cauchy}}|(e^{i\phi} f) = |e^{i\phi}|\,|W_{\text{Cauchy}}|f = |W_{\text{Cauchy}}|f.$$

It is then hopeless to try to exactly recover functions from their wavelet transforms modulus: we can at best hope to reconstruct them *up to a global phase.*

Second, we observe that for any functions $f, g \in L^2(\mathbb{R})$ whose Fourier transforms are equal on $\mathbb{R}^+$, since Cauchy wavelets are analytic, f and g have the same wavelet transform, and hence the same wavelet transform modulus.

To avoid this source of non-uniqueness, we restrict our analysis to analytic functions $f \in L^2_+(\mathbb{R})$, and we have the following theorem.

Theorem 3.1. *Let $f, g \in L^2_+(\mathbb{R})$ be such that*

$$|W_{\text{Cauchy}}|f = |W_{\text{Cauchy}}|g.$$

There exists $\phi \in \mathbb{R}$ such that

$$f = e^{i\phi} g.$$

In audio applications, the functions that we encounter are not analytic, but real-valued. Fortunately, we can extend our theorem from analytic functions to real-valued ones.

Corollary 3.2. *Let $f, g \in L^2(\mathbb{R})$ be real-valued. Let g_+ be the analytic part of g, that is,*

$$\forall \omega \in \mathbb{R}, \quad \hat{g}_+(\omega) = \begin{cases} 2\hat{g}(\omega) & \text{if } \omega \geq 0, \\ 0 & \text{if } \omega < 0. \end{cases}$$

If

$$|W_{\text{Cauchy}}|f = |W_{\text{Cauchy}}|g,$$

then there exists $\phi \in \mathbb{R}$ such that

$$f = \text{Re}\left(e^{i\phi} g_+\right).$$

3.2. Proof sketch of Theorem 3.1

The proof of Theorem 3.1 crucially relies on a particular property of Cauchy wavelets: the Cauchy wavelet transform of a function $f \in L^2(\mathbb{R})$ is the restriction to a set of horizontal lines of (a variant of) the holomorphic extension of f to the complex upper half plane. This property allows us to apply to our problem classical harmonic analysis techniques, as has also been done in [1, 7].

More specifically, if we denote by $\mathbb{H}$ the Poincaré half-plane,

$$\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\},$$

then, for any $f \in L^2_+(\mathbb{R})$, we can define a holomorphic function

$$\begin{array}{rccc} F & : & \mathbb{H} & \to & \mathbb{C} \\ & & z & \to & \frac{1}{2\pi}\int_{\mathbb{R}} \omega^p \hat{f}(\omega) e^{i\omega z} d\omega. \end{array} \tag{3.1}$$

When f is sufficiently regular, and p is an integer, F is the holomorphic extension to the Poincaré half-plane of the p-th derivative of f (up to multiplication by i^p).

Proposition 3.3. *Let $f \in L^2_+(\mathbb{R})$ be any analytic function. We define F as in Equation (3.1). Then,*

$$\forall j \in \mathbb{Z}, \forall x \in \mathbb{R}, \quad f \star \psi_j^{\text{Cauchy}}(x) = 2^{jp} F(x + i2^j).$$

Using the previous proposition, we can rephrase our problem in terms of holomorphic functions. Indeed, let $f, g \in L^2_+(\mathbb{R})$ be two functions. Let F, G be defined as in Equation (3.1). From the proposition, the following two properties are equivalent:

(1) $|W_{\text{Cauchy}}|f = |W_{\text{Cauchy}}|g$;

(2) $\forall j \in \mathbb{Z}, \forall x \in \mathbb{R}, |F(x + i2^j)| = |G(x + i2^j)|$.

With standard tools from harmonic analysis, notably the decomposition into Blaschke products, it can be shown that, when two holomorphic functions $F, G : \mathbb{H} \to \mathbb{C}$ (satisfying a simple condition) have the same modulus on at least two horizontal lines, they are equal up to a global phase. This proves Theorem 3.1. More details can be found in [9].

Remark 3.4. *From our numerical experiments, we expect Theorem 3.1 to hold for relatively general families of wavelets, and not only for the Cauchy family. However, since Proposition 3.3 only holds for Cauchy wavelets, the proof does not extend to more general families.*

4. Local stability for Cauchy wavelets

In the previous section, we have seen that any analytic function is uniquely determined by its Cauchy wavelet transform modulus, up to a global phase. In applications, we never have an exact knowledge of the modulus, so we also need to understand which signals have "almost equal" wavelet transform modulus (instead of "exactly equal").

In this section, we first describe a method to construct pairs of functions with almost the same wavelet transform modulus (Paragraph 4.1). This method consist in multiplying the wavelet transform of an arbitrary function by "slow-varying" phases. Then we prove a partial converse result: any two functions with almost the same wavelet transform modulus have wavelet transforms that are equal up to multiplication by "slow-varying" phases, except around points where the modulus is small (Paragraph 4.2).

As in the previous section, we assume the wavelets to be Cauchy. The construction described in the first paragraph does not necessitate it, but the converse statement does.

4.1. Functions with almost identical wavelet transform modulus

First, let us see how to construct functions $f, g \in L^2_+(\mathbb{R})$ such that

$$||\,|W_{\text{Cauchy}}|f - |W_{\text{Cauchy}}|g||_2 \ll 1 \quad \text{but} \quad \inf_{\phi\in\mathbb{R}} ||f - ge^{i\phi}||_2 \gtrsim 1.$$

Here, we have defined $||\,|W_{\text{Cauchy}}|f - |W_{\text{Cauchy}}|g||_2$ by

$$||\,|W_{\text{Cauchy}}|f - |W_{\text{Cauchy}}|g||_2^2 = \sum_{j\in\mathbb{Z}} ||\,|f \star \psi_j^{\text{Cauchy}}| - |g \star \psi_j^{\text{Cauchy}}|\,||_2^2.$$

Let us fix any function $f \in L^2(\mathbb{R})$ such that $||f||_2 = 1$. Let us choose functions $(\phi_j)_{j\in\mathbb{Z}} \in \left(\mathcal{C}^1(\mathbb{R},\mathbb{R})\right)^{\mathbb{Z}}$, such that $(j,t) \in \mathbb{Z}\times\mathbb{R} \to \phi_j(t)$ varies "slowly": for some $\epsilon > 0$,

$$\forall j \in \mathbb{Z}, \quad 2^j||\phi_j'||_\infty < \epsilon \quad \text{and} \quad ||\phi_j - \phi_{j+1}||_\infty < \epsilon.$$

We set

$$g = W^\dagger\left(\left((f\star\psi_j^{\text{Cauchy}})e^{i\phi_j}\right)_{j\in\mathbb{Z}}\right) = \sum_{j\in\mathbb{Z}}\left((f\star\psi_j^{\text{Cauchy}})e^{i\phi_j}\right)\star\tilde{\psi}_j^{\text{Cauchy}}.$$

Let us recall that $W^\dagger$ is the pseudo-inverse of W, and $(\tilde{\psi}_j^{\text{Cauchy}})_{j\in\mathbb{Z}}$ is the family of dual wavelets (Equation (2.1)).

Because $(j,t) \to \phi_j(t)$ varies slowly, the multiplication by $(j,t) \to (e^{i\phi_j(t)})$ approximately commutes with the operator $WW^\dagger$, so for any $j \in \mathbb{Z}$,

$$g\star\psi_j^{\text{Cauchy}} \approx (f\star\psi_j^{\text{Cauchy}})e^{i\phi_j} \quad \Rightarrow \quad |g\star\psi_j^{\text{Cauchy}}| \approx |f\star\psi_j^{\text{Cauchy}}|. \tag{4.1}$$

More precisely,

$$||\,|W_{\text{Cauchy}}|f - |W_{\text{Cauchy}}|g||_2 = O(\epsilon). \tag{4.2}$$

However, unless the function $(j,t) \in \mathbb{Z}\times\mathbb{R} \to \phi_j(t)$ is approximately constant on a subset of $\mathbb{Z}\times\mathbb{R}$ where most of the norm of $(f\star\psi_j^{\text{Cauchy}})_{j\in\mathbb{Z}}$ is concentrated,

$$\inf_{\phi\in\mathbb{R}} ||f - ge^{i\phi}||_2 \text{ is of the order of 1.} \tag{4.3}$$

To summarize, we have seen that the wavelet transform of any function f, multiplied by slow-varying phases, is approximately equal to the wavelet transform of another function g (Equation (4.1)). Functions f and g may not be approximately equal, but they have almost the same wavelet transform modulus (Equations (4.2) and (4.3)).

4.2. Local stability theorem

Conversely, when functions $f, g \in L^2_+(\mathbb{R})$ have almost the same wavelet transform modulus, $W_{\text{Cauchy}}g$ is approximately equal to $W_{\text{Cauchy}}f$ up to multiplication by slow-varying phases, except maybe in regions where $W_{\text{Cauchy}}f$ has values close to zero.

Theorem 4.1 (Simplified version of [9, Thm 5.1]).
Let $f, g \in L^2_+(\mathbb{R})$ be two non-zero analytic functions. For any $j \in \mathbb{Z}$, we define

$$N_j = \max\left(||f \star \psi_j^{\text{Cauchy}}||_\infty, ||g \star \psi_j^{\text{Cauchy}}||_\infty \right).$$

Let $\epsilon, c \in]0;1[$ such that $c \geq \epsilon$ be fixed. Let $M > 0$ be fixed. We assume that, for some $j_0 \in \mathbb{Z}$,

$$\sup_{x\in[-M2^{j_0};M2^{j_0}]} \left| |f \star \psi_{j_0}^{\text{Cauchy}}(x)| - |g \star \psi_{j_0}^{\text{Cauchy}}(x)| \right| \leq \epsilon N_{j_0}; \tag{4.4a}$$

$$\sup_{x\in[-M2^{j_0};M2^{j_0}]} \left| |f \star \psi_{j_0+1}^{\text{Cauchy}}(x)| - |g \star \psi_{j_0+1}^{\text{Cauchy}}(x)| \right| \leq \epsilon N_{j_0+1}; \tag{4.4b}$$

and

$$\inf_{x\in[-M2^{j_0};M2^{j_0}]} \min\left(|f \star \psi_{j_0}^{\text{Cauchy}}(x)|, |g \star \psi_{j_0}^{\text{Cauchy}}(x)| \right) \geq c N_{j_0}; \tag{4.5a}$$

$$\inf_{x\in[-M2^{j_0};M2^{j_0}]} \min\left(|f \star \psi_{j_0+1}^{\text{Cauchy}}(x)|, |g \star \psi_{j_0+1}^{\text{Cauchy}}(x)| \right) \geq c N_{j_0+1}. \tag{4.5b}$$

Then

$$\sup_{x\in\left[-\frac{M}{2}2^{j_0};\frac{M}{2}2^{j_0}\right]} \left| \frac{f \star \psi_{j_0+2}^{\text{Cauchy}}}{f \star \psi_{j_0+1}^{\text{Cauchy}}}(x) - \frac{g \star \psi_{j_0+2}^{\text{Cauchy}}}{g \star \psi_{j_0+1}^{\text{Cauchy}}}(x) \right| \leq A \left(\frac{N_{j_0-1}}{N_{j_0+1}} \right)^{4/3} \epsilon^\alpha, \tag{4.6}$$

where A is a constant that depends only on c, and α is a constant that depends only on M, and is strictly positive when M is large enough.

At first sight, it might not be obvious why Equation (4.6) implies what we previously said: if $|W_{\text{Cauchy}}|f \approx |W_{\text{Cauchy}}|g$, then $W_{\text{Cauchy}}f \approx W_{\text{Cauchy}}g$ up to multiplication by slow-varying phases, in regions of $\mathbb{Z} \times \mathbb{R}$ where $W_{\text{Cauchy}}f$ does not take values close to zero.

But observe that, when Equations (4.4a) and (4.4b) are satisfied for all values of j, and not only for $j = j_0, j_0+1$, then, provided that Equations (4.5a)-(4.5b) hold for all $j \in \{j_0-1, j_0, \dots, j_0+K\}$, for some $K \in \mathbb{N}^*$, Equation (4.6) tells us that

$$\begin{aligned} &\forall j \in \{j_0, \dots, j_0+K-1\}, \quad \frac{f \star \psi_{j+1}^{\text{Cauchy}}}{f \star \psi_j^{\text{Cauchy}}} \approx \frac{g \star \psi_{j+1}^{\text{Cauchy}}}{g \star \psi_j^{\text{Cauchy}}}; \\ &\Rightarrow \quad \text{phase}\left(\frac{f \star \psi_j^{\text{Cauchy}}}{g \star \psi_j^{\text{Cauchy}}} \right) \approx \text{phase}\left(\frac{f \star \psi_{j+1}^{\text{Cauchy}}}{g \star \psi_{j+1}^{\text{Cauchy}}} \right). \end{aligned} \tag{4.7}$$

Combining these approximate equalities from $j = j_0$ to $j = j_0 + K - 1$, we get

$$\text{phase}\left(\frac{f \star \psi_{j_0}^{\text{Cauchy}}}{g \star \psi_{j_0}^{\text{Cauchy}}} \right) \approx \text{phase}\left(\frac{f \star \psi_{j_0+K}^{\text{Cauchy}}}{g \star \psi_{j_0+K}^{\text{Cauchy}}} \right). \tag{4.8}$$

Hence, from Equation (4.7), $(j,t) \to \text{phase}\left(\frac{f\star\psi_j^{\text{Cauchy}}}{g\star\psi_j^{\text{Cauchy}}}(t) \right)$ varies slowly in j when j stays in $\{j_0, \dots, j_0+K-1\}$.

It also varies slowly in t, at least when $j = j_0$: $f\star\psi_{j_0+K}^{\text{Cauchy}}$ and $g\star\psi_{j_0+K}^{\text{Cauchy}}$ have lower characteristic frequencies than $f \star \psi_{j_0}^{\text{Cauchy}}$, and satisfy Equation (4.5a), so $\text{phase}(f \star \psi_{j_0+K}^{\text{Cauchy}} / g \star \psi_{j_0+K}^{\text{Cauchy}})$ varies slowly, compared to $f \star \psi_{j_0}^{\text{Cauchy}}$ and $g \star \psi_{j_0}^{\text{Cauchy}}$. Equation (4.8) then implies that

$$\text{phase}\left(\frac{f \star \psi_{j_0}^{\text{Cauchy}}}{g \star \psi_{j_0}^{\text{Cauchy}}} \right) \text{ varies slowly compared to } f \star \psi_{j_0}^{\text{Cauchy}}, g \star \psi_{j_0}^{\text{Cauchy}}.$$

Related results can be found in [2] and [6].

5. Scattering transform

For complex data analysis tasks, more sophisticated representations than the wavelet transform modulus are generally required. Towards this goal, [8] has introduced the scattering transform, that is computed by iteratively applying to a function several wavelet transform modulus.

Our long-term goal is to study the same question for the scattering transform as we did for the wavelet transform modulus: to what extent is it possible to reconstruct a function from its scattering transform? Since this question is a priori difficult, we simply present a first step in this direction: we show that there is a relation between the decay of the Fourier transform of a signal, and the decay of its scattering transform.

Paragraph 5.1 contains the definition of the scattering transform. The main result is in Paragraph 5.2. In this section, we do not need to assume that the wavelets are Cauchy wavelets. We do not even need them to be analytic anymore.

5.1. Definition

Let us assume that a family of wavelets $(\psi_j)_{j\in\mathbb{Z}}$, as defined in Section 2, is fixed, as well as some $J \in \mathbb{Z}$. Let us also fix some low-pass filter $\phi_J \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, that is, a function with real positive values, such that

$$\int_{\mathbb{R}} \phi_J(t)dt = 1.$$

Let us now consider any function $f \in L^2(\mathbb{R})$. The 0*-th order scattering coefficient* of f is defined as

$$f \star \phi_J.$$

Intuitively, $f \star \phi_J$ is a "blurred" version of f. It is insensitive to small modifications of f (notably small deformations), which is good for applications, but does not contain much information about middle or high frequencies of f. We recover this information by computing, in addition to $f \star \phi_J$, the wavelet transform modulus of f: $|W|f = (|f \star \psi_j|)_{j\in\mathbb{Z}}$. Since we are only interested in recovering the high-frequency content of f, we discard the components corresponding to frequencies lower than 2^{-J}, which yields a truncated wavelet transform modulus $(|f \star \psi_j|)_{j\leq J}$.

If we convolve all functions of this latter set with the low-pass filter, we obtain the 1*st order scattering coefficients* of f:

$$(|f \star \psi_j| \star \phi_J)_{j\leq J}.$$

Because of the convolution with ϕ_J, these coefficients are also robust to small deformations of f, and, compared to the 0-th order coefficients, they contain some information on the middle and high frequencies of f.

Iterating this process leads to the following definition, illustrated by Figure 5.1.

Definition 5.1. *Let*

$$\mathcal{P} = \{(j_1, \dots, j_k), k \in \mathbb{N}, j_1 \leq J, \dots, j_k \leq J\}.$$

For any $p = (j_1, \dots, j_k) \in \mathcal{P}$, we call k the length of p, and denote it by $|p|$.

Let $f \in L^2(\mathbb{R})$ be any function.

For any $p = (j_1, \dots, j_k) \in \mathcal{P}$, we define, if $|p| > 0$,

$$S[p]f = |\dots||f \star \psi_{j_1}| \star \psi_{j_2}| \cdots \star \psi_{j_k}| \star \phi_J,$$

and, if $|p| = 0$,

$$S[p]f = f \star \phi_J.$$

The scattering transform of f is

$$Sf = (S[p]f)_{p\in\mathcal{P}}.$$

Since its introduction in [8], the scattering transform has found successful applications in, for example, audio genre, visual textures or medical data classification [3, 11, 12].

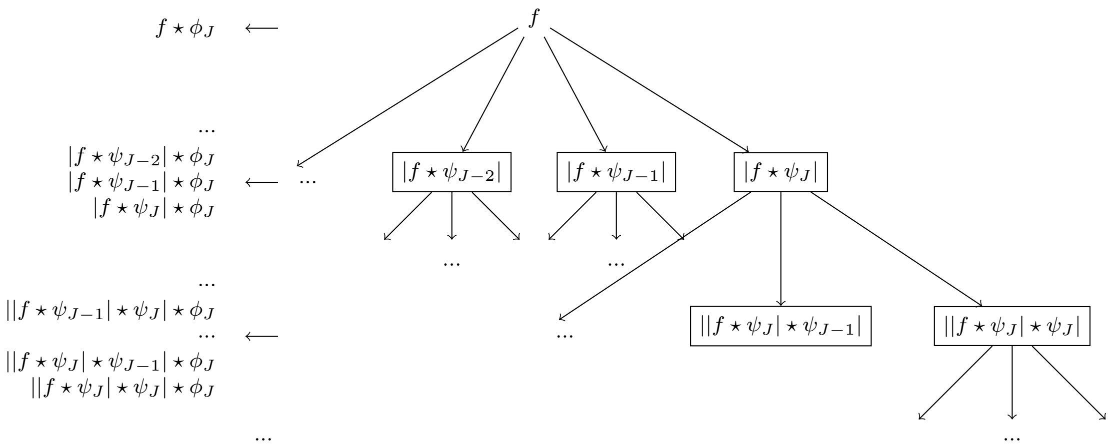

Figure 5.1. Schematic representation of the scattering transform of a function $f \in L^2(\mathbb{R})$.

5.2. Decay of scattering coefficients

To what extent can we reconstruct $f \in L^2(\mathbb{R})$ from Sf?

This problem is a priori more involved as in the case of the wavelet transform modulus. Instead of directly solving it, we can first try to determine a few properties of f that can be recovered from Sf, as well as a few properties of f that cannot be recovered from Sf.

From [8], we already know that, under suitable conditions on the wavelet family, the scattering transform preserves the L^2-norm:

$$\forall f \in L^2(\mathbb{R}), \quad \sum_{p\in\mathcal{P}} ||S[p]f||_2^2 = ||f||_2^2.$$

On the other hand, it is insensitive to small deformations: if $T \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$ is such that $||T' - 1||_\infty$ is "small", then

$$\forall f \in L^2(\mathbb{R}), \quad Sf \approx S(f \circ T).$$

(We refer to [8] for a formal statement.)

The next theorem follows this line of work: it describes a relation between the decay of the Fourier transform of f, and the decay in norm of scattering coefficients as a function of the order. Its interest is twofold. First, it shows that the scattering transform contains some information on the smoothness of the initial signal: non-negligible high-order scattering coefficients are the sign of a highly irregular function f.

Second, it explains a phenomenon observed in numerical experiments: for the classes of signals that appear in most applications, for typical families of wavelets and values of J, the norm of scattering coefficients decays very rapidly, to the point that only zero-th, first- and second-order coefficients are usable. This fact implies that further studies of the scattering transform should probably focus on these orders only, which avoids considering the whole cascade, as could have been expected from Definition 5.1.

Theorem 5.2 ([13, Thm III.2]). *We assume that the family of wavelets is such that there exists* $c_0 > 0$ *satisfying*

$$\forall \omega \in \mathbb{R}, \quad c_0 \leq |\hat{\phi}_J(\omega)|^2 + \frac{1}{2}\sum_{j\leq J}\left(|\hat{\psi}_j(\omega)|^2 + |\hat{\psi}_j(-\omega)|^2\right) \leq 1.$$

Additionally, we assume that there exists $\epsilon > 0$ *such that*

$$|\hat{\psi}(\omega)| = O(|\omega|^{1+\epsilon}) \text{ when } \omega \to 0.$$

Then there exists $a_0 > 0, c > 1$ such that, for any real-valued function $f \in L^2(\mathbb{R})$, and any $n \geq 2$,

$$\sum_{\substack{p\in\mathcal{P}\\|p|\geq n}} ||S[p]f||_2^2 \leq \int_{\mathbb{R}} |\hat{f}(\omega)|^2 \left(1 - \frac{1}{\left(1+\sqrt{\left|\frac{2^J\omega}{c^n a_0}\right|}\right)^4} \right) d\omega. \tag{5.1}$$

To understand this theorem, let us note that

$$\omega \to 1 - \frac{1}{\left(1+\sqrt{\left|\frac{2^J\omega}{c^n a_0}\right|}\right)^4}$$

is a high-pass filter with cut-off frequency of the order of $2^{-J}c^n a_0$. So Equation (5.1) means that the norm of scattering coefficients with order at least n is (essentially) upper bounded by the norm of $\hat{f}$, restricted to frequencies of order of at least $2^{-J}c^n a_0$.

From numerical experiments, it seems that for the wavelet families that we usually consider, a_0 is typically of the order of 1, and c of the order of 2.

As a corollary, as soon as the Fourier transform of f decays at least as fast as $O\left(|\omega|^{-1}\right)$, the norm of scattering coefficients decays exponentially fast when their order increases.

Corollary 5.3. *For any real-valued function $f \in L^2(\mathbb{R})$, if there exists $C > 0$ such that*

$$\forall \omega \in \mathbb{R}, \quad |\hat{f}(\omega)| \leq \frac{C}{\sqrt{1+\omega^2}},$$

then $\left(\sum_{|p|\geq n} ||S[p]f||_2^2\right)_{n\in\mathbb{N}}$ decays exponentially fast.

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